Element–Free Galerkin Method for Electromagnetic Field Computations

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Reprinted from
IEEE TRANSACTIONS ON MAGNETICS
Vol. 34, No. 5, September 1998

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Abstract—**Although numerically very efficient the finite element method exhibits difficulties whenever the remeshing of the analysis domain must be performed. For such problems utilizing meshless computation methods is very promising. In this paper, a kind of meshless method called the element-free Galerkin method is introduced for electromagnetic field computation. The mathematical background for the moving least square approximation employed in the method is given, and the numerical implementation is briefly discussed. Application of the proposed method for electromagnetic field computation and verification of the obtained results using theoretically known solution is also presented.**

Index terms—**Element-free Galerkin methods, least squares methods, finite element methods, electrostatic analysis, voltage transformers.**

I. Introduction

Over the past two decades, the finite element method (FEM) has been established as a very powerful numerical technique for solving various engineering problems. Its main power results from the fundamental idea of replacing a given continuous function defined over the entire analysis domain by piecewise approximations, usually polynomials, over a set of finite number of geometrically simple domains called finite elements. Although the idea of domain division is really ingenious, meshing of the analysis domain can sometimes be very laborious and time consuming. Moreover, there are certain classes of problems for which the FEM is difficult, or even impossible to be applied. For example, in electromagnetic field computation, problems which involve large geometrical changes or deformations of an analysis model such as inverse shape optimization, melting and metal casting, moving conductors, cracks, etc., the finite element method usually requires remeshing in order to insure equality between finite element boundaries and the moving discontinuities. Therefore, not only decreasing of the accuracy, but huge computation time can be experienced during numerical analysis. For such problems, development of advanced methods which do not rely on meshing are very advantageous.

The main objective for the development of meshless methods is to eliminate even to a small degree the complexity of the analysis method by making approximation which is entirely based in terms of nodes, not elements. Thus, by utilizing these methods it becomes possible to enlarge the area of applications without very laborious and time consuming remeshing process.

Meshless methods originated about twenty years ago having their roots in the smooth particle hydrodynamics method [1]. They only become more attractive much later with the usage of the moving least square method (MLSM) [2]. First, with the work of Nayroles who for the first time used the MLS approximation in connection with the Galerkin method called diffuse element method [3], and later with the work of Belytschko et al. who modified and improved the diffuse element method developing the element-free Galerkin (EFG) method [4]. Recently, a large scale of modifications and application of this method have been proposed, however, mainly in the area of structure analysis.

In this paper, for the first time the authors present successful implementation of EFG method for electromagnetic field computations. First, the basic idea and the mathematical background for the method are given. Next, the numerical implementation of the method is addressed. An application of the EFG method for a simple 1-D model with known exact solution is presented and the numerical results are compared with the analytical solution. The accuracy of the results and the convergence of the EFG method are also briefly discussed. Finally, a 2-D application of the element-free Galerkin method for electrostatic field analysis of a voltage transformer (VT) is also given.

II. Element-free Galerkin Method

The element-free Galerkin (EFG) method is considered as a meshless method, because to obtain an approximated solution it requires only the definition of a set of nodes distributed over the entire analysis region and the definition of the boundary conditions [5]. However, for the numerical evaluation of the integrals generated using the Galerkin procedure, some kind of mesh must be considered. In the EFG method, this mesh does not depend on the node disposition and it is far simpler than the ordinary finite element mesh. This, so called integration mesh is usually simple rectangular mesh which overlaps the entire analysis region as shown in Fig. 1. In the finite element analysis the integration is performed at several integration points inside each finite element, and the element matrix is defined by the element nodes, e.g. in Fig. 1a nodes: 7, 8, 12 and 13.

On the other side, in the EFG method the integration is performed at each integration point of a simple integration cell also called a bucket. The domain of influence of each integration point is defined by the radius of influence r, as shown in Fig. 1b. Therefore, for each integration point a list of nodes that lies inside the domain of influence is indispensable. For example, the list of nodes for the

Manuscript received November 3, 1997.

