Wavelet-Galerkin solution of some ordinary differential equations

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***Abstract*—In this paper we apply the wavelet-Galerkin method to some special types of Sturm-Liouville differential equation. We use a scaling function that allows us to find the numerical solutions of nonhomogeneous differential equation such as Van der Pol equation.**

I. Introduction

Wavelets have generated significant interest from both theoretical and applied researchers over the last few decades. The concepts for understanding wavelets were provided by Meyer, Mallat, Daubechies, and many others, [1], [2]. Since then, the number of applications where wavelets have been used has exploded. In areas such as time-series analysis, approximation theory and numerical solutions of differential equations, wavelets are recognized as powerful weapons not just tools.

In general it is not always possible to obtain exact solution of an arbitrary differential equation. This necessitates either discretization of differential equations leading to numerical (approximate) solutions, or their qualitative study which is concerned with deduction of important properties of the solutions without actually solving them. The Galerkin method is one of the best known methods for finding numerical solutions of ordinary and partial differential equations. Its simplicity makes it perfect for many applications. The wavelet-Galerkin method is an improvement over the standard Galerkin methods by using a compactly supported orthogonal functional basis, [4], [5], [6], [8]. The translates of a wavelet for all dilations form an unconditional orthonormal basis of $L^2(R)$ and the translates of a scaling function for all dilations form an unconditional orthonormal bases for $V_j \subset L^2(R)$, which is a great improvement over the standard polynomial basis or a trigonometric basis which not necessary have to be unconditional. In many cases, the wavelets also provide better basis of the approximation spaces than other basis in the following sense. First, the representations of the differential operators are almost diagonal on the wavelet bases, that improves the conditioning of the discrete algebraic equations. Second, the wavelet representations are effective in the adaptive procedures so that the complexity of calculations can be reduced. Furthermore, when the solution has a certain singularity, its wavelet representation can automatically capture the singularity.

In this paper we apply the wavelet-Galerkin method to some special types of Sturm-Liouville differential equation. We use a scaling function that allows us to find the numerical solutions of nonhomogeneous differential equation such as Van der Pol equation.

This article is organized as follows. Section II is of preliminary character; we describe the spaces of functions that we use throughout this paper; we also recall some basic wavelet tools such as multiresolution analysis (MRA) and describe the classical Galerkin method for numerical solving of ordinary differential equations. In Section III we apply the wavelet-Galerkin method to the one dimensional second order differential equation $u''(t) + \alpha u(t) = f(t)$, $t \in [0, 1]$, α is a constant, with Dirichlet boundary conditions $u(0) = u(1) = 0$. We solve a differential equation of this type, whose exact solution is known and compute the absolute error of our numerical solution. We also applied the wavelet-Galerkin method to the differential equation of the form $u''(t) + g(t)u'(t) = f(t)$, $t \in [0, 1]$ with the same Dirichlet boundary conditions $u(0) = u(1) = 0$, where $g(t)$ and $f(t)$ are a real-valued continuous functions on $[0, 1]$. After that, we find a numerical solution of the known Van der Pol equation which depends on one parameter and which does not have analytic solution.

II. Preliminaries and Notations

A. *Spaces of functions*

$L^2(R)$ is the Hilbert space of square integrable functions on R with the inner product

$$< f, g >= \int_R f(t)\overline{g(t)}dt,$$

where $\overline{g(t)}$ is a complex conjugate of $g(t)$. The Hilbert space of square integrable functions on $[0, 1]$ with the inner product

$$< f, g >= \int_0^1 f(t)\overline{g(t)}dt,$$

is denoted by $L^2([0, 1])$. The space of twice differentiable functions on $[0, 1]$ is denoted by $C^2([0, 1])$.

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B. Galerkin method for ordinary differential equations

We consider the class of ordinary differential equations (known as *Sturm-Liouville equations*) of the form

$$Lu(t) \equiv -\frac{d}{dt}\Big(a(t)\frac{du}{dt}\Big) + b(t)u(t) = f(t),\ 0 \le t \le 1 \quad (1)$$

with Dirichlet boundary conditions

$$u(0) = u(1) = 0. \quad (2)$$

Let $a(t), b(t)$ and $f(t)$ be a real-valued functions, such that $f(t)$ and $b(t)$ are continuous functions and the function $a(t)$ has a continuous derivative on $[0, 1]$.

