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Comparison between Nested and Non-Nested Multigrid Methods for Magnetostatic Field Analysis

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Abstract. A comparison of the efficiency of the multigrid solution method for magnetostatic field analysis using nested and non-nested finite element meshes is presented. Two types of multigrid algorithms are investigated; the V-cycle and the W-cycle multigrid method. For the generation of nested meshes a simple halving procedure is employed resulting in fine meshes strongly dependent on the initial coarse mesh and with uniform mesh densities. Non-nested meshes are generated using adaptive meshing with a suitable error estimator. The results show that both methods provide faster solution than the ordinary ICCG method. The speed-ups range from 2 to 10 mainly depending on the type of the cycle and the number of unknowns.

1. Introduction

Decreasing the computation time required for the solution of large algebraic systems such as those usually arising during numerical solution of boundary value problems becomes a very important task, especially in 3D. For the solution of such large systems of simultaneous algebraic equations a variety of numerical methods have already been proposed with various success. In the fields of structure analysis and computational fluid dynamics, the so-called multigrid solution methods have been widely used, however, mainly in connection with the finite difference method [1], [2]. However, for electromagnetic field analysis, due to the complexity of the analyzed domains and inter-material boundary conditions, the finite element method finds a huge application area. Although computationally efficient and accurate, the finite element method is very often a time consuming procedure, especially for complex 3D field problems. At the same time, with the recent tremendous developments in the computer hardware and software technology, the computers required for large scale electromagnetic analysis and simulation became available. For such problems, it is very desirable to investigate the efficiency of the multigrid solution method, not only as a preconditioning method, but also as a full solution method in order to decrease the overall computing time.

Recently, we implemented and successfully applied several multigrid solution algorithms to electrostatic and magnetostatic field analysis using nested finite element meshes [3]. However, although development of nested meshes is a very straightforward procedure that can be easily applied to any finite element mesh, nested meshes have several disadvantages such as:

- Dependence of fine meshes on the initially constructed coarse mesh
- Generation of uniformly dense meshes inside the entire analysis domain, and
- Generation of unnecessary dense meshes around areas that are not of main interest in the analysis (mainly areas close to the boundaries) [3].

To overcome these problems, we present a multigrid solution procedure based on so-called non-nested finite element meshes. First, we briefly describe the difference between nested and non-nested finite element meshes. Next, we shortly address the main idea that lies behind the high efficiency of the multigrid method. Two models are treated and the numerical data obtained during analysis are given.

2. Nested and Non-nested Meshes

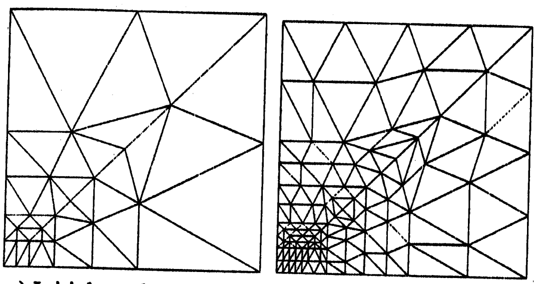
Multigrid methods, as the name suggests, are methods for the numerical solution of a system of algebraic equations generated by means of numerical discretization of a physical problem defined by partial differential equations over several grids or meshes with various mesh densities. In general, these grids (usually associated with the finite difference method), or meshes (associated with the finite element method) can be nested or non-nested. Nested meshes are those meshes that have common nodes, i.e. all nodes of a coarse mesh are at the same time nodes for any finer developed mesh too. The generation of nested meshes is usually trivial, and can be executed by simple subdivision. A typical example is shown in Figure 1. Here, each finite element of the coarse mesh (a) is subdivided into four smaller elements for the finer mesh (b) using midpoints along each side of the triangle. Although easy, it is obvious that this procedure generates meshes with low quality of the elements. Additionally, from Figure 1, it is readily apparent that nested meshes exhibit several other disadvantages, among which, the strong dependency on the initial coarse mesh is probably the most unpleasant.

On the other hand, non-nested meshes do not require coincidence between nodes of two successive meshes. These meshes can be constructed without any constraints and they do not depend on the initial coarse mesh. Consequently, with the development of non-nested meshes, we can ensure the generation of optimal meshes with graded mesh densities. Even more, non-nested meshes can be generated using any type of adaptive mesh generation, therefore increasing the accuracy of the results and optimizing the computational resources. Figure 2 shows two successive non-nested meshes generated adaptively. As can be seen, non-nested meshes have only few common nodes between two successive meshes. These meshes have a mesh density in correlation with the problem, i.e. denser meshes are generated around areas with vigorous changes of the unknown variable, and opposite. The quality of the finite elements for these meshes is also better than in the case of nested meshes. Each mesh is independent of the previous one, and only the accuracy of the solution that is implicitly defined with the error estimation is important for the mesh generation, thus enabling an optimal solution regarding the computation time, resources and accuracy.

