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Degrees of Freedom Analysis of the Non-coherent Block Fading MAC Channel

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Abstract

1 Introduction

We study a two-user multiple-input multiple-output (MIMO) fading multiple-access channel (MAC), where two terminals wish to communicate with a third one. The channels between the terminals are MIMO block fading channels with channel coefficients which are constant in a block of length T, and then change in an independent realization. The system model follows the model introduced by Hochwald and Marzetta in [1]. We consider a non-coherent channel model, where all terminals are aware of the statistics of the fading but not of its realization. Instead of using pilots for channel estimation, we assume communication where the information is communicated by the use of subspaces, following the geometric approach introduced in [2]. We focus on the high signal-to-noise ratio (SNR) regime. In particular, we study the pre-log region, defined as the limiting ratio of the achievable-rate region to log SNR as the SNR tends to infinity.

2 System Model

We are interested in the two-user multiple access (MAC) channel. We assume the block Rayleigh model where the channel is constant in a certain time block of length T and then changes in an independent realization. User 1 has M_1 transmit antennas, user 2 M_2 transmit antennas and the receiver has N antennas. We assume a joint transmission scheme and address the case where $M_1 + M_2 \leq N$ and $T \geq M_1 + M_2 + N$. The results can be extended to the other cases as well, but we identify this case as particularly relevant.

We denote by SNR the average signal-to-noise-ratio at each transmit antenna. The system model is the following

$$Y = H_1 X_1 + H_2 X_2 + W \tag{1}$$

where $H_1 \in \mathbb{C}^{N \times M_1}$ and $H_2 \in \mathbb{C}^{N \times M_2}$, with i.i.d $CN(0,1)$ entries. The noise matrix $W \in \mathbb{C}^{N \times T}$ has coefficients which are i.i.d. $CN(0,\sigma^2)$. The matrix X_1 is a $T \times M_1$ matrix normalized such that $\mathrm{E}[\mathrm{tr}(X_1^{\mathrm{H}} X_1)] = P_1 T$. Accordingly, X_2 is a $T \times M_2$ matrix, normalized such that $\mathrm{E}[\mathrm{tr}(X_2^{\mathrm{H}} X_2)] = P_2 T$.

3 Rate Region Analysis for the MAC Channel

The achievable-rate region for the block fading MIMO MAC channel is given by the closure of the convex hull of the set [3]

$$\begin{aligned}\mathcal{R} = \{R_1(SNR), R_2(SNR) : R_1(SNR) &\leq I(X_1; Y|X_2),\\ R_2(SNR) &\leq I(X_2; Y|X_1),\\ R_1(SNR) + R_2(SNR) &\leq I(X_1, X_2; Y)\}\end{aligned} \tag{2}$$

We are interested in the pre-logs of $R_1(SNR)$ and $R_2(SNR)$, defined as the limiting ratios of $R_1(SNR)$ and $R_2(SNR)$ to the logarithm of the SNR as the SNR tends to infinity.

We do not know the optimal distributions of X_1 and X_2 in general. Motivated by the results in [1] and [2] for the point-to-point case, we assume isotropic distributions for the transmit signals, i.e $X_1 = A_1\Theta_1$, $X_2 = A_2\Theta_2$, where $\Theta_1\Theta_1^H = I_{M_1}$ and $\Theta_2\Theta_2^H = I_{M_2}$.

3.1 Analysis of $I(X_1; Y|X_2)$ and $I(X_2; Y|X_1)$

For the mutual information $I(X_1; Y|X_2)$ we have

$$I(X_1; Y|X_2) = h(Y|X_2) - h(Y|X_1, X_2) \tag{3}$$

The calculation of $h(Y|X_1, X_2)$ is as follows. First, we observe that given X_1 and X_2, the row vectors y_j, $j = 1, \dots, N$ of Y are independent Gaussian vectors with identical covariance matrix

$$\begin{aligned}R &= X_1^H X_1 + X_2^H X_2 + \sigma^2 I_T\\ &= \Theta^H \Lambda \Theta + \sigma^2 I_T,\end{aligned} \tag{4}$$

where Λ includes the non-zero eigenvalues, whose number is denoted as M and $M \leq M_1 + M_2$ (maximum of the rank of $X_1^H X_1 + X_2^H X_2$). Hence, we have the following relation for the conditional entropy $h(Y|X_1, X_2)$

$$\begin{aligned}h(Y|X_1, X_2) &= N\mathrm{E}\left[\log(\pi e)^T \det R\right]\\ &\leq NT\log\pi e + N\mathrm{E}\left[\log\det(\Theta^H\Lambda\Theta^{\mathrm{H}} + \sigma^2 I_T)\right]\end{aligned} \tag{5}$$