For the Galerkin method, we suppose that $\{v_j\}$ is a complete orthonormal system (orthonormal basis) for $L^2([0, 1])$, and that every v_j is $C^2([0, 1])$ function that satisfies

$$v_j(0) = v_j(1) = 0.$$

We select some finite set Λ of indices j and consider the subspace

$$S = span\{v_j, j \in \Lambda\},$$

i.e. the set of all finite linear combination of the elements $\{v_j\}$, $j \in \Lambda$.

We look for an approximation u_s of the exact solution u of the equation (1) in the form

$$u_s = \sum_{k\in\Lambda} x_k v_k \in S \quad (3)$$

where the coefficients x_k, $k \in \Lambda$ are unknown. Our criterion for determining the coefficients x_k is that u_s should behave like the true solution u on the subspace S, i.e.

$$< Lu_s, v_j >=< f, v_j >, \forall j \in \Lambda. \quad (4)$$

If we substitute the equation (3) in (4) we obtain

$$\sum_{k\in\Lambda} \langle Lv_k, v_j\rangle x_k = \langle f, v_j\rangle, \forall j \in \Lambda. \quad (5)$$

Let X denote the vector $(x_k)_{k\in\Lambda}$ and let Y be the vector $(y_k)_{k\in\Lambda}$ where $y_k = \langle f, v_k\rangle$. Let $A = [a_{j,k}]_{j,k\in\Lambda}$ where $a_{j,k} = \langle Lv_k, v_j\rangle$. Thus, (5) is a linear system of equations

$$\sum_{k\in\Lambda} a_{j,k}x_k = y_j,\ j \in \Lambda, \quad (6)$$

or

$$AX = Y. \quad (7)$$

For each subset Λ we obtain an approximation $u_s \in S$ to the true solution u by solving the linear system (7) for X, and then we determine u_s by equation (3).

C. Wavelets

Let $\psi_{a,b}, a > 0, b \in R$ be a family of functions defined as translation (or shifting) by a factor b and dilatation (or scaling) by a factor a of a function $\psi \in L^2(R)$

$$\psi_{a,b}(t) = \frac{1}{\sqrt{a}}\psi\left(\frac{t-b}{a}\right).$$

The function ψ (called a *wavelet* or a *mother wavelet*) is assumed to satisfy the admissibility condition

$$C_\psi = \int_{-\infty}^{\infty} \frac{|\hat{\psi}(\omega)|^2}{|\omega|}d\omega < \infty,$$

where $\hat{\psi}(\omega)$ is the Fourier transform of $\psi(t)$. The admissibility condition implies that

$$\hat{\psi}(0) = \int_{-\infty}^{\infty} \psi(t)dt = 0. \quad (8)$$

One can prove that, if $\int_{-\infty}^{\infty} \psi(t)dt = 0$ and $\int_{-\infty}^{\infty}(1 + |t|^\alpha)|\psi(t)|dt < \infty$ for some $\alpha > 0$, then $C_\psi < \infty$.

In most situations, it is usefully to restrict ψ to be well localized both in time and frequency domains, i.e. $\psi(t)$ and its derivatives must decay very rapidly. For frequency localization, $\hat{\psi}(\omega)$ must decay sufficiently fast as $|\omega| \to \infty$ and $\hat{\psi}(\omega)$ must become flat in the neighborhood of $\omega = 0$. The flatness is associated with the number of vanishing moments of $\psi(t)$ since

$$\int_{-\infty}^{\infty} t^k\psi(t)dt = 0 \Leftrightarrow \hat{\psi}^{(k)}(0) = 0 \quad (9)$$

for $k = 0, 1, ..., n$. It means that larger number of vanishing moments more is the flatness when ω is small.