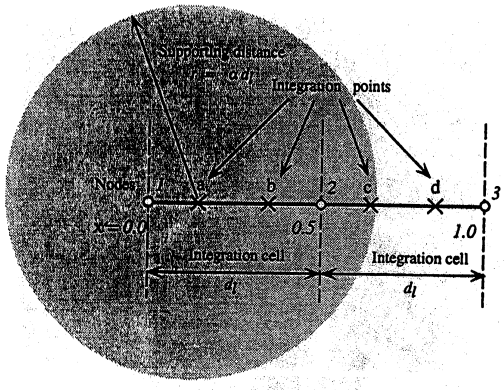


Fig. 3. Numerical implementation.

where n is the number of nodes inside the domain of influence, and N is the shape function given by

$$N = \mathbf{p}^T(\mathbf{x})\,\mathbf{A}^{-1}(\mathbf{x})\,\mathbf{B}(\mathbf{x}). \tag{12}$$

B. Numerical Implementation

In this paragraph we will briefly address some topics on numerical implementation of the EFG method. The most important point is that the EFG method can be easily developed in the same manner as the FEM. To easily demonstrate the main procedure, we will address a simple 1-D model shown in Fig. 3. After definition of the analysis domain and the number and position of nodes, we have to determine the number and size of the integration cells and the number of integration points inside each integration cell. In this example we have only two integration cells, each with two integration points, or in total four integration points: a, b, c and d. After definition of the weighted function as in (5), generation of a list of nodes inside the domain of influence for each integration node respectively must be performed. In our example, the radius of influence is set to $r = \alpha d_l$, where $\alpha = 1.5$, and d_l is the length of the integration cell. The list of nodes for integration point a consists of only nodes 1 and 2. Next, discretization of the governing equation for the analyzed problem is performed according to the Galerkin method which results in a local matrix for the integration point a. The same procedure is repeated for each integration point inside the analysis domain utilizing updated list of nodes for each integration point, respectively. Finally, each local contribution from integration points is summarized to obtain the global matrix of the system, which after inclusion of the boundary conditions is solved using direct or iterative method obtaining the unknown function values at each node inside the analyzed domain.

III. Applications

A. 1-D Test Problem

The usefulness of the EFG method was verified using a simple 1-D electrostatic model with known exact solution which is given in Fig. 4. It has constant electric charge density $\rho = 2500[C/m]$, and constant relative permitivity $\varepsilon_r = 3.6$. Three runs were performed in order to obtain the convergence characteristic of the EFG method using

TABLE I
Accuracy vs. number of nodes for 1-D test model

	6 nodes	*11 nodes*	*21 nodes*
Maximum error [%]	3.41	1.21	0.139

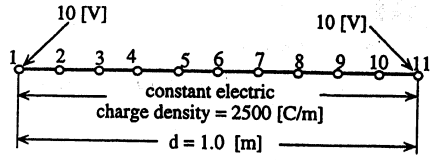


Fig. 4. 1-D test model.

six, eleven and twenty-one nodes with constant number of integration cells equal to ten and two integration points in each cell. The obtained solutions together with the exact one are given in Fig. 5. The relative error for different number of nodes is given in Table I. From the obtained results it is apparent that: (1) the EFG method provides very accurate results even with a small number of nodes, and (2) the results converge very fast with increasing the number of the nodes. This opens new perspectives for the EFG method as an adaptive technique where by simply dropping a large number of nodes in the area where a large gradient of the unknown function, it is possible to increase the accuracy of the results significantly. Finally, we would like to point out that from computational point of view, it is advisable to increase the accuracy of the results by increasing the number of cells and keeping the number of integration points to the modest level, instead of using small number of large integration cells with large number of integration points.

