Review of models used to price options: Stock-price vs option price and Implied volatility vs actual volatility comparison for different option pricing methods

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Abstract

This paper will review LVM models, GP-LVM, CEV model, DD model, Crank-Nicolson, and other finite difference methods, Greeks, SABR, martingales, and LSMC option pricing. The effects of changing the volatility on paths generated by Bachelier, Black-Scholes proved no difference between these two models. Implied volatility for all the models was higher when compared to actual volatility for:BS,BSM, and Bachelier. Crank-Nicolson method for ATM, ITM,OTM showed higher intrinsic value for the price after the initial stock price, but with diminishing returns, intrinsic minus extrinsic value is zero at the last price. In Greeks analysis t was observed no put and call parity for different values of :Delta,Gamma,rho. GP-LVM forecast proved to be closest to the actual stock price.

Keywords: Option pricing, local volatility, stochastic volatility, finite difference methods, Greeks

JEL: G12, G13

1.Introduction

The market implied volatilities of stock index options have skewed structure, called volatility smile[1].This has been problem in option pricing literature, how to reconcile skewed volatility smile with Black-Scholes model (see Black-Scholes(1973)). B-S model assumes that the index level is a random walk[2] with constant volatility, but this seems to be false since if it is true then the index distribution at any option expiration is log-normal[3], and all options on the index must have same implied volatility. Implied volatility is calculated by taking the observed option price in the market and a pricing formula such as the Black-Scholes formula that will be introduced below and backing out the volatility that is consistent with the option price given other input parameters such as the strike price of the option, for example ,see Kosowski,Neftci (2015).But Derman,Kani (1994) conclude that "87 crash (1987 The first

[1] The relationship between implied volatility and exercise price is not constant and may look like a smile, a skew, etc. (for simplicity are all called "smiles") see Orlando,G., Taglialatela(2017).

[2] Definition for Random walk: $X_1, \dots, X_n$ is a sequence pf $\mathbb{R}^d$ valued independent and i.i.d random variables. A random walk started at $z \in \mathbb{R}^d$ is the sequence $(S_n)_{n\geq 0}$ where $S_0 = z \wedge \mathrm{S_n} = \mathrm{S_{n-1}} + \mathrm{X_n}, \mathrm{n} \geq 1, X_n$ are referred as steps of the random walk.

[3] The stock evolution is described simple as: $\frac{dS}{S} = \mu dt + \sigma dW$, where μ is the expected return $\mu = r - q$,risk free rate minus dividend, S is the stock price, dW is a Wiener process $W \sim (0, dt)$

contemporary global financial crisis unfolded on October 19, 1987, a day known as “Black Monday,” when the Dow Jones Industrial Average dropped 22.6 %), the market’s implied Black Scholes volatilities for index options have shown a negative relationship between implied volatilities and strike prices – out-of-the-money puts trade at higher implied volatilities than out-of-the-money calls”. Schwert (1987) has done a time series analysis of the volatility of U. S. stock prices 1859- 1986 and compared this volatility through time with other macroeconomic variables[4],see Shiller (1988). The first explicit general equilibrium solution to the option pricing problem for simple puts and call was presented in Black,Scholes(BS) (1973) and Merton (BSM) (1973) all these four papers by Merton 1973 a,Merton 1973 b, Merton1973 c ,Merton 1975 , provide, within the Capital Asset Pricing Model(CAPM) framework, an elegant answer to the problem of assigning price to every option by identifying a relation between the value of the stock and its option. LVM or Local volatility models can be traced back to the work by Dupire (1994),and Derman,Kani (1994). They (previous papers) have realized that under the assumption of risk neutrality a unique state-dependent diffusion process can be constructed which is consistent with market prices for European options, where those prices are quoted[5].As we will see later when we will introduce this model in this paper σ_l represents diffusion coefficient which is a function of space and time. This paper will summarize main formulae for LVM models and the relationship between local volatilities and implied volatilities, for further discussions see Gatheral, J. (2006), Rebonato, R. (2004),see Kienitz, Wetterau (2012). These models in order to be consistent with the Efficient market hypothesis (EMH) see Fama (1970) (i.e. that asset prices fully reflect the information), the unanticipated part of the stock price movements should be a martingale (conditional expectation of the next value of the sequence, given all prior information, is equal to the present value). With the papers by LeRoy and Porter (1981) and Shiller(1981), a literature has emerged arguing that financial markets may be too volatile to be accounted for in terms of efficient markets hypothesis (EMH).Later we will study Stochastic local volatility (SVM) models resented by SABR model. The concept of an SVM applies the idea of a second source of randomness (these models are capable of modeling not only the skew but the smile too). Therefore, we are adding another source risk to the modelling. It is a proper assumption that the randomness of volatility is modelled dependent on the asset. This is called leverage. CEV model[6] if $\beta = 2$ degenerates to B-S model. Empirical evidence (see Beckers (1980)) has shown that the CEV diffusion process could be a better candidate for describing the actual stock price behavior than the BS model. The exponent β is called CEV exponent. The CEV model furthermore can be seen as an arithmetic average of normal and logarithmic normal model[7]. CEV and displaced diffusion models are related. As a rule of thumb as closer $\beta \approx 1$ better DD model approximates the CEV model. The CEV and displaced diffusion processes have been posited as suitable alternatives to a lognormal process in modelling the dynamics of market variables such as stock prices and interest rates, see Svoboda-Greenwood, S. (2009). Some authors first of them would be Marris

[4] Schwert (1987) found that the volatility of inflation, money growth, industrial production and business failures is high during war periods, yet the volatility of stock returns is not particularly high during those periods.