D. Multiresolution analysis

The notion of multiresolution analysis (MRA) was introduced in 1988/89 by Mallat and Meyer as a natural approach to the wavelet orthonormal basis. One can easily obtain a wavelet basis associated to the particular multiresolution approximation as follows.

A *multiresolution analysis* of the space $L^2(R)$ consists of a sequence of closed subspaces $\{V_j\}_{j=-\infty}^{\infty}$ (called approximation spaces) with the following properties:

1. $V_j \subset V_{j+1}$
2. $\overline{\cup_{j\in Z}V_j} = L^2(R)$
3. $\cap_{j\in Z}V_j = \{0\}$
4. $f(t) \in V_j \Leftrightarrow f(2t) \in V_{j+1}$
5. $f(t) \in V_j \Leftrightarrow f(t-k) \in V_j, \forall k \in Z$
6. there exists a function ϕ (called *scaling function* or *father wavelet*) such that $\phi_{j,k}(t) = 2^{j/2}\phi(2^jt - k)$, $k \in Z$ constitute orthonormal basis for corresponding subspace V_j.

Let $\phi \in L^2(R)$ be compactly supported scaling function of MRA (the support of a function is the closure of the set of points where the function is not zero-valued). Then,

$$\int_{-\infty}^{\infty} \phi(t)dt \neq 0, \quad (10)$$

and satisfies a dilation equation

$$\phi(t)=\sum_{k\in Z}a_k\phi(2t-k) \tag{11}$$

where a_k are real coefficients and $a_k\neq 0$ for only finitely many $k\in Z$ (the number of nonzero coefficients a_k in the series (11) is denoted by L). Since $\phi_{j,k}(t)=2^{j/2}\phi(2^jt-k)$, $j,k\in Z$ are orthonormal in $L^2(R)$, we have

$$\int_{-\infty}^{\infty}\phi(t-n)\phi(t-k)dt=\delta_{k,n}, \tag{12}$$

where $\delta_{n,k}$ is the Kronecker delta function defined by

$$\delta_{n,k}=\begin{cases}0, & n\neq k\\ 1, & n=k\end{cases}. \tag{13}$$

If $\phi\in L^2(R)$ is compactly supported scaling function of MRA, one can construct the wavelet ψ such that $\psi_{j,k}(t)=2^{j/2}\phi(2^jt-k)$, $j,k\in Z$ constitute an orthonormal basis for $L^2(R)$.

In 1988, Ingrid Daubechies defined scaling function as

$$\phi(t)=\sum_{k=0}^{L-1}a_k\phi(2t-k), \tag{14}$$

where L is a positive even integer and denotes the genus of the Daubechies wavelet. The functions generated with the coefficients a_k have support $[0,L-1]$ and $(\frac{L}{2}-1)$ vanishing wavelet moments. The wavelet function $\psi(x)$ is given by

$$\psi(t)=\sum_{k=2-L}^{1}(-1)^k a_{1-k}\phi(2t-k). \tag{15}$$

Daubechies wavelets are compactly supported functions and therefore there are useful for representing the solution of differential equation with boundary conditions. A complete wavelet theory can be found in [1], [2], [7], [9], [11].

III. Wavelet-Galerkin method for ordinary differential equations

We will consider two special types of Sturm-Liouville differential equation (1). The first one is of the form

$$u''(t)+\alpha u(t)=f(t),\quad t\in[0,1], \tag{16}$$

with Dirichlet boundary conditions

$$u(0)=u(1)=0, \tag{17}$$

where α is a constant and $f(t)$ is a continuous function on $[0,1]$. This equation is the one dimensional counterpart of Helmholtz's equation that arises in many problems of electromagnetic radiation, seismology and acoustics.

Let the solution $u(t)$ of the equation (16) be approximated by its j-th level scaling function expansion on the interval $(0,1)$

$$u_j(t)=\sum_{k=1-L}^{2^j}c_k\phi_{j,k}(t), k\in Z, \tag{18}$$

where ϕ is a scaling function of MRA and c_k are unknown coefficients that should be determined. It is clear that the larger integer j is used, the more accurate solution is obtained.