B. 2-D Application model

A simplified vertical cross section of a voltage transformer (VT) which was used as an application problem treated by the EFG method is given in Fig. 6a. It is constructed of primary, secondary and tertiary windings isolated by PVC with relative dielectric constant $\varepsilon_r = 3.6$,

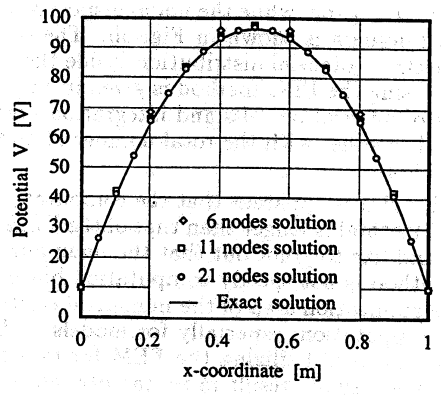
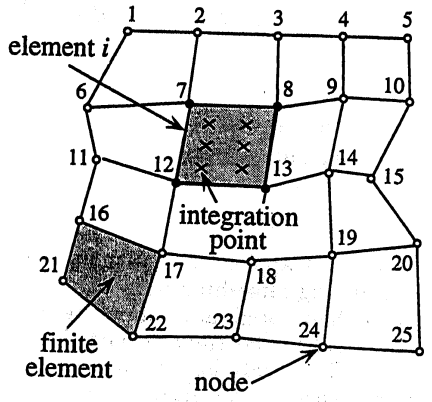
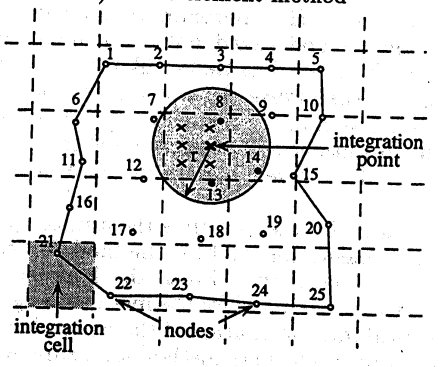


Fig. 5. Exact solution vs. computational results.



a) Finite element method



b) Element-free Galerkin method

Fig. 1. Finite element method vs. Element-free Galerkin method

integration point shown in Fig. 1b, consists of nodes 8, 13 and 14, shown in the same figure as black circles.

A. Moving least square approximation

For approximation of the unknown function $\mathbf{u}$, the EFG method utilizes the MLSM which encompasses the following three components: a weighted function associated to each node, a polynomial basis, and a set of coefficients which depend on the position. In the MLS approximation, the interpolant function $u^h(\mathbf{x})$ of the unknown function $u(\mathbf{x})$ inside the domain Ω is defined by [4]

$$u^h(\mathbf{x}) = \sum_{i=1}^{m} p_i(\mathbf{x})\, a_i(\mathbf{x}) \equiv \mathbf{p}^T(\mathbf{x})\, \mathbf{a}(\mathbf{x}), \qquad (1)$$

where m is the number of terms in the basis, $p_i(\mathbf{x})$ are the monomial basis functions, and $a_i(\mathbf{x})$ are the unknown coefficients which are functions of the spatial coordinates $\mathbf{x}$. A common linear and quadratic bases in 1-D and 2-D space are:

- *Linear Bases:*

$$\begin{aligned} \mathbf{p}^T_{(m=2)} &= \{1, x\}, \quad \text{1-D}, \\ \mathbf{p}^T_{(m=3)} &= \{1, x, y\}, \quad \text{2-D}. \end{aligned} \qquad (2)$$

- *Quadratic Bases:*

$$\begin{aligned} \mathbf{p}^T_{(m=3)} &= \{1, x, x^2\}, \quad \text{1-D}, \\ \mathbf{p}^T_{(m=6)} &= \{1, x, y, x^2, xy, y^2\}, \quad \text{2-D}. \end{aligned} \qquad (3)$$

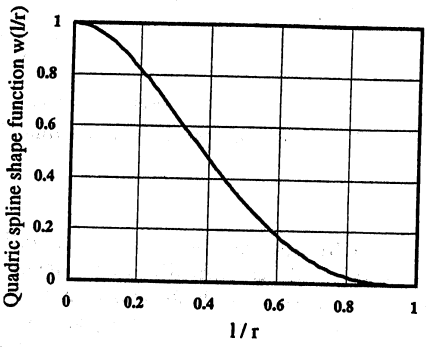


Fig. 2. Quartic spline shape function.