3. Multigrid Methods

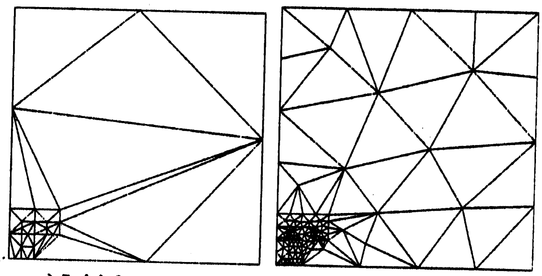
Multigrid methods are a set of techniques for solving systems of algebraic equations using several finite element meshes with different mesh density [1]. They keep the number of iterations almost independent of the number of unknowns providing the solution of elliptic partial equations discretized on $\mathcal{N}$ nodal points in $O(\mathcal{N})$ operations, which is much faster than any other rapid iterative method which could go as far as $O(\mathcal{N} \log \mathcal{N})$. Of large importance is to acknowledge the main reason why multigrid iterative methods are so superior over other iterative methods. As described lengthily in [3], the main advantage of the multigrid method is its property to utilize the smoothing characteristic of some iterative procedures using several meshes with different mesh density. It is well known that for dense meshes where a large amount of smooth error components exists, we usually have slow convergence rates. However, if we somehow project such smooth error components onto another coarser mesh, than they become less smooth and can be diminished easily. This process is called *restriction*. Later on, solution improvements have to be interpolated from the coarser mesh toward the finer mesh, a procedure that is called *prolongation*. Multigrid methods extensively use these two procedures alternately to increase the computational speed and to improve the convergence rate of the iterative solution method. There is a large number of widely established multigrid algorithms with names set according to the shape of the iterative cycles that each of them performs and the number of restriction and prolongation steps, such as the V–cycle, the W–cycle, the F–cycle, etc., [3]. In this paper we will only treat two iterative cycles: the V–cycle and the W– cycle as the two most commonly used (see Figure 3). A description of the cycles can be found in various references such as [3] and goes beyond the scope of this paper.

What is important to say is how to obtain the restriction and prolongation operators between coarse and fine meshes. In the case of nested meshes, these two operators can be easily defined taking into account that each new node is placed exactly at the centre of an edge, therefore, a simple halving procedure works fine [3]. However, in the case of non–nested meshes, this procedure is not valid and a new method for constructing the restriction and prolongation operators (matrices) must be found. In our research, we obtained very good results utilizing the ordinary area coordinates method. The main idea is given in Figure 4. Unknown values at each node of a coarse and of a fine mesh are computed using linear interpolation from the values at finite element vertices, very similar with the ordinary finite element approximation. Therefore, for the restriction operator we use a simple injection method:



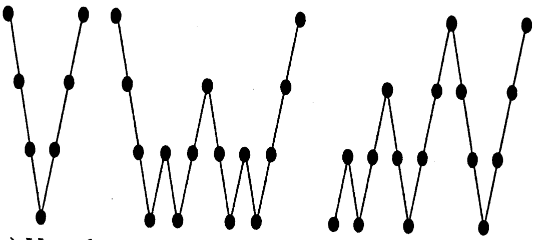
a) Initial mesh division b) Mesh #2

Figure 1. Nested finite element meshes.



a) Initial mesh division b) Mesh #2

Figure 2. Non – nested finite element meshes.



(a) V-cycle (b) W-cycle (c) Full multigrid

Figure 3. Several types of multigrid algorithms.

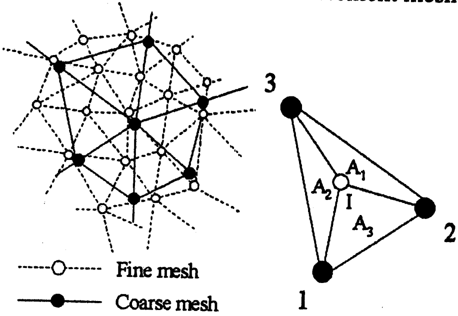


Figure 4. Correlation between coarse and fine non – nested meshes.