The boundaries of the support of $u_j(t)$ given by (18) are $\dfrac{1-L}{2^j}$ and $\dfrac{L-1+2^j}{2^j}$. Subsequently, the original boundaries 0 and 1 are now changed to fictitious boundaries, i.e. the boundaries on both sides of 0 and 1 are extended by an amount $\dfrac{L-1}{2^j}$ without affecting the solution within $[0,1]$, so the affected solution is within the intervals $\left[\dfrac{1-L}{2^j},0\right]$ and $\left[1,\dfrac{L-1+2^j}{2^j}\right]$.

In this paper, we do not use the Daubechies scaling functions (the explanation is given in Remark 1). We will work with the function

$$\phi(t)=\begin{cases}\frac{1}{6}(2+t)^3, & t\in[-2,-1]\\ \frac{1}{6}(4-6t^2-3t^3), & t\in[-1,0]\\ \frac{1}{6}(4-6t^2+3t^3), & t\in[0,1]\\ \frac{1}{6}(2-t)^3, & t\in[1,2]\\ 0, & t\notin[-2,2]\end{cases}. \tag{19}$$

This function satisfies the following dilatation equation

$$\phi(t)=\frac{1}{8}\phi(2t+2)+\frac{1}{2}\phi(2t+1)+\frac{3}{4}\phi(2t)+\frac{1}{2}\phi(2t-1)+\frac{1}{8}\phi(2t-2),$$

so we conclude that $L=5$. We take $j=0$ and look for the solution u_0 of differential equation (16) in the form

$$u_0(t)=\sum_{k=-4}^{1}c_k\phi(t-k), t\in[0,1]. \tag{20}$$

Substitute (20) in (16) we get

$$\frac{d^2}{dt^2}\sum_{k=-4}^{1}c_k\phi(t-k)+\alpha\sum_{k=-4}^{1}c_k\phi(t-k)=f(t). \tag{21}$$

Without any loss of generality, let $\alpha=-1$. Taking inner product with $\phi(t-n)$, $n\in\{-4,-3,-2,-1,0,1\}$ we have

$$\begin{aligned}&\sum_{k=-4}^{1}c_k\int_{\frac{1-L}{2^j}}^{\frac{L-1+2^j}{2^j}}\phi''(t-k)\phi(t-n)dt-\\ -&\sum_{k=-4}^{1}c_k\int_{\frac{1-L}{2^j}}^{\frac{L-1+2^j}{2^j}}\phi(t-k)\phi(t-n)dt=\\ =&\int_{\frac{1-L}{2^j}}^{\frac{L-1+2^j}{2^j}}f(t)\phi(t-n)dt,\end{aligned}$$

i.e.

$$\sum_{k=-4}^{1}c_k\Omega_{n-k}-\sum_{k=-4}^{1}c_k a_{n,k}=b_n, \tag{22}$$

$n\in\{-4,-3,-2,-1,0,1\}$, where

$$\Omega_{n-k}=\int_{-4}^{5}\phi''(t-k)\phi(t-n)dt, \tag{23}$$

$$a_{n,k}=\int_{-4}^{5}\phi(t-k)\phi(t-n)dt, \tag{24}$$

$$b_n = \int_{-4}^{5} f(t)\phi(t-n)dt. \tag{25}$$

We should note that we work only with the scaling function ϕ and not the actual wavelet ψ. The problem can arise from the formula (25) for the coefficients b_n if $f(t)$ is a polynomial since the wavelet ψ has $(\frac{L}{2}-1)$ vanishing moments, (9). So, according to (10), it is much more suitable to work with the scaling function.