[5] In the financial markets, a quoted price is the last price at which a trade took place. Quoted prices is the lowest price at which the holder of a security is willing to sell it. In other sales transactions, the quoted price is the estimate given to provide goods or services.

[6] $dS = \mu S_t dt + \sigma S_t^{\frac{\beta}{2}} dB_t$

[7] We take $v = \frac{1}{2(1-\beta)}; \tau = T - t, x = S(t)\,; p(X,t;x,t) = \frac{\sqrt{xX^{1-4\beta}}}{(1-\beta)\sigma_{cev}^2} \exp\left(-\frac{x^{2(1-\beta)}+X^{2(1-\beta)}}{2(1-\beta)\sigma_{cev}^2}\right) I_v\left(\frac{(xX)^{1-\beta}}{(1-\beta)^2\sigma_{cev}^2\tau}\right)$, where I_v is a Bessel function which are functions that serve as solutions to difference equation: $x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + (x^2 - n^2)y = 0$,but in the first kind Bessel function is given as:$I_v(x) = \left(\frac{x}{2}\right)^v \sum_{k=0}^{\infty} \frac{\left(\frac{x^2}{4}\right)^k}{k!\Gamma(v+k+1)}$,see Kienitz, Wetterau (2012).

(1999), stated that for a linear parametrization option prices by CEV and DD model display similar close correspondence across many ranges of strikes and maturities. The Crank-Nicolson method was the approximation of the implicit method and the explicit method, and we are applying it later in this paper as part of finite difference methods. This method approximation is more accurate than either implicit or explicit method finite difference method approximations[8]. It has faster convergence than implicit and explicit finite difference methods, see Kiprop,K.G.,Langata set2019).Finite difference[9] methods are used to price options by approximating continuous time differential equation that is used to describe how an options price evolves over time, by a set of difference equations. Greeks measure a different dimension to the risk in an option position and the aim of a trader is to manage the Greeks so that all risks are acceptable, see Hull (2012). They are financial measures of the sensitivity of an option's price to its underlying determining parameters, such as volatility or the price of the underlying asset. They are collectively called: risk measures, hedge parameters. SABR model or stochastic α, β, ρ model is a stochastic volatility model, which attempts to capture the volatility smile[10] in derivatives markets. The SABR model is an extension of the CEV model in which the volatility parameter follows a stochastic process[11].Martingale option pricing formulae are in form of expectations which can be efficiently solved numerically by using a Monte Carlo approach. This paper will investigate models of Local volatility (Bachelier and Black-Scholes) introduced by Bachelier (1900),and Black-Scholes (1973)[12],versus Gaussian Process volatility model or GP-Vol is presented as an instance of GP-SSM or Gaussian process state-space model see Tegner, Roberts (2019), and a Wu, Y. Lobato, J. M. H., Ghahramani,Z.(2014), and Wu, Y. Lobato, J. M. H. , Ghahramani,Z.(2014) .Next, we will introduce CEV (Constant elasticity of variance) introduced by Cox (1975). And DD (Displaced diffusion model) introduced by Rubisnstein (1983), also see Schroder, M. (1989) and Andersen, L. ,Andreasen, J.(2000), see Kienitz, Wetterau (2012). Later we will see how CEV and DD model are related and will investigate CEV approximation exposed by Merino (2020). This will be followed by Crank-Nicolson finite difference method that represents an average of the implicit method and the explicit method, see (Crank,Nicolson (1947)).Later we will investigate Greeks this material draws heavily from Hull (2012).Next, we will introduce SABR (Stochastic, α, β, ρ model) introduced by Hagan, P.S., Kumar, D., Lesniewski, A.S., Woodward, D.E. (2002). Martingale option pricing will be reviewed this method was introduced in option pricing by Ross (1976), Lèvy process, followed by Monte-Carlo methods of option pricing due to Boyle (1977),and LSMC- Least squares Monte Carlo method proposed by Longstaff and Schwartz (2001).

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[8] As described in Fadugba, SE and Nwozo, CR. (2013), it can be shown that by equating the central difference and the symmetric central difference at $F_{n+\frac{1}{2},m} \equiv F(t+\frac{\Delta t}{2}, S)$, we would end up with the Crank-Nicolson method with an accuracy of $\mathcal{O}((\Delta t)^2, (\Delta S)^2)$.

[9] Finite forward difference is defined as:$\Delta f_p \equiv f_{p+1} - f_p$ and finite backward difference is :$\nabla \mathrm{f_p} \equiv f_p - f_{p-1}$

[10] Volatility smiles are implied volatility patterns that are present in pricing financial options. It is a parameter implied volatility that is needed to be modified for the Black–Scholes formula to fit market prices

[11] $dF(t) = \sigma(t)F(t)^{\beta} dW(t); d\sigma(t) = \alpha\sigma(t)dZ(t); E\big(dW(t), dZ(t)\big) = \rho dt$ where ρ is assumed constant.

