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MATHEMATICAL THEORY OF HEAT FLOW

BLAGICA DONEVA[1], RADMILA KARANAKOVA STEFANOVSKA[1]

Abstract. This paper presents the mathematical theory of heat flow in the Earth's crust, with emphasis on conductive heat transfer in homogeneous and isotropic media. Starting from Fourier's law, the fundamental differential equations governing one-dimensional and three-dimensional heat flow are derived and extended to include temperature variation with time. The study incorporates the effects of physical parameters such as thermal conductivity, density, and specific heat, leading to the classical heat equation. Special attention is given to geothermal applications, including the interpretation of subsurface temperature distributions and their relation to geological structures. The influence of porosity and moisture content on thermal conductivity is also analysed. Furthermore, periodic temperature variations caused by solar radiation, including diurnal and annual cycles, are mathematically modelled to evaluate temperature changes with depth. The presented theory provides a basis for understanding geothermal processes and their applications in geophysics, engineering geology, and subsurface exploration.

1. Introduction

The formation and release of heat governs the life history of millions of stars distributed throughout the universe. After the initial period of condensation of the nebula from which our Earth was formed, followed a period in which the loss of internal energy by radiation was much more rapid than its formation [1]. From this period onwards, radioactive decay in the Earth became a source of energy of ever-increasing importance. If any of the reasonable assumptions about the distribution of radioactive matter with depth are accepted, it can be shown that 3/4 to 4/5 of the total heat flux through the Earth's surface originates from radioactivity. During the estimated 3,000 million years of our Earth's existence, the production of heat from radioactivity has decreased by about 50 %. Accordingly, the internal heating of the Earth has reached a fairly stable stage and will continue to decline at a slower rate, i.e. exponentially with time [2].

The present temperature of the Earth is influenced by many factors [3]. From a geophysical point of view, one factor is interesting - the dissipation of heat from the interior of the Earth. The heat from the interior of the Earth must pass through the outer part of the crust and thus affect the temperature of the layers near the surface. This temperature is not uniform and depends on the thermal conductivity of the various materials in the surface and near-surface parts of the crust and their geometric arrangement. By studying these temperature variations, the nature and subsurface distribution of these materials can often be predicted.

Temperature measurements of the outer part of the Earth's crust are used to obtain basic data on the origin and history of the studied areas. In addition, temperature

[93]**Keywords:** heat flow, Fourier's law, heat equation, thermal conductivity, temperature distribution, geothermal gradient

measurements, in suitable cases, can provide useful data on the zonal distribution of ores, the configuration of intrusives, the contacts of sedimentary and igneous rocks, the strike of faults, the distribution of groundwater and their local and regional underground flow. The practical application of geothermal investigations for solving problems of economic geology usually comes down to one of two types: a) temperature measurements near the surface, for the study of lateral temperature variations; b) deep measurements in boreholes, for the study of the vertical distribution of temperatures along the hole.

2. Mathematical theory of heat flow

As with the flow of electric current through solids, it is usual to derive a differential equation for one-dimensional heat flow and then generalize the equation to three-dimensional flow. Physically, one-dimensional flow can be represented as flow through a sheet or plate whose two dimensions (y and z) significantly exceed the third (x).

The theoretical discussion will be limited to a homogeneous and thermally isotropic medium.

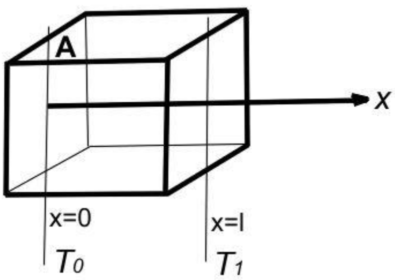

Fig.1. Sketch with one - dimensional heat flow in x direction

Referring to Figure 1, we assume that the temperatures in the planes x = 0 and x = l are T_0 and T_1 respectively, and that $T_0 > T_1$. According to Fourier's law [4, 5], the amount of heat (q) flowing in the x direction through a unit area, and for a time t, is proportional to the product of time and the temperature gradient.

$$q = kt\,\frac{T_0 - T_1}{l} \tag{1}$$

where

q – heat amount

t - time

l - plate thickness

$\frac{T_0 - T_1}{l}$ - temperature gradient

k – constant, thermal conductivity of the material.

k can be a function of position; in general, it is determined at any point when the temperature of a homogeneous medium is known and within reasonable limits.