By using Dirichlet boundary conditions (17), we obtain the next two equations

$$u_0(0) = \sum_{k=-4}^{1} c_k\phi(-k) = 0, \text{ i.e.}$$

$$c_{-4}\phi(4)+c_{-3}\phi(3)+c_{-2}\phi(2)+c_{-1}\phi(1)+c_0\phi(0)+c_1\phi(-1) = 0, \tag{26}$$

and

$$u_0(1) = \sum_{k=-4}^{1} c_k\phi(1-k) = 0, \text{ i.e.}$$

$$c_{-4}\phi(5)+c_{-3}\phi(4)+c_{-2}\phi(3)+c_{-1}\phi(2)+c_0\phi(1)+c_1\phi(0) = 0. \tag{27}$$

The equations (26) and (27) give the relation between the coefficients c_k, $k \in \{-4,-3,-2,-1,0,1\}$. Now, we eliminate the first and the last equation of a system (22) and in that places equations (26) and (27) are included respectively. So, we get the following matrix equation

$$TC = B, \tag{28}$$

where

$$T = \begin{bmatrix} \phi(4) & \phi(3) & \phi(2) & \phi(1) & \phi(0) & \phi(-1) \\ \Omega_1 - a_{-3,-4} & \Omega_0 - a_{-3,-3} & \Omega_{-1} - a_{-3,-2} & \Omega_{-2} - a_{-3,-1} & \Omega_{-3} - a_{-3,0} & \Omega_{-4} - a_{-3,1} \\ \Omega_2 - a_{-2,-4} & \Omega_1 - a_{-2,-3} & \Omega_0 - a_{-2,-2} & \Omega_{-1} - a_{-2,-1} & \Omega_{-2} - a_{-2,0} & \Omega_{-3} - a_{-2,1} \\ \Omega_3 - a_{-1,-4} & \Omega_2 - a_{-1,-3} & \Omega_1 - a_{-1,-2} & \Omega_0 - a_{-1,-1} & \Omega_{-1} - a_{-1,0} & \Omega_{-2} - a_{-1,1} \\ \Omega_4 - a_{0,-4} & \Omega_3 - a_{0,-3} & \Omega_2 - a_{0,-2} & \Omega_1 - a_{0,-1} & \Omega_0 - a_{0,0} & \Omega_{-1} - a_{0,1} \\ \phi(5) & \phi(4) & \phi(3) & \phi(2) & \phi(1) & \phi(0) \end{bmatrix}$$

and

$$C = \begin{bmatrix} c_{-4} \\ c_{-3} \\ c_{-2} \\ c_{-1} \\ c_0 \\ c_1 \end{bmatrix}, B = \begin{bmatrix} 0 \\ b_{-3} \\ b_{-2} \\ b_{-1} \\ b_0 \\ 0 \end{bmatrix}.$$

By Gaussian elimination algorithm we get the coefficients c_k, $k \in \{-4,-3,-2,-1,0,1\}$ and the approximate solution u_0 using (20).

Remark 1. The authors of [6] used the Daubechies scaling functions for the wavelet-Galerkin method. Because there are no closed-form formulas for the Daubechies wavelets and scaling functions, the element of the matrix B can not be computed. Therefore, the wavelet-Galerkin method based on the Daubechies scaling functions can be applied only to a homogeneous differential equations when $f(t) = 0$. In that case B is a null-matrix and $a_{n,k} = \delta_{n,k}$ since the scaling functions $\phi_{j,k}$, $j,k \in Z$ are orthonormal. Our main goal is application of this method to nonhomogeneous differential equations, and therefore we use the function ϕ given by (19).

Example 1. Let us consider the differential equation

$$u''(t) - u(t) = t - 1, \qquad 0 \le t \le 1, \tag{29}$$

with Dirihlet boundary conditions $u(0) = u(1) = 0$.

The exact solution of this equation is

$$u(t) = -\frac{1}{1-e^2}e^t + \frac{e^2}{1-e^2}e^{-t} - t + 1.$$

Using the formulas (23) and (25) we obtain the elements of the matrix T and B respectivly.