[12] The Black-Scholes(1973) model is the simplest formulation for derivative pricing and is still utilized, there is a flaw of that model when volatility surfaces, a situation which implies different underlying parameters for every quoted option, so in that situation Black-Scholes(1973) model is unable to correctly predict the evolution of prices of the underlying asset, see Hirsa (2012). Despite its popularity, it is well known that the BS model suffers from several deficiencies, such as inconsistencies with the market-observed implied volatility smile (or skew). The Black-Scholes theory relies on two assumptions: the values of contingent claims do not depend on investor preferences; therefore, the option can be valued as though the underlying stock's expected return is riskless. The risk neutral valuation is allowed because the option can be hedged with stock to create instantaneously riskless portfolio.

2. Local volatility models (LVM)

GBM model (Geometric Brownian motion) for stock prices states[13]:

equation 1

$$dS_t = \mu S_t dt + \sigma S_t dW_t$$

In previous σ, μ are constants. Success in applying LVM models can be attributed to Dupire, B. (1994) and Derman, E. and Kani, I. (1994).Now, we can summarize the main formulae illustrating the relationship between local volatilities, implied volatilities and European Call option prices. Following results with proofs can be found in Gatheral, J. (2006), Rebonato, R. (2004) .

Theorem 1. The Dupire Formula. Now, Let $C = C(K, T)$ be the price of a call option as a function of strike and time to maturity. Hence the Local volatility function (LVF) satisfies;

equation 2

$$\sigma_l^2(T, K) = \frac{\frac{\partial C}{\partial T} + (r - q)K \frac{\partial C}{\partial K} + qC}{\frac{K^2}{2}\frac{\partial^2 C}{\partial K^2}}$$

Proof: from forward price $C(T, K) = \int_K^\infty (x - K)\phi(T, x)dx$,where K is strike and T is exercise time, and $\phi(T,\cdot)$ is the density function. The actual market price at $T = 0$ would be $p = C(T, K)e^\wedge - (\int_t^T r(s)ds)$ and $\mu(t) = r(t) - q(t)$, where $r(\cdot), q(\cdot)$ are possibly time varying. From forward equation $\frac{\partial \phi}{\partial t} = \frac{1}{2}\frac{\partial^2}{\partial x^2}(\sigma^2(t, x)x^2\phi)$.So now from $C(T, K) = \int_K^\infty (x - K)\phi(T, x)dx$ we have:

equation 3

$$\frac{\partial C}{\partial T}(T, K) = \int_K^\infty (x - K)\frac{\partial \phi}{\partial T}(T, x)dx = \frac{1}{2}\int_K^\infty (\sigma^2 x^2 \phi)''(x - K)dx$$
$$= -\frac{1}{2}\int_K^\infty (\sigma^2 x^2 \phi)' 1 dx = \frac{1}{2}\sigma^2(T, K)K^2\phi(T, K) - \frac{1}{2}\sigma^2\,(T, K)K^2\frac{\partial^2 C}{\partial K^2}(T, K)$$

So this gives Dupire formula which is:

equation 4

$$\sigma(T, K) = \frac{1}{K}\sqrt{\frac{2\frac{\partial C}{\partial T}(T, K)}{\frac{\partial^2 C}{\partial x^2}(T, K)}}\ \blacksquare$$

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[13] GBM motion is defined as:$y(t) = e^{x(t)}$ where $\{y(t), t \geq 0; x(t), t \geq 0\}$,one may also write previous as:$\log\big(Y(t)\big) = X(t) = \mu t + \sigma B(t)$.The solution to this model is given as: $Y(t) = Y(0)\exp\left\{\left[\mu - \frac{1}{2}\sigma^2\right]t + W(t)\right\}$,previously $Y(t) = e^{X(t)}; X(t) = \mu t + \sigma B(t)$; and we define: $f\big(Y(t)\big) = \ln Y(t) = X(t)$ so now:$\frac{\partial f}{\partial t} = 0; \frac{\partial f}{\partial Y} = \frac{1}{Y}; \frac{\partial^2 f}{\partial Y^2} = -\frac{1}{Y^2}$,since $\mu(Y(t), t) = \mu Y(t), \sigma(Y(t), t) = \sigma Y(t)$,from Ito's formula we have: $df(Y(t), t) = \left\{\mu - \frac{1}{2}\sigma^2\right\}dt + \sigma dB(t)$,we now know that $d[\ln\{Y(t)\}] = \frac{dY(t)}{Y(t)} = \left[\mu - \frac{1}{2}\sigma^2\right]dt + \sigma dB(t)$ from previous we obtain that : $\int_0^t d\ln Y(u) = \int_0^t \left\{\mu - \frac{1}{2}\sigma^2\right\}du + \int_0^t \sigma dB(t)$. Now since $B(0) = 0$ we have that $\ln\frac{Y(t)}{Y(0)} = \left\{\mu - \frac{1}{2}\sigma^2\right\}t + \sigma B(t)$ .Form here we get solution for $Y(t) = Y(0)\exp\left\{\left[\mu - \frac{1}{2}\sigma^2\right]t + W(t)\right\}$ and $X(t) = \left[\mu - \frac{1}{2}\sigma^2\right]t + \sigma B(t) \Rightarrow dX(t) = \left[\mu - \frac{1}{2}\sigma^2\right]t + \sigma B(t)$.