From $\det(I_T + AB) = \det(I_M + BA)$, where $A \in \mathbb{C}^{T\times M}$, $B \in \mathbb{C}^{M\times T}$, we have

$$\begin{aligned}\det(\Theta\Lambda\Theta^{\mathrm{H}} + \sigma^2 I_T) &= \sigma^{2T}\det\left(\frac{1}{\sigma^2}\Theta^{\mathrm{H}}\Lambda\Theta + I_M\right)\\ &= \sigma^{2(T-M)}\prod_{i=1}^{M}\left(\lambda_i + \sigma^2\right).\end{aligned} \tag{6}$$

After some rewriting, for the differential entropy we have

$$h(Y|X_1, X_2) = N\mathrm{E}\left[\log\prod_{i=1}^{M}(\lambda_i + \sigma^2)\right] + (M)^2\log\pi e + N(T-M)\log\pi e\sigma^2, \tag{7}$$

We recall that we have the following power constraint

$$\mathrm{E}\left[\mathrm{tr}\,(\Lambda)\right] \leq (P_1 + P_2)T.$$

Hence, it holds

$$\mathrm{E}\left[\prod_{i=1}^{M}\left(\lambda_i+\sigma^2\right)\right] \leq \left(\frac{(P_1+P_2)T+\sigma^2}{M}\right)^M, \tag{8}$$

where the equality is achieved when all eigenvalues are equal. Additionally, $M \leq M_1 + M_2$. Thus, for the differential entropy we have the following bound

$$\begin{aligned} h(Y|X_1,X_2) \leq{} & N(M_1+M_2)\log\left[(P_1+P_2)T+\sigma^2\right] \\ & + N(M_1+M_2)\log\frac{\pi e}{M_1+M_2} + N(T-M_1-M_2)\log \pi e\sigma^2, \end{aligned} \tag{9}$$

What remains is to evaluate $h(Y|X_2)$. A bound can be obtained by conditioning on H_2, since conditioning does not increase the entropy

$$h(Y|X_2) \geq h(Y|X_2,H_2) = h(Y_1), \tag{10}$$

where

$$Y_1 = X_1H_1 + W. \tag{11}$$

For the differential entropy $h(Y_1)$ we have [2]

$$h(Y_1) \approx h(H_1A_1Q_1) + \log|G(T,M_1)| + (N-M_1)(T-M_1)\log \pi e\sigma^2 + (T-M_1)\mathrm{E}\left[\log\det(H_1H_1^{\mathrm{H}_1})\right]. \tag{12}$$

where $Q_1 \in \mathbb{C}^{M\times M}$ is unitary i.d. matrix that is independent on H_1 and A_1.

Combining this result with the previous one for $h(Y|X_1,X_2)$, we obtain the following bound on $I(X_1;Y|X_2)$

$$I(X_1;Y|X_2) \geq M_1\left(T-M_1+M_2\right)\log\left(\frac{P_1}{\sigma^2}\right) + c_1 + o(1) \tag{13}$$

where c_1 is a term which does not depend on the SNR and $o(1)$ is a term which tends to 0 as SNR tends to infinity.

The factor $M_1\left(T-M_1-M_2\right)$ is the pre-log factor of interest and is achievable with the assumed isotropic distributions of X_1 and X_2, since the derivations yield a lower bound on $I(X_1;Y|X_2)$

In an analogous fashion, for $I(X_2;Y|X_1)$ we have the following bound

$$I(X_2;Y|X_1) \geq M_2\left(T-M_1-M_2\right)\log\left(\frac{P_2}{\sigma^2}\right) + c_2 + o(1) \tag{14}$$

As in the previous case, the pre-log factor $M_2\left(T-M_1-M_2\right)$ is achievable with the assumed input distributions.

3.2 Analysis of $I(X_1,X_2;Y)$

For the mutual information $I(X_1,X_2;Y)$ we have

$$I(X_1,X_2;Y) = h(Y) - h(Y|X_1,X_2) \tag{15}$$

For the purpose of the analysis of $h(Y)$, we will write the received signal Y in the following form

$$\mathbf{Y} = \begin{pmatrix} H_1 & H_2 \end{pmatrix}\begin{pmatrix} X_1 \\ X_2 \end{pmatrix} + W = HX + W \tag{16}$$

This model corresponds to a non-coherent MIMO point-to-point block fading channel with $M_1 + M_2$ transmit, N receive antennas and coherence time T. Hence, the capacity of high-SNR capacity of this channel, as given in [2] is an obvious upper bound to the achievable sum-rate.