$$T = \begin{bmatrix} 0 & 0 & 0 & \frac{1}{6} & \frac{2}{3} & \frac{1}{6} \\ \frac{1}{8} - \frac{397}{1680} & -\frac{2}{3} - \frac{1}{252} & \frac{1}{8} - \frac{397}{1680} & \frac{1}{5} - \frac{1}{42} & \frac{1}{120} - \frac{1}{5040} & 0 \\ \frac{1}{5} - \frac{1}{42} & \frac{1}{8} - \frac{397}{1680} & -\frac{2}{3} - \frac{151}{630} & \frac{1}{8} - \frac{397}{1680} & \frac{1}{5} - \frac{1}{42} & \frac{1}{120} - \frac{1}{5040} \\ \frac{1}{120} - \frac{1}{5040} & \frac{1}{5} - \frac{1}{42} & \frac{1}{8} - \frac{397}{1680} & -\frac{2}{3} - \frac{599}{1260} & \frac{1}{8} - \frac{397}{1680} & \frac{1}{5} - \frac{1}{42} \\ 0 & \frac{1}{120} - \frac{1}{5040} & \frac{1}{5} - \frac{1}{42} & \frac{1}{8} - \frac{397}{1680} & -\frac{2}{3} - \frac{151}{315} & \frac{1}{8} - \frac{397}{1680} \\ 0 & 0 & 0 & 0 & \frac{1}{6} & \frac{2}{3} \end{bmatrix},$$

$$B^T = \left[\ 0\ -\tfrac{7}{60}\ -\tfrac{19}{15}\ -\tfrac{28}{15}\ -1\ 0\ \right].$$

Using the Gaussian elimination algorithm we solve the matrix equation $TC = B$ and get

$$c_{-4} = -\frac{2215101437733534}{43246859986663}, \quad c_{-3} = \frac{499827906129216}{43246859986663},$$

$$c_{-2} = -\frac{463553467007022}{43246859986663}, \quad c_{-1} = \frac{183867171805800}{43246859986663},$$

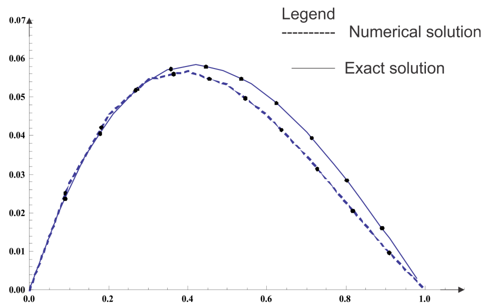
$$c_0 = -\frac{49031245814880}{43246859986663}, \quad c_1 = \frac{12257811453720}{43246859986663}.$$

So the approximate solution is

$$u_0(t) = c_{-1}\phi(t+1) + c_0\phi(t) + c_1\phi(t-1), \quad t \in [0,1].$$

TABLE I
COMPARISON OF RESULTS

Case t	numerical solution u_0	exact solution u	absolute error
0	0	0	0
0.1	0.0276352	0.0265183	0.00111691
0.2	0.0453501	0.0442945	0.00105559
0.3	0.0545619	0.0545074	0.0000544741
0.4	0.0566876	0.0582599	0.00157229
0.5	0.0531447	0.0565906	0.0034459
0.6	0.0453501	0.0504834	0.00513329
0.7	0.0347212	0.0408782	0.00615698
0.8	0.0226751	0.0286795	0.00600449
0.9	0.0106289	0.0147663	0.00413736
1	0	0	0



We now consider the differential equation of the form

$$u''(t) + g(t)u'(t) = f(t), \quad t \in [0,1], \tag{30}$$

with the same Dirichlet boundary conditions

$$u(0) = u(1) = 0, \tag{31}$$

where $g(t)$ and $f(t)$ are continuous functions on $[0,1]$.

At the similar way, we want to find the approximate solution of this equation of the form

$$u_0(t) = \sum_{k=-4}^{1} c_k \phi(t-k), t \in [0,1]. \tag{32}$$

Substitute (32) in (30) we get

$$\frac{d^2}{dt^2} \sum_{k=-4}^{1} c_k \phi(t-k) +$$

$$+ g(t)\frac{d}{dt} \sum_{k=-4}^{1} c_k \phi(t-k) = f(t). \tag{33}$$

Taking the inner product with $\phi(t-n)$, $n \in \{-4,-3,-2,-1,0,1\}$ we obtain

$$\sum_{k=-4}^{1} c_k \Omega_{n-k} + \sum_{k=-4}^{1} c_k d_{n,k} = b_n, \tag{34}$$