Previously if we differentiate $C(T,K) = \int_K^\infty (x-K)\phi(T,x)dx$ twice we get :

equation 5

$$\frac{\partial C}{\partial K} = -\int_K^\infty \phi(x)dx = \Phi(T,K) - 1$$
$$\frac{\partial^2 C}{\partial K^2} = \phi(T,x)$$

Where in previous $\Phi(T,\cdot)$ is the distribution function of S_T,

*Theorem 2. Gyöngy theorem (*see Gyöngy, I. (1986)) Now, let W be and r-dimensional Brownian motion and let :

equation 6

$$dX(t) = \mu_t dt + \sigma_t dW(t)$$

Is a d-dimensional Itô process where μ is bounded d-dimensional adapted process and σ is a bounded $d \times r$-dimensional adapted process such that $\sigma\sigma^T$ is uniformly positive definite.∃ deterministic measurable function $\hat{\mu}$ and $\hat{\sigma}$ such that:

equation 7

$$\hat{\mu}(t, X(t)) = E[\mu_t | X(t)]\ \forall t,$$
$$\hat{\sigma}\hat{\sigma}^T(t, X(t)) = E[\sigma_t\sigma_t^T | X(t)]\ \forall t$$

And there exist solution to differential equation:

equation 8

$$d\hat{X}_t = \hat{\mu}(t,\hat{X}_t)dt + \hat{\sigma}(t,\hat{X}_t)d\widehat{W}_t$$

So that $\mathcal{L}(\hat{X}_t) = \mathcal{L}X(t) \forall t \in \mathbb{R}_+$. Gyöngy's theorem is only valid for continuous Itô processes. It is important to extend the result to processes with jumps.

Proof: Let following applies: $\mu(A) = E\left[\int_0^\infty 1_A(X(t))e^{-\int_0^\infty \gamma(s,\omega)ds}dt\right]$ where ∀ Borel set $A \subset R^n$,where 1_A denotes indicator function of the set,γ is a non-negative t adapted stochastic process. Denote the mimicking process by $Y(t)$. Gyöngy showed that the Green measure[14] of $(t, X(t))$ is identital to the Gren measure of $(t, Y(t))$ .Now let $\gamma(t)$ killing rate $\gamma(t) \equiv 1$ .so now we have:

equation 9

$$\mathrm{E}\left[\int_0^\infty e^{-t} f(t, X(t)dt\right] = \mathrm{E}\left[\int_0^\infty e^{-t} f(t, Y(t))dt\right]$$

Gyögy proved his theorem by extending a result of Krylov (1984).Now, taking $f(t,x) = e^{-\lambda t}g(x)$ with this arbitrary non-negative constant λ and functions $g \in C_0(\mathbb{R}^n)$ we get:

[14] The Green formula: $E^x[T_D] < +\infty, \forall x \in D$, and let $f: R^n \to R$ be a C^2 with compact support ,then: $f(x) = E^x[f(X_{\tau_D})] - \int_D L_x f(y)G(x,dy)$. In particular C^2 functions ⊂⊂ in D. And, $f(x) = -\int_D L_x f(y)G(x,dy)$ .The proof of Green formula is done by Dynkin's formula and the definition of the Green measure: $E^x[f(X_{\tau_D})] = f(x) + E^x[\int_0^{\tau_D} L_x f(X_s)ds = f(x) + \int_D L_x f(y)G(x,dy)]$.The infinitesimal generator or Fourier multiplier operator A of X is $Af(x) = \lim_{t \downarrow 0} \frac{E^x[f(X_t)] - f(x)}{t}$, see Dynkin E.B.(1965).

equation 10

$$\int_0^{\infty} e^{-\lambda t} e^{-t} E[gX(t)]dt = \int_0^{\infty} e^{-\lambda t} e^{-t} E[gY(t)]dt$$

So previous gives us $E[gX(t)] = E[gY(t)]$■ . Or in general lets consider n-dimensional Itô process X_t that satisfies:

equation 11

$$dX_t = \alpha(t,\omega)dt + \beta(t,\omega)dW_t$$
$$\exists\ (dY_t = a(t,Y_t)dt + b(t,Y_t)dW_t) - according\ to\ \text{Gyögy theorem}$$

Where X_t, Y_t have the same marginal distributions. $X_t. Y_t$ have the same distribution $\forall t$.Moreover, Y_t can be constructed by setting this:

equation 12

$$a(t,y) = E_0[\alpha(t,\omega)|X_t = y]$$
$$b(t,y)b(t,y)^T = E_0[\beta(t,\omega)\beta^T(t,\omega)|X_t = y\,]$$

In a financial setting, X_t might be risk-neutral dynamics of a particular security. Then $b(t,y)/y$ represents the local volatility function $\sigma(t,\cdot)$,because X_t, Y_t have same marginal distributions then we know that European option prices can be priced correctly if we assume price dynamics are given by Y_t .