Following the approach in [2], we decompose the signal HX in two parts: the subspace Ω_X spanned by the row vectors of HX and the matrix C_{HX} which specifies the position of the N row vectors inside Ω_X. Under the assumption of isotropic input distributions of X_1 and X_2, we can easily show that HX is isotropically distributed in Ω_X. Following [2], we can write the $h(Y)$ in the following form

$$h(Y) = h(C_{HX}) + \log|G(T, M_1 + M_2)| + (T - M_1 - M_2)E[\log \det HXX^H H] \tag{17}$$

Combining with the result for $h(Y|X_1, X_2)$, we can write the following

$$I(X_1, X_2; Y) \approx (M_1 + M_2)(T - M_1 - M_2) \log \frac{P_1 + P_2}{\sigma^2} + c + o(1) \tag{18}$$

Hence, for the sum-rate, the pre-log factor $(M_1 + M_2)(T - M_1 - M_2)$ is achievable. Additionally, this is the maximal achievable pre-log factor, i.e. the sum-rate bound is tight.

With the above analysis, and after a normalization (in order to have the result in terms the rate per channel use), the following region is achievable for the pre-log of the MAC channel

$$\begin{aligned}
\Pi_R = \{\Pi_{R_1}, \Pi_{R_2} : \Pi_{R_1} &\leq M_1 \left(1 - \frac{M_1 + M_2}{T}\right), \\
\Pi_{R_2} &\leq M_2 \left(1 - \frac{M_1 + M_2}{T}\right), \\
\Pi_{R_1} + \Pi_{R_2} &\leq (M_1 + M_2) \left(1 - \frac{M_1 + M_2}{T}\right)\}
\end{aligned} \tag{19}$$

With exception to the last one, it is possible that the achievable rates (and thus the pre-log terms) can be improved. In order to assess this question in depth, we present an alternative analysis based on a more involved geometric approach.

4 Rate Analysis: Geometric Approach

We recall the system model

$$Y = H_1X_1 + H_2X_2 + W \tag{20}$$

According to [2], we can decompose the signal H_1X_1 in two parts: the subspace Ω_{X_1} spanned by the row vectors of H_1X_1 and the matrix $C_{H_1X_1}$ which specifies the position of the N row vectors inside Ω_{X_1}. Hence, we can see H_1X_1 as an object on a submanifold of $\mathbb{C}^{N\times T}$, $\mathcal{M}_1$ with dimension

$$D_1 \doteq dim(\mathcal{M}_1) = M_1(T - M_1) + NM_1 \tag{21}$$

Similarly, H_2X_2 can be seen as an object on a submanifold of $\mathbb{C}^{N\times T}$, $\mathcal{M}_2$ with dimension

$$D_2 \doteq dim(\mathcal{M}_2) = M_2(T - M_2) + NM_2 \tag{22}$$

Hence, the sum $H_1X_1 + H_2X_2$ is an object on $\mathcal{M} = \mathcal{M}_1 \bigcup \mathcal{M}_2$ with dimension

$$D \doteq dim(\mathcal{M}_1 + \mathcal{M}_2) = D_1 + D_2 - D_{12} \tag{23}$$

where D_{12} is the dimension of the intersection of $\mathcal{M}_1$ and $\mathcal{M}_2$. We observe that for given X_1, X_2, the dimension of the intersection D_{12} is a random variable which depends on H_1 and H_2. The probability distribution of D_{12} is obtained in a combinatorial way and is given by

$$P(D_{12} = d) = \frac{\binom{D_1}{d}\binom{NT-D_1}{D_2-d}}{\binom{NT}{D_2}} \tag{24}$$

The dimension of the intersection is on average

$$\bar{D}_{12} = \sum_{d=0}^{min(D_1,D_2)} d \cdot P(D_{12} = d) \tag{25}$$

4.1 Analysis of $I(X_1; Y|X_2)$ and $I(X_2; Y|X_1)$

First, we need to include the dimension of the intersection, D_{12} in the analysis of the mutual information. Given D_{12}, for the mutual information of interest we have

$$I(X_1; Y|X_2, D_{12}) = h(Y|X_2, D_{12}) - h(Y|X_1, X_2, D_{12}) \tag{26}$$

Regarding the calculation of $h(Y|X_2, D_{12})$, we have the following observation. The randomness of $Y|X_2, D_{12}$ is affected by the noise in $[M_2(T - M_2) - L] + [NT - D_1 - D_2 + D_{12}]$ dimensions. The term $M_2(T - M_2) - L$ is a consequence of the fact that given X_2 we know the subspace Ω_{X_2}. However, a fraction of the dimensions L of $M_2(T - M_2)$ are affected by H_1X_1 as well. The distribution of L is given by

$$P(L = l) = \frac{\binom{M_2(T-M_2)}{l}\binom{NT-M_2(T-M_2)}{M_1(T-M_1)-l}}{\binom{NT}{M_1(T-M_1)}} \tag{27}$$

On average, L is

$$\bar{L} = \sum_{l=0}^{M^*(T-M^*)} l \cdot P(L = l) \tag{28}$$

where $M^* = min(M_1, M_2)$. Note that in the remaining dimensions of $\mathcal{M}_2$, there is still randomness which is dominated by $H_1X_1 + H_2X_2$.