$n \in \{-4,-3,-2,-1,0,1\}$, where

$$\Omega_{n-k} = \int_{-4}^{5} \phi^{''}(t-k)\phi(t-n)dt, \tag{35}$$

$$d_{n,k} = \int_{-4}^{5} g(t)\phi'(t-k)\phi(t-n)dt, \tag{36}$$

$$b_n = \int_{-4}^{5} f(t)\phi(t-n)dt. \tag{37}$$

We get the following matrix equation

$$TC = B, \tag{38}$$

where

$$T = \begin{bmatrix} \phi(4) & \phi(3) & \phi(2) & \phi(1) & \phi(0) & \phi(-1) \\ \Omega_1 - d_{-3,-4} & \Omega_0 - d_{-3,-3} & \Omega_{-1} - d_{-3,-2} & \Omega_{-2} - d_{-3,-1} & \Omega_{-3} - d_{-3,0} & \Omega_{-4} - d_{-3,1} \\ \Omega_2 - d_{-2,-4} & \Omega_1 - d_{-2,-3} & \Omega_0 - d_{-2,-2} & \Omega_{-1} - d_{-2,-1} & \Omega_{-2} - d_{-2,0} & \Omega_{-3} - d_{-2,1} \\ \Omega_3 - d_{-1,-4} & \Omega_2 - d_{-1,-3} & \Omega_1 - d_{-1,-2} & \Omega_0 - d_{-1,-1} & \Omega_{-1} - d_{-1,0} & \Omega_{-2} - d_{-1,1} \\ \Omega_4 - d_{0,-4} & \Omega_3 - d_{0,-3} & \Omega_2 - d_{0,-2} & \Omega_1 - d_{0,-1} & \Omega_0 - d_{0,0} & \Omega_{-1} - d_{0,1} \\ \phi(5) & \phi(4) & \phi(3) & \phi(2) & \phi(1) & \phi(0) \end{bmatrix}$$

and

$$C = \begin{bmatrix} c_{-4} \\ c_{-3} \\ c_{-2} \\ c_{-1} \\ c_0 \\ c_1 \end{bmatrix}, B = \begin{bmatrix} 0 \\ b_{-3} \\ b_{-2} \\ b_{-1} \\ b_0 \\ 0 \end{bmatrix}.$$

Example 2. Let us consider the Van Der Pol equation

$$u^{''}(t) + \mu(t^2-1)u^{'}(t) = -t, \quad 0 \le t \le 1, \mu \in R \tag{39}$$

with Dirihlet boundary conditions

$$u(0) = u(1) = 0. \tag{40}$$

The Van der Pol [10] oscillator was originally proposed by the Dutch electrical engineer and physicist Balthasar van der Pol while he was working at Philips. Van der Pol found stable oscillations, which he called relaxation-oscillations and are now known as limit cycles, in electrical circuits employing vacuum tubes. The Van der Pol equation has a long history of being used in both the physical and biological sciences. The equation has also been utilized in seismology to model the two plates in a geological fault.

The goal here will be, in absence of the exact analytic solution, to find numerical solution using the wavelet-Galerkin method. In the limits or small or large values of the parameter μ, the reduced equations are amenable to asymptotic analysis. For the case of large values of the parameter μ (relaxation oscillations) an analytic solution to the problem is provided that is exact up to $O(\mu^{-2})$.

Two interesting regimes for the characteristics of the unforced oscillator are:

1. When $\mu = 0$, i.e. there is no damping function, the equation becomes $u^{''}(t) + t = 0$. This is a form of the simple harmonic oscillator and there is always conservation of energy.
2. When $\mu > 0$, the system will enter a limit cycle. Near the origin $u = du/dt = 0$ the system is unstable, and far from the origin the system is damped.