2.1 Bachelier and Black-Scholes model

Pricing in Bachelier model is given as:

equation 13

$$C(K,T) = \begin{cases} (S-K)^+ + \dfrac{|S-K|}{4\sqrt{\pi}}\tilde{\gamma}\left(-\dfrac{1}{2}, \dfrac{(S-K)^2}{2\sigma_B^2 T}\right) & S \neq K \\ (S-K)^+ + \dfrac{\sigma_B\sqrt{T}}{\sqrt{2\pi}} & \text{otherwise} \end{cases}$$

In previous $\tilde{\gamma}(a,b)$ denotes incomplete gamma Γ function[15].In Black, F. ,Scholes, M. (1973) the authors consider the following model for an asset price:

equation 14

$$dS(t) = \mu S(t)dt + \sigma_{BS} S_t dW_t$$
$$S(0) = S_0$$

See Kienitz,J. Wetterau,D.(2012). In Bachelier model [16]previous is the same except$\sigma = \sigma_B$.Pricing European call and put option prices in the case of Bachelier model is:

[15] Incomplete Gamma function., i.e. the regularized incomplete gamma function P and the regularized upper incomplete gamma function Q are defined by : $P(x,a) = \frac{1}{\Gamma(a)}\int_0^x t^{a-1}e^{-t}dt$, $Q(x,a) = \frac{1}{\Gamma(a)}\int_x^{\infty} t^{a-1}e^{-t}dt$,gamma function is defined by : $\Gamma(a) = \int_0^{\infty} t^{a-1}e^{-t}dt.$

[16] The Bachelier model is a model of an asset price under Brownian motion presented by Louis Bachelier in his PhD thesis : The Theory of Speculation

equation 15

$$C(K,T) = \left(S(0)e^{-dT} - e^{-rT}K\right)\mathcal{N}(d_1) + \sigma_B\sqrt{T_n}(d_1)$$
$$P(K,T) = \left(S(0)e^{-dT} - e^{-rT}K\right)\mathcal{N}(-d_1) + \sigma_B\sqrt{T_n}(d_1)$$

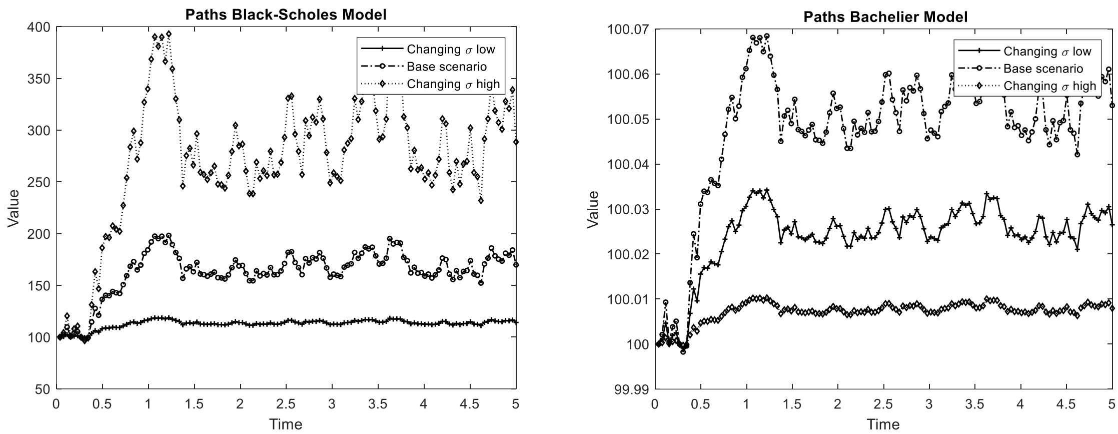
Where : $d_1 = \frac{S(0)\exp((r-d)T)-K}{\sigma\sqrt{T}}$ where $\mathcal{N}(\cdot)$ denotes CDF normal distribution and $n(\cdot)$ corresponding PDF.For the B-S model we have:

equation 16

$$C(K,T) = \left(S(0)e^{-dT} - e^{-rT}K\right)\mathcal{N}(d_1) - e^{-rT}K\mathcal{N}(d_2)$$
$$P(K,T) = \left(S(0)e^{-dT} - e^{-rT}K\right)\mathcal{N}(-d_1) - e^{-rT}K\mathcal{N}(-d_2)$$

Where $d_1 = \frac{\log\left(\frac{S(0)}{K}\right)+\left(r-d+\frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}$; $d_2 = d_1 = \frac{\sigma^2}{2}T$ .Next, we plot the effects of changing volatility on paths generated by Bachelier and Black-Scholes model.