In differential form, the equation for one-dimensional heat flow in the x direction is:

$$\left(\frac{\partial q}{\partial t}\right)_x = k\frac{\partial T}{\partial x} \tag{2}$$

Equation (2) shows that the amount of heat per unit time transferred through a unit area of a plane perpendicular to the x-direction is equal to the product of the thermal conductivity and the temperature gradient in the x-direction.

The equation for three-dimensional heat flow is obtained from equation (2) by replacing - with the temperature gradient corresponding to the three dimensions. Thus, it is obtained:

$$\frac{\partial q}{\partial t} = k\left(\frac{\partial T}{\partial x} + \frac{\partial T}{\partial y} + \frac{\partial T}{\partial z}\right) \tag{3}$$

From this equation it is seen that the heat flow can be represented by a vector whose x, y and z components are respectively $k\frac{\partial T}{\partial x}, k\frac{\partial T}{\partial y}, k\frac{\partial T}{\partial z}$. Hence it is obvious that the flow field T is constant, because of its gradient that is always perpendicular to the isothermal surfaces.

3. Temperature change per unit time

Equation (3) can be transformed into a second-order differential equation in which only T is an independent variable. To perform this transformation, it is necessary to introduce the **specific heat c** of the material and assume that a heat flow occurs. By definition, the specific heat is the amount of heat, expressed in calories, required to heat a unit mass of a given substance for 1°C.

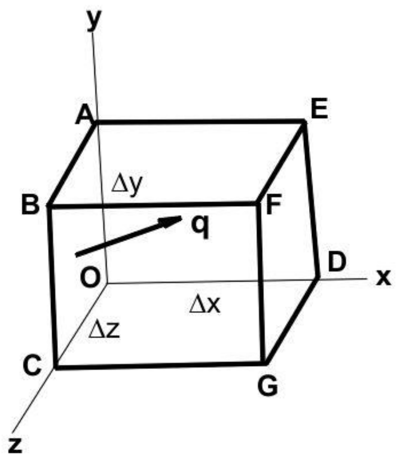

Fig. 2. Sketch showing three – dimensional heat flow

A volume element is considered in the shape of a rectangular parallelepiped with sides Δx, Δy and Δz located in a completely homogeneous and isotropic space of density σ, specific heat c, and thermal conductivity k (fig. 2). If the temperature of the body increases by ΔT, the amount of heat absorbed by this volume element changes by an amount equal to the product of the specific heat, mass, and temperature change. Therefore, the change in absorption per unit time is

$$c\sigma\,\Delta x\,\Delta y\,\Delta z\,\frac{\partial T}{\partial t}$$

If it is assumed that the volume element has neither a source nor a sink of heat, the amount of heat that enters the volume element per unit time must be equal to the amount of heat that leaves the volume element per unit time. The amount of heat that enters the volume element per unit time through the surface OABC is:

$$\left(\frac{\partial q}{\partial t}\right)_x \Delta y \Delta z$$

From the equation (2)

$$\left(\frac{\partial q}{\partial t}\right)_x \Delta y \Delta z = k\frac{\partial T}{\partial x}\,\Delta y \Delta z$$

Let us say it is

$$k\frac{\partial T}{\partial x} = U_x; \quad k\frac{\partial T}{\partial y} = U_y; \quad k\frac{\partial T}{\partial z} = U_z \qquad (4)$$

Expressed in terms of a new variable, the amount of heat that flows through the surface OABC per unit time is

$$U_x\,\Delta y\,\Delta z$$

The amount of heat that flows out from the parallelepiped through the surface, DEFG, per unit time is:

$$\left(U_x + \frac{\partial U_x}{\partial x}\,\Delta x\right)\Delta y\,\Delta z$$

Therefore, the amount of heat that flows out from the element in the x direction in a unite time is

$$U_x\,\Delta y\,\Delta z - \left(U_x + \frac{\partial U_x}{\partial x}\,\Delta x\right)\Delta y\,\Delta z = -\frac{\partial U_x}{\partial x}\,\Delta x\,\Delta y\,\Delta z$$

A similar result is obtained for the amount of heat that flows out in the y and z directions per unit time.

$$-\frac{\partial U_y}{\partial y}\,\Delta x\,\Delta y\,\Delta z \;\; and \;\; -\frac{\partial U_z}{\partial z}\,\Delta x\,\Delta y\,\Delta z$$

And the total amount of heat that flows out per unit of time from that element is

$$-\left(\frac{\partial U_x}{\partial x}+\frac{\partial U_y}{\partial y}+\frac{\partial U_z}{\partial z}\right)\Delta x\,\Delta y\,\Delta z$$