The second term in the sum is a result of the effect of the noise in the normal space of $\mathcal{M}_1 \bigcup \mathcal{M}_2$, which has dimension $NT - D_1 - D_2 + D_{12}$. Hence, the noise affects the randomness of $h(Y|X_2, D_{12})$ in the form

$$h(Y|X_2, D_{12}) \approx c - [M_2(T - M_2) - L + NT - D_1 - D_2 + D_{12}] \log(SNR) \tag{29}$$

where c is a constant which does not depend on the SNR.

The analysis of $h(Y|X_1, X_2, D_{12})$ is as follows. As already argued, the sum $H_1X_1 + H_2X_2$ is an object in a submanifold $\mathcal{M}$ of $\mathbb{C}^{NT}$ of dimension $D = D_1 + D_2 - D_{12}$. The randomness of $Y|X_1, X_2, D_{12}$ in the normal space of $\mathcal{M}$ comes from the noise. The dimension of this space is $NT - D_1 - D_2 + D_{12}$. The subspaces Ω_{X_1} and Ω_{X_2} are known and they reside in manifolds of dimensions $M_1(T - M_1)$ and $M_2(T - M_2)$ respectively. However, their intersection is of random dimension L, as discussed, and in their intersection the randomness is dominated by the contribution of $H_1X_1 + H_2X_2$ rather then the noise. Hence, the noise contributes to the randomness of $Y|X_1, X_2, D_{12}$ in $M_1(T - M_1) + M_2(T - M_2) - 2L + NT - D_1 - D_2 + D_{12}$ dimensions.

Finally, we can represent $I(X_1;Y|X_2,D_{12})$ in the form

$$\begin{aligned} I(X_1;Y|D_{12}) &= h(Y|X_2,D_{12}) - h(Y|X_1,X_2,D_{12}) \\ &\approx c + [M_1(T-M_1) - L]\log SNR \end{aligned} \tag{30}$$

where c is a constant which does not depend on the SNR. We note that $I(X_1;Y|X_2,D_{12})$ is a function of L, which is random. Thus, for the mutual information $I(X_1;Y|X_2)$ we have

$$\begin{aligned} I(X_1;Y|X_2) &= \sum_{L=0}^{M^*(T-M^*)} P(L=l)\cdot I(X_1;Y|X_2,L) \\ &\geq I(X_1;Y|X_2,\bar{L}) \approx c + [M_1(T-M_1) - \bar{L}]\log SNR \end{aligned} \tag{31}$$

where the inequality comes from Jensen's inequality. The achievable pre-log factor is thus $\Pi_{R_1} = M_1(T-M_1) - \bar{L}$. In an analogous fashion, for user 2 we have,

$$I(X_2;Y|X_1) \geq c + [M_2(T-M_2) - \bar{L}]\log SNR \tag{32}$$

leading to an achievable pre-log factor of $[M_2(T-M_2) - \bar{L}]$

As for the analysis of $I(X_1,X_2;Y)$ we already argued that the bound derived in the previous section is tight. With all this in mind, and after a normalization (in order to have the result in terms the rate per channel use), the following region is achievable for the pre-log of the MAC channel

$$\begin{aligned} \Pi_R = \{\Pi_{R_1},\Pi_{R_2} : \Pi_{R_1} &\leq M_1\left(1-\frac{M_1}{T}\right) - \frac{\bar{L}}{T}, \\ \Pi_{R_2} &\leq M_2\left(1-\frac{M_2}{T}\right) - \frac{\bar{L}}{T}, \\ \Pi_{R_1} + \Pi_{R_2} &\leq (M_1+M_2)\left(1-\frac{M_1+M_2}{T}\right)\} \end{aligned} \tag{33}$$

where

$$\bar{L} = \sum_{l=0}^{M^*(T-M^*)} l\cdot P(L=l)$$

with

$$P(L=l) = \frac{\binom{M_2(T-M_2)}{l}\binom{NT-M_2(T-M_2)}{M_1(T-M_1)-l}}{\binom{NT}{M_1(T-M_1)}}$$

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