Figure 1 The effects of changing the volatility on paths generated by Bachelier, Black-Scholes $\sigma_B = 0.01, 0.02, 0.03$ and $\sigma_{BS} = 0.05, 0.2, 0.4$



Source :Authors' calculations based on a code available at: https://www.mathworks.com/matlabcentral/fileexchange/36966-risk-neutral-densities-for-financial-models?s_tid=FX_rc2_behav

3.Gaussian process Local volatility model

In this part of the paper presented models are from Tegner, Roberts (2019), and a Wu, Y. Lobato, J. M. H. , Ghahramani,Z.(2014). In the second reference model by Wu, Y. Lobato, J. M. H. , Ghahramani,Z.(2014) Gaussian Process volatility model or GP-Vol is presented as an instance of GP-SSM or Gaussian process state-space model:

equation 17

$$x_t \sim \mathcal{N}(0, \sigma_t^2); v_t := \log(\sigma_t^2) = f(v_{t-1}, x_{t-1}) + \epsilon_t, \epsilon_t \sim \mathcal{N}(0, \sigma_n^2)$$

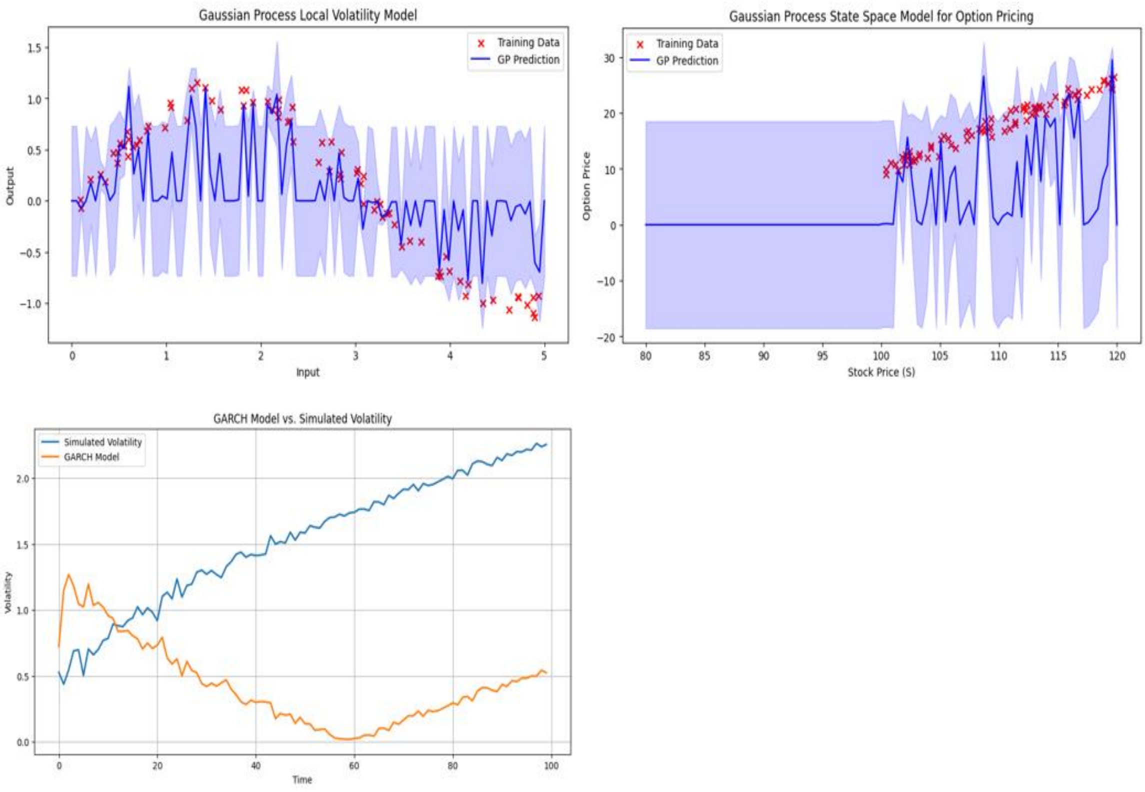
The previous equation defines GP-SSM. Advantages of GP-Vol model are: GP-Vol Models the unknown transition function in a non-parametric manner, GP-Vol reduces the risk of overfitting by following a full Bayesian approach. Most popular volatility model is GARCH:

equation 18

$$x_t \sim \mathcal{N}(0, \sigma_t^2); \sigma_t^2 = \alpha_0 + \sum_{i=1}^{q} \alpha_j x_{t-j}^2 + \sum_{i=1}^{p} \beta_j \sigma_{t-i}^2$$

Next figure graphically depicts these models.