If we replace U_x, U_y, U_z with their equivalents from equation (4), the total amount of heat that flows outs from the element per unit time will be:

$$-k\left(\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}+\frac{\partial^2 T}{\partial z^2}\right)\Delta x\,\Delta y\,\Delta z$$

The amount of heat that flows out per unit time is equal in magnitude and opposite in sign to the heat absorption per unit time in the elementary volume; therefore

$$k\left(\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}+\frac{\partial^2 T}{\partial z^2}\right)\Delta x\,\Delta y\,\Delta z = c\sigma\,\frac{\partial T}{\partial t}\,\Delta x\,\Delta y\,\Delta z$$

or

$$\frac{k}{c\sigma}\left(\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}+\frac{\partial^2 T}{\partial z^2}\right)=\frac{k}{c\sigma}\,\nabla^2 T=\frac{\partial T}{\partial t} \qquad (5)$$

Equation (5) gives the temperature change per unit time at any point in an isotropic homogeneous medium and is equal to $\frac{k}{c\sigma}$ times Laplace's expression for temperature. Thus, the temperature distribution, expressed by equation (5), includes the specific heat, thermal conductivity, and density of the studied material [6]. Therefore, changes in any of these parameters cause changes in the temperature and heat flow distribution. Because of that, measuring temperature changes allows mapping of the contact of rocks with different thermal coefficients and densities.

Equation (5) does not cover cases where sources or sinks of heat are located within a volume element. Therefore, if chemical and radioactive processes play a significant role in the temperature distribution of an area investigated by geothermal methods, the equation (5) can be modified to take into account such heat sources and sinks in the vicinity of geothermal stations [7, 8].

At constant flow and constant conductivity, the amount of heat flowing per unit time is constant, and the analogy with other fields, e.g. Newton's force field, is complete. An example of this is a rod with constant temperature at the ends, without sources or sinks in it, and when the temperature at each point is independent of time. In this case:

$$\frac{\partial T}{\partial t}=0 \quad and \quad \nabla^2 T=\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}+\frac{\partial^2 T}{\partial z^2}=0 \qquad (6)$$

An even closer analogy exists with the steady flow of electricity, where

$$i=\frac{\partial Q}{dt}=\sigma\,grad\,u \qquad \text{[compare with equation (3)]}$$

$$div\,i=\nabla^2 u=0 \qquad \text{[compare with equation (6)]}$$

where
Q – charge
i – current density (compared with heat flow)
u – potential (compared with temperature)
σ – electrical conductivity (compared with heat conductivity)

It is obvious from the assumption $\frac{\partial T}{\partial t} = 0$ that Laplace's equation can be applied to the Earth where temperature is not a function of time. Neglecting all internal sources of heat, such as radioactive decay, and neglecting the upper part of the Earth's crust which is exposed to the radiation from the Sun, what remains of the Earth that is measurable is subject to Laplace's equation.
If we consider that the Earth is a homogeneous and isotropic sphere, Laplace's equation can be most easily solved by introducing spherical coordinates with the obvious assumption that temperature is a function only of radial distance. If we perform the transformation

$$\nabla^2 T = \frac{1}{r}\frac{d^2(rT)}{dr^2} = 0 \tag{7}$$

and integration

$$T = A + \frac{B}{r} \tag{8}$$

we obtain an expression showing the temperature T as a function of the radial distance r (A and B are constants). It is obvious that the temperature increases as r decreases, measured from the centre of the Earth as always in spherical coordinates. The Earth is not a homogeneous sphere and the deviation, observed in the measurements, from the theoretical temperature - depth curve must be attributed to this inhomogeneity.

4. Porosity and temperature

The diagram in Figure 3 allows a comparison of the depth-temperature curve with the porosity-depth curve in a borehole. These data show that the increase in temperature gradient with depth can depend on the moisture content in the rocks. The product of the temperature gradient and the thermal conductivity represents the amount of heat that flows through a vertical column of unit cross-section in time t.

$$q = kt\,\frac{\Delta T}{l}$$

where
q – heat amount
t - time
l - plate thickness
ΔT - temperature gradient
k – constant, thermal conductivity of the material.

If it is assumed that there are no sources or sinks nearby (q = const.), then this product must be constant for a certain time t, and it is

$$\frac{q}{t} = const. = k\ \frac{\Delta T}{l}$$

Therefore, if the gradient *ΔT/l* increases with depth, the conductivity k decreases with depth. The porosity decreases with depth in the same way. Therefore, the conductivity becomes smaller as the moisture content decreases, and the temperature gradient changes inversely with the conductivity.