Coefficients $a_i(\mathbf{x})$ can be obtained by performing a weighted least square fit by minimizing the difference between the local approximation and the function which yields to the following quadratic form

$$J = \sum_{I}^{n} w(l) \left[\sum_{i}^{m} p_i(\mathbf{x}_I)\, a_i(\mathbf{x}) - u_I \right]^2, \qquad (4)$$

where $w(l)$ is a weighted function with compact support [6], n is the total number of nodes inside the domain of influence, and $l = x - x_I$ is the distance between each integration point and node I inside the domain of influence. Several types of functions can be used as a weighted functions such as exponential or conic functions, and quartic or cubic spline functions. In our research we employed the following quartic spline function given in Fig. 2

$$w(l) = \begin{cases} 1 - 6(\frac{l}{r})^2 + 8(\frac{l}{r})^3 - 3(\frac{l}{r})^4 & \text{for } \frac{l}{r} \le 1 \\ 0 & \text{for } \frac{l}{r} > 1 \end{cases}, \qquad (5)$$

where r is the radius of influence.

Using matrix notation, (4) can be rewritten in the following form

$$J = (\mathbf{P}\,\mathbf{a} - \mathbf{u})^T\, \mathbf{W}(\mathbf{x})\, (\mathbf{P}\,\mathbf{a} - \mathbf{u}), \qquad (6)$$

where $\mathbf{u}$ is the vector of unknowns, $\mathbf{P}$ is the matrix of the nodal coordinates, and $\mathbf{W}(\mathbf{x})$ is the diagonal weighted matrix. The coefficients $\mathbf{a}(\mathbf{x})$ can be easily found by finding the extremum of the quadratic form J

$$\frac{\partial J}{\partial \mathbf{a}} = 2\,(\mathbf{A}(\mathbf{x})\,\mathbf{a}(\mathbf{x}) - \mathbf{B}(\mathbf{x})\,\mathbf{u}) = 0, \qquad (7)$$

where

$$\mathbf{A} = \mathbf{P}^T\, \mathbf{W}(\mathbf{x})\, \mathbf{P}, \qquad (8)$$

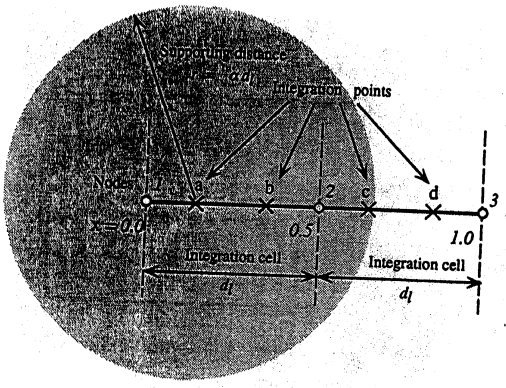
$$\mathbf{B} = \mathbf{P}^T\, \mathbf{W}(\mathbf{x}). \qquad (9)$$

Therefore, for the coefficients $\mathbf{a}(\mathbf{x})$ we have

$$\mathbf{a}(\mathbf{x}) = \mathbf{A}^{-1}(\mathbf{x})\, \mathbf{B}(\mathbf{x})\, \mathbf{u}, \qquad (10)$$

and for the approximation $u^h(\mathbf{x})$, very similar with the ordinary finite element method, we have

$$u^h(\mathbf{x}) = \sum_{I=1}^{n} N_I(\mathbf{x})\, u_I, \qquad (11)$$



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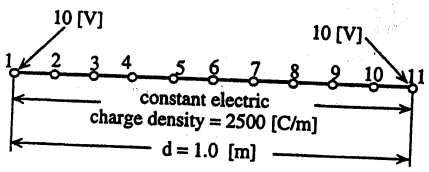