Figure 2 GP-Vol,GP-SSM,GARCH model vs simulated volatility



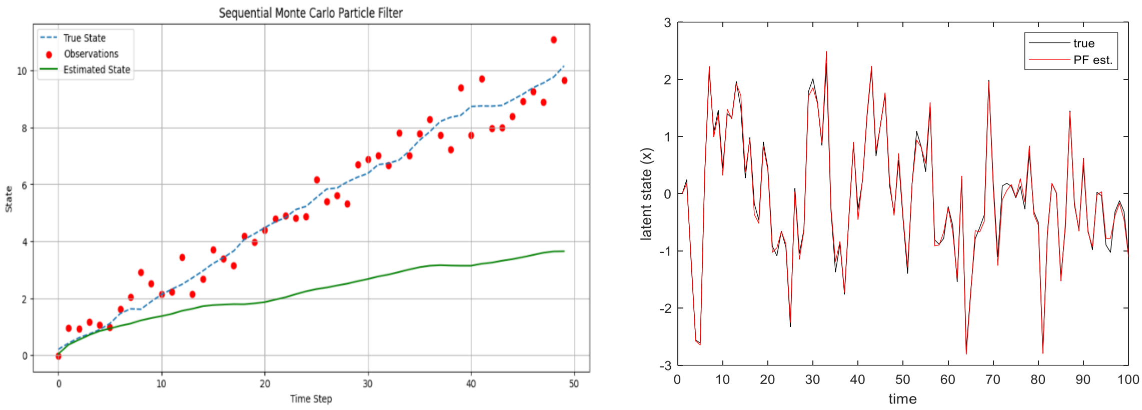
Source: Author's own calculations

Next, we will move on Bayesian inference in GP-Vol model. In the standard Gaussian process regression setting, the inputs and targets are fully observed and f can be learned using exact Bayesian inference, see Rasmussen, Williams (2006). This is reported not to be the case in GP-vol model, where the unknown parts of v_t form part of the inputs and all targets. Now, let θ denote the model hyper-parameters[17] and let $f = f(v_1), \ldots, f(v_T)$,while directly learning the joint posterior of

[17] Not to be confused with machine learning where in machine learning hyperparameter is a parameter whose value is used to control the learning process. In Bayesian statistics, a hyperparameter is a parameter of a prior distribution; the term is used to distinguish them from parameters of the model for the underlying system under analysis.In Beta distribution p of Bernouli distribution $0 \le p \le 1; q = 1 - p$; p is a parameter of the underlying system, while $\alpha; \beta$ are parameters of prior distribution beta distr.which is $f(x; \alpha, \beta) = \frac{1}{B(\alpha,\beta)} x^{\alpha-1} (1 - x)^{\beta-1} = \frac{x^{\alpha-1}(1-x)^{\beta-1}}{\int_0^1 u^{\alpha-1}(1-u)^{\beta-1} du}$

the unknown variables $f, v_{1:T}$ and θ seems to be a challenging task. But the posterior $p(v_t|\theta, x_{1:T})$ can be approximated with particles[18], see [Andrieu, C., Doucet, A. and Holenstein, R. (2010)](#).

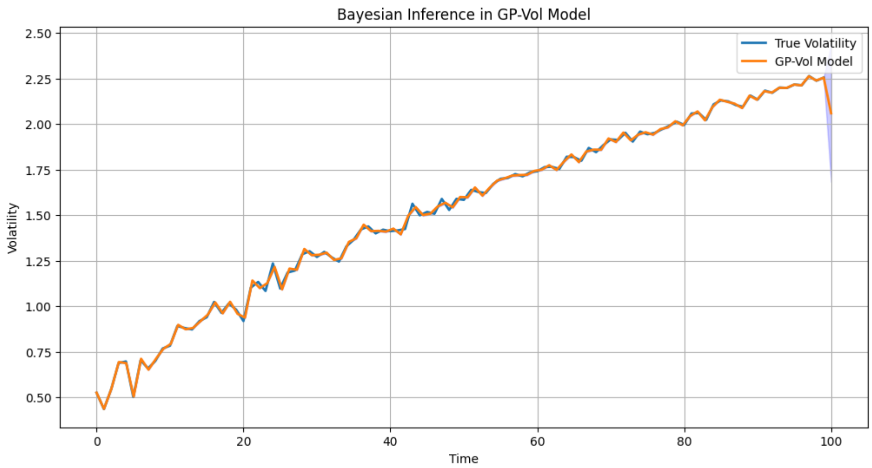
Figure 3 SMC particle filter and l Particle smoother (Forward-filtering Backward-Smoother) (FFBSm) for linear Gaussian state space model



Source:Author's wn calculatons and MATLAB code available at: https://user.it.uu.se/~thosc112/research/sequential-monte-carlo-smc.html

For these models more can be found in papers by : Moral,D.P.,Doucet,A.(2014), Schön , T.B., Lindsten, F. (2014),Doucet , A. Johansen, A. M. (2011), Briers, M. , Doucet A., Maskell, S. (2010).Next plot shows Bayesian inference and GP-Vol model.

Figure 4 Bayesian inference and GP-Vol model

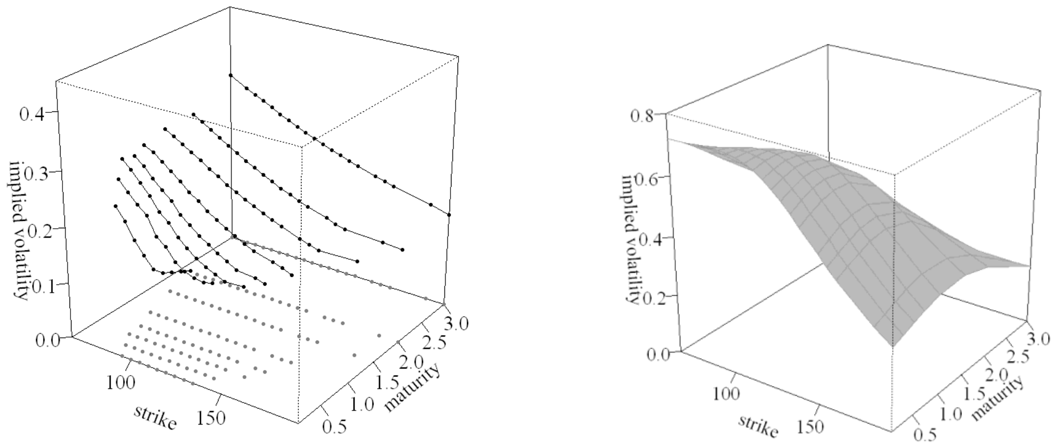


Source: Author's own calculations

Next,GP-LV model will be plotted,R code was executed in Jupyter with added R notebook.