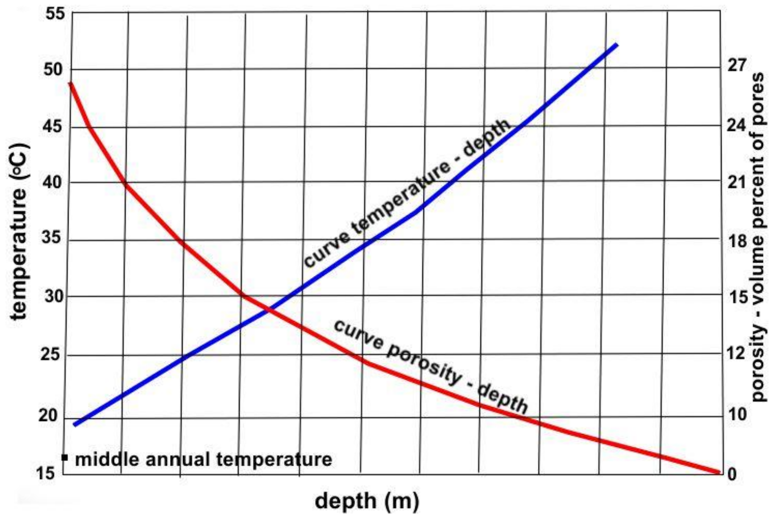

Fig. 3. Temperature and porosity curves as a function of depth. [9]

6. Periodical heat flow

Solar radiation causes periodic changes in the heat flux through points near the Earth's surface. The periodicity of temperature changes is twofold: one is related to the position of the Sun during the day relative to the horizon, and the other is related to the change in the apparent path of the Sun during the year. These two periodicities are called diurnal variation and annual variation respectively [10, 11].

The heat of the sun reaches the surface of the earth mostly by radiation. The conductivity of the atmosphere is very low. The sun's spectrum has a peak in the yellow part, and the absorption of this radiation in the atmosphere is smaller than that of longer wave lengths. During the night, the Earth, which has a much lower temperature and therefore a spectrum with a peak far in the infrared part, loses a great part of the heat received during the day. Atmosphere absorbs a large part of this radiation and therefore acts as a protective screen against loss of radiation, and this factor, combined with the low conductivity of the atmosphere, allows the earth in the summer to retain during the night a part of the heat received from the sun. Hence, in the summer, the outer part of the earth’s crust gains heat over a 24-hour period. In the winter, however, the losses during the night exceed the gains during the day, and the balance is negative over a 24-hour period.

During the day, the temperature is at maximum between 2 and 3 p.m. and at minimum just before sunrise. The plotting of temperature against time during a 24-hour period affords a means for evaluating the gains and losses over the whole or part of the period.

The diurnal temperature variations manifest themselves only to a depth of a few meters below the surface. The depth to which the changes are measurable depends on the character of the rocks close to the surface. In solid rock formations, the diurnal variations usually become imperceptible at a depth of about 1 meter. In loose sand and alluvial fill material, the variations escape measurement at a depth of 30 to 60 cm. In swamps and porous materials containing water and in areas where the water table is near the surface of the earth, the variations become imperceptible at a depth of 0,5 to 1,5 m.

The annual variation for a given area is determined from data on measurements made intermittently throughout the year, or daily, at the same time of day.

The annual variation can be observed to a depth of between 23 and 300 m, depending on the thermal properties of the rocks. The periodic variations in temperature may show local fluctuations due to changes in meteorological conditions and to differences in topography and overburden.

Below 30 m, the temperature depends, in general, only on the flow of heat from the centre of the earth.

Diurnal variations (when they have not been excluded from measurements by a sufficient depth of bore hole) are corrected for in a manner very similar to that used in correcting for the diurnal magnetic variations, or by calculation.

7. Calculating periodic temperature variations

The theory which is applied in this case is based on Equation 5, the general equation excluding internal sources or sinks

$$\frac{k}{c\sigma}\,\nabla^2 T = \frac{\partial T}{\partial t} \tag{5}$$

The one-dimensional form of Equation 5 may be used without introducing any appreciable error. This implies the assumption that the surface of the earth is presented as a plane in the region of investigation. The surface is thus uniformly radiated, and the temperature is a function only of the depth z.

$$\frac{k}{c\sigma}\,\frac{\partial^2 T}{\partial z^2} = \frac{\partial T}{\partial t} \tag{5a}$$