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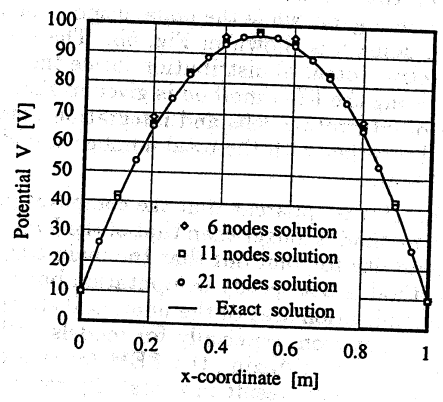

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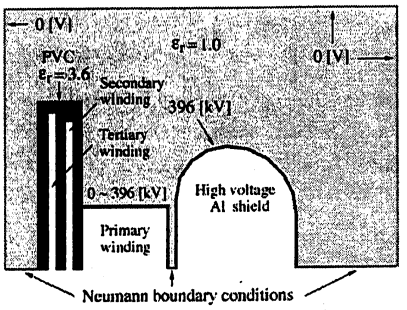
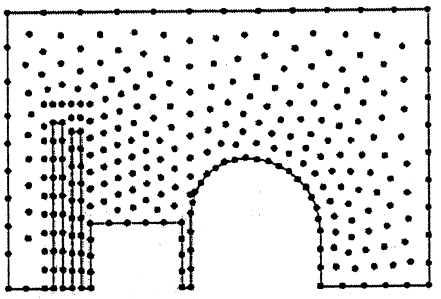


a) Analysis domain and boundary conditions



b) Nodal disposition

Fig. 6. Model of voltage transformer.

TABLE II
Comparison between the EFG method and the FEM - Voltage transformer

	Element-free	*Finite element*
Nodes	316	316
Elements	-	516
Integration cells	345	-
Integration points per cell	9	-
Computation time [sec]	1.312	1.298

and high voltage Al shield with constant potential of 396 [kV]. The potential distribution along primary winding linearly changes from 0 to 396 [kV]. The boundary conditions for the performed electrostatic field analysis are also given in Fig. 6a, while the nodal disposition inside the analyzed domain is shown in Fig. 6b. The obtained result for electric potential distribution inside the voltage transformer using the EFG method is given in Fig. 7, and the number of integration cells and integration points inside each cell together with the total number of nodes are given in Table II.

From Table II it is apparent that the computation time for the EFG method is longer than that of the FEM. However, we would like to point out that the main purpose of the EFG method is not speedy computation but enlargement of the application area of the numerical electromagnetic field computation, especially for models with large geometrical changes. Utilizing the FEM for this class of problems would usually result in repeatedly remeshing of the analysis domain increasing strongly the overall computation time.

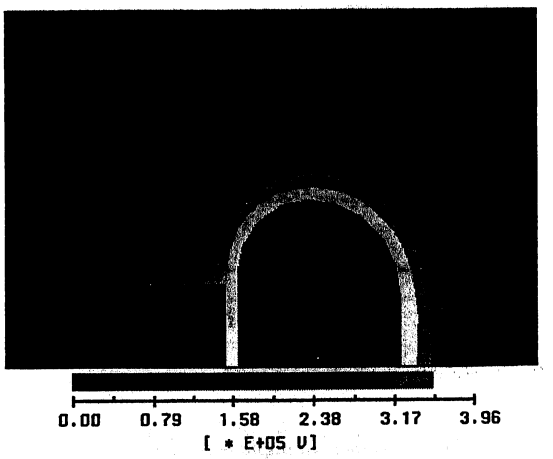


Fig. 7. Obtained results for electric potential

IV. Conclusions

We introduced a new element-free Galerkin method for electromagnetic field computation in one and two dimensional space. The element-free Galerkin method does not require usually laborious and time consuming mesh generation, and only depends on the distribution of a set of nodes inside the analysis domain. Therefore, it is very suitable for numerical analysis whenever the necessity of remeshing of the analysis region is indispensable, such as in case of inverse shape optimization, melting, moving conductors, crack, etc. It was shown that the EFG method easily and accurately deals with electromagnetically homogeneous and inhomogeneous domains, and provides results with the same accuracy as the ordinary finite element method for the slightly longer computation time. Depending only on the number of nodes and their position, the EFG method could be very attractive as an adaptive technique for solutions of various electromagnetic field problems.

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