[18] Interacting particle methods are a class of Monte Carlo methods to sample from complex high-dimensional probability distributions and to estimate their normalizing constants,see Moral,D.P.,Doucet,A.(2014).

Figure 5 Implied volatility strike price and maturity



Source: Author's own calculations based on a code available at: https://github.com/martnj/GP-LV.git

Previous code and figures are based on a paper by Tegner, Roberts (2019).

4.CEV model

The model has been introduced by Cox (1975) as one of the early alternatives to the geometric Brownian motion to model asset prices, see ,Linetsky, Mendoza(2010). The constant elasticity of variance (CEV) model is a one-dimensional diffusion process that solves a stochastic differential equation (SDE) of following type:

equation 19

$$dS_t = \mu S_t dt + aS_t^{\beta+1} dB_t$$

Instantaneous volatility is $\sigma(S) = aS^{\beta}$,specified to be power function of the spot price. The model has been introduced by Cox (1975) as one of the early alternative processes to the geometric Brownian motion to model asset prices. Here β is the elasticity parameter of the local volatility, or $\frac{d\sigma}{dS} = \frac{\beta\sigma}{S}$,and a is volatility scale parameter. When $\beta = 0$,CEV model is reduced to constant volatility geometric Brownian motion process employed in the Black-Scholes- Merton model. And when $\beta = -1$ the volatility specification is that of Bachelier[19], and for $\beta = \frac{1}{2}$ this model reduces to square-root model of Cox,Ross (1976). Beckers, S. (1980),writes that the option pricing formula for CEV class contains infinite summation which makes evaluation difficult. This problem is solved by Cox,Ross (1976) , who introduce alternative formulation for he option price :

equation 20

$$C(S,\tau) = (S - Ke^{-r\tau})N(y_1) + (S + Ke^{-r\tau})N(y_1) + v\big(n(y_1) - n(y_2)\big)$$

Where $N(\cdot)$ is CDF of Normal distribution, $n(\cdot)$ is unit normal density distribution, also:

[19] The asset price has the constant diffusion coefficient, while the logarithm of the asset price has the $\frac{a}{S}$ volatility.

equation 21

$$v = \sigma\left(\frac{1 - e^{-2r\tau}}{2r}\right)^{\frac{1}{2}}$$
$$y_1 = \frac{S - Ke^{-r\tau}}{v}$$
$$y_2 = \frac{-S - Ke^{-r\tau}}{v}$$

4.1 Displaced diffusion model (DD)

DD model had been presented by Rubisntein (1983).DD model can be presented in following manner:

equation 22

$$DS_t = \mu(S_t + a)dt + \sigma_{DD_t}(S_t + a)dW_t; \quad S(0) = S_0$$

The only parameter different than the standard Black-Scholes(1973) model is $a > 0$.This is called displacement parameter hence the name of the pricer.Pricng formulae in DD model is given as see also Rebonato (2002):

equation 23

$$C(K,T) = e^{-rT}\big((S(0) + a)\mathcal{N}(d_1) - K^*\mathcal{N}(d_2)\big)$$
$$P(K,T) = e^{-rT}\big(K^*\mathcal{N}(-d_2) - (S(0) + a)\mathcal{N}(-d_1)\big)$$

Where $K^* = K + a$ and where :

equation 24

$$d_1 = \frac{\log\left(\frac{S(0) + a}{K^*}\right) + \frac{\sigma_{DD}^2}{2}T}{\sigma_{DD}\sqrt{T}}; d_2 = d_1 - \sigma_{DD}^2 T$$

For time dependent volatility we replace $\sigma_{DD}^2$with $v_{DD}^2(t_0, t_1) \coloneqq \int_{t_0}^{t_1} \sigma_{DD}^2 u(du)$,parity between Black-Scholes and DD model means : $C_{DD}(K,T) = C_{BS}(K,T)$.Rebonato (2004) shows that European call option ATM (at the money) prices can be recovered reasonably:

equation 25

$$\sigma_{DD} \approx \frac{S_0}{S_0 + a}; \ \sigma_{BS} \frac{1 - \frac{1}{24}\sigma_{BS}^2 T}{1 - \frac{1}{24}\left(\frac{S_0}{S_0 + a}\sigma_{BS}\right)^2 T}$$

DD dynamics is applied to del the skew in Libor markets see Joshi, Rebonato (2001).Now, the dynamics of forward rate $F_i(t)$ in the spot measure is:

equation 26

$$d(F_i(t) + a) = (F_i(t) + a)\sigma_i^{DD}(t)\sum_{k=0}^{i} \frac{\tau_k(F_k(t) + a)\left(\sigma_k^{DD}\right)^T}{1 + \tau_k F_k(t)}$$

The corresponding volatility in previous