Since the sun's radiation is roughly periodic, the change in heat flow through points close to the surface of the Earth is likewise periodic in both diurnal and annual variations. At the surface, then, the following periodic boundary condition must be imposed

$$T = T_0 \sin \omega t \qquad \text{for} \qquad z = 0 \tag{9}$$

Equation (5a) is linear and homogeneous, and a particular solution has the form

$$T = Ae^{\alpha t + \beta z} \tag{10}$$

which, by substitution into Equation 5a, is a solution if, and only if,

$$\alpha = \frac{k}{c\sigma} \beta^2$$

With proper substitutions, and noting that Equation 13 is a particular solution, a total of four particular solutions may be written as follows:

$$T = Be^{-R} sin(\gamma t - R) \tag{11}$$

$$T = Be^{R} sin(\gamma t + R) \tag{12}$$

$$T = Ce^{-R} cos(\gamma t - R) \tag{13}$$

$$T = Ce^{R} cos(\gamma t + R) \tag{14}$$

where

$$R = \frac{z}{\sqrt{k/c\sigma}} \sqrt{\frac{\gamma}{2}}$$

and

$$\alpha = \pm r\gamma$$

Equations 12 and 14 demand that the temperature increases indefinitely as z increases, which is impossible. Equation 13 is excluded by the boundary conditions of Equation 9. Equation 11 satisfies 9 if

$$B = T_0 \quad and \quad \gamma = \omega$$

With these substitutions, the solution of Equation 5a is

$$T = T_0 e^{-\frac{z}{\sqrt{k/c\sigma}}\sqrt{\frac{\omega}{2}}} \sin\left(\omega t - \frac{z}{\sqrt{k/c\sigma}} \sqrt{\frac{\omega}{2}}\right) \tag{15}$$

which yields the temperature at any time t, at any depth z from the surface.

The range of temperature for any point below the surface may be calculated from the maximum variation of the temperature at the point. Since $sin\, \theta = \pm 1$ for a maximum and minimum respectively,

$$R_T = 2T_0 e^{-\frac{z}{\sqrt{k/c\sigma}}\sqrt{\frac{\omega}{2}}} = 2T_0 e^{-\frac{z}{\sqrt{k/c\sigma}}\sqrt{\frac{\pi}{P}}} \tag{16}$$

where is $\omega = \frac{2\pi}{P}$, and P is the period of variation of the solar radiation. From the equation (16) it is seen that T_0 is the amplitude or half range at the surface (z = 0).

Consider the diurnal wave as an example. Let us suppose that at a certain season of the year, the surface temperature of the soil $\left(\frac{k}{c\sigma} = 0{,}0049\right)$ varies from + 16° C. to - 4° C. The surface half range value (T_0) is then $\frac{16-(-4)}{2} = 10℃$, P = 24 hours = 86 400 seconds. The mean surface temperature is 6°.

Equation 16 should be used to determine the temperature range at 30 cm and at 1 m, as follows:

$$z = 30\ cm \qquad R_T = 2\,(10)e^{-\frac{30}{\sqrt{0{,}0049}}\sqrt{\frac{\pi}{86400}}} = 2\,(10)\,(0{,}07) = 1{,}4\ ℃$$

$$z = 100\ cm \qquad R_T = 2\,(10)e^{-\frac{100}{\sqrt{0{,}0049}}\sqrt{\frac{\pi}{86400}}} = 2\,(10)\,(0{,}00019) = 0{,}0038\ ℃$$

Conclusions

The mathematical treatment of heat flow demonstrates that temperature distribution within the Earth is governed by well-defined physical laws and material properties. The derived heat equation shows that variations in thermal conductivity, density, and specific heat significantly influence heat transfer and temperature gradients. These relationships enable the use of temperature measurements as an effective tool for identifying subsurface structures and geological features. The analysis confirms that porosity and moisture content play an important role in controlling thermal conductivity and, consequently, heat flow behaviour. Additionally, periodic temperature variations due to solar radiation are shown to diminish with depth, allowing deeper regions to reflect primarily internal geothermal conditions. Overall, the study highlights the importance of mathematical modelling in geothermal investigations and provides a theoretical framework applicable to practical problems in geophysics and engineering.

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Blagica Doneva
University Goce Delcev
Faculty of Natural and Technical Sciences,
“Goce Delcev” No. 89,
Stip, Republic of North Macedonia
e-mail address: blagica.doneva@ugd.edu.mk

Radmila Karanakova Stefanovska
University Goce Delcev
Faculty of Natural and Technical Sciences,
“Goce Delcev” No. 89,
Stip, Republic of North Macedonia
e-mail address: radmila.karanakova@ugd.edu.mk