Using the formulas (35) and (37) we obtain the elements of the matrix T and B respectively.

$$T = \begin{bmatrix} 0 & 0 & 0 \\ \frac{1}{8} - \frac{16553\mu}{5040} & -\frac{2}{3} + \frac{1359\mu}{1120} & \frac{1}{8} + \frac{8203\mu}{3360} \\ \frac{1}{5} - \frac{353\mu}{630} & \frac{1}{8} - \frac{1411\mu}{1120} & -\frac{2}{3} + \frac{302\mu}{315} \\ \frac{1}{120} - \frac{23\mu}{3360} & \frac{1}{5} - \frac{41\mu}{210} & \frac{1}{8} - \frac{1361}{10080} \\ 0 & \frac{1}{120} - \frac{\mu}{672} & \frac{1}{5} + \frac{\mu}{70} \\ 0 & 0 & 0 \end{bmatrix}$$

$$\left.\begin{matrix} \frac{1}{6} & \frac{2}{3} & \frac{1}{6} \\ \frac{1}{5}+\frac{61\mu}{210} & \frac{1}{120}+\frac{\mu}{480} & 0 \\ \frac{1}{8}+\frac{8507\mu}{10080} & \frac{1}{5}+\frac{\mu}{30} & \frac{1}{120}-\frac{\mu}{1120} \\ -\frac{2}{3}+\frac{151\mu}{315} & \frac{1}{8}-\frac{7\mu}{96} & \frac{1}{5}-\frac{43\mu}{630} \\ \frac{1}{8}+\frac{1039\mu}{3360} & -\frac{2}{3} & \frac{1}{8}-\frac{1039\mu}{3360} \\ 0 & \frac{1}{6} & \frac{2}{3} \end{matrix}\right],$$

$$B = \left[\ 0\ \tfrac{113}{40}\ 2\ 1\ 0\ 0\ \right].$$

By Gaussian elimination algorithm we solve the matrix equation $TC = B$ and obtain the coefficients c_k, $k \in \{-4,-3,-2,-1,0,1\}$ as functions of parametar μ.

$$c_{-4} = -\frac{p_{-4}}{q_{-4}},$$

where

$$p_{-4} = 252(8542300253760 - 19415138950032\mu + 13072514451716\mu^2 + 19758040532699\mu^3)$$

and

$$q_{-4} = -124122644448768 + 1319754746705472\mu - 3666921988214352\mu^2 + 797356612349572\mu^3 + 1184632629273253\mu^4 \quad .$$

The other coefficients c_k, $k \in \{-3,-2,-1,0,1\}$ are not given here since their complexity. In a case when $\mu = 0.05$, we obtain $c_{-1} = -2.15117, c_0 = 0.573645$, $c_1 = -0.143411$ and the approximate solution is

$$u_0(t) = c_{-1}\phi(t+1)c_0\phi(t) + c_1\phi(t-1), \quad t \in (0,1).$$

TABLE II
NUMERICAL SOLUTION u_0 OF THE VAN DER POL EQUATION FOR DIFFERENT VALUES OF PARAMETAR μ

Case	μ=0.05	μ=0.1	μ=1
0	0	0	0
0.1	0.0838955	0.218264	0.240269
0.2	0.137675	0.358176	0.394288
0.3	0.16564	0.430931	0.474378
0.4	0.172093	0.44772	0.49286
0.5	0.161338	0.419738	0.462057
0.6	0.137675	0.358176	0.394288
0.7	0.105407	0.274229	0.301877
0.8	0,0688374	0.179088	0.197144
0.9	0.0322675	0.0839475	0.0924113
1	0	0	0

IV. CONCLUSION

As a result of an increased utilization of the wavelet analysis to solving mathematical and engineering problems, the numerical wavelet methods for differential and integral equations are becoming increasingly important research field. In this paper, we used multioresolution analysis and scaling functions when studding the wavelet-Galerkin method and we applied the method to some special types of Sturm-Liouville differential equation. In example 1, we can see how this method is applied to concrete differential equation (29), and there we make comparison between the exact solution of this equation and our numerical solution. Also, in example 2, we found numerical solutions of nonhomogeneous differential known as Van der Pol equation which does not have analytic solution.

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