$$
\begin{aligned}
\mathbf{r}_{coarse}(1) &= \mathbf{r}_{coarse}(1) + A_1 \cdot \mathbf{r}_{fine}(I) \\
\mathbf{r}_{coarse}(2) &= \mathbf{r}_{coarse}(2) + A_2 \cdot \mathbf{r}_{fine}(I) \\
\mathbf{r}_{coarse}(3) &= \mathbf{r}_{coarse}(3) + A_3 \cdot \mathbf{r}_{fine}(I) \quad ,
\end{aligned}
\tag{1}
$$

while for the prolongation operator:

$$
\mathbf{e}_{fine}(I) = A_1 \cdot \mathbf{e}_{coarse}(1) + A_2 \cdot \mathbf{e}_{coarse}(2) + A_3 \cdot \mathbf{e}_{coarse}(3) \quad , \tag{2}
$$

where, A is the area coordinate values, while $\mathbf{r}$ and $\mathbf{e}$ are the residual and error (defect) vector at the vertices 1, 2 and 3 and at the node I, respectively (see Figure 4).

4. Analyzed Model and Obtained Results

To investigate the efficiency of the multigrid solution method for magnetostatic field analysis, a simple model of a C–type electromagnet was investigated. Two sets of meshes were developed: a set of nested meshes as shown in Figure 1, and a non-nested adaptively generated set of meshes shown in Figure 2. For adaptive mesh generation we used the well-established Bank–Weiser error estimator [4]. From the generated meshes it is visible that, the error estimator correctly represents the amount of the generated computational error – the error is lerger around singular points of the model. Relevant data for both sets of meshes are given in Table I. As can be seen, we intentionally developed two sets of meshes with approximately the same number of nodes and elements in order to correctly evaluate the influence of the type of meshes, nested or non–nested.

Figure 5 shows the comparison of the obtained convergence rates for both nested and non–nested meshes using the V–cycle and the W–cycle multigrid method. Two things are visible: (1) the W–cycle always converges faster than the V–cycle, and (2) the mesh type (nested or non – nested) has large influence on the convergence rates for the V–cycle, and almost no influence for the W–cycle. The computation time is given in Figure 6 in comparison with the computation time for the ICCG solution method (for the densest mesh in Table I only). From Figure 6, one can easily see that although for nested meshes the computation time for both, the V– and the W–cycle is almost the same, in the case of non–nested meshes the W–cycle should be preferred over the V–cycle. It is also visible that both multigrid algorithms are superior over the ordinary ICCG iteration method, which for solving the same problem using only the finest mesh (level 5) needed 145 and 137 iterations for nested and non–nested mesh, respectively. Speedup coefficients for both multigrid methods over the ICCG method range from 2 up to 10 depending on the size and number of used mesh levels.

Table I. Mesh data for the analyzed model

Nested Meshes			Non – nested Meshes		
Mesh Level	Nodes	Elements	Mesh Level	Nodes	Elements
1	30	44	1	30	44
2	106	180	2	103	176
3	391	720	3	381	704
4	1501	2880	4	1465	2814
5	5881	11520	5	5745	11264

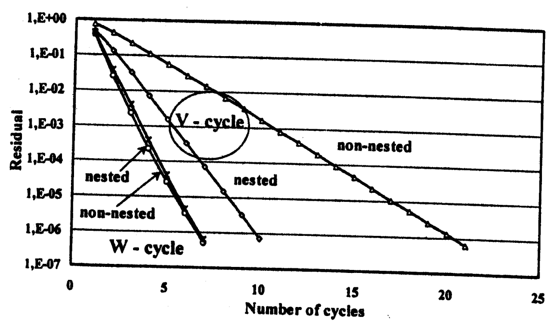


Figure 5. Convergence rates.

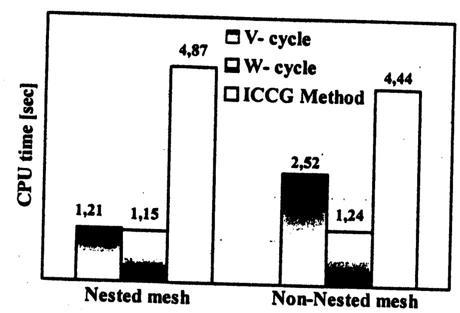


Figure 6. Computation time.

5. Conclusions

We presented results of the efficiency investigation of the multigrid solution method as a numerical solution method in 2D magnetostatic field analysis. It was shown that multigrid methods show very fast and stable convergence rates, which are independent of the number of mesh levels and type of meshes. We also show that although computationally easier and faster (only for the V–cycle) nested meshes have additional disadvantages and should be avoided. Non–nested meshes could be a little bit more difficult to generate, but they are very promising tools for fast and accurate solution in electromagnetic field analysis, especially in 3D, in connection with adaptive mesh generation and multigrid solution algorithms. Additional research and comparison between results presented here and similar ones using so–called full multigrid method could be very interesting for a wide range of researchers in this area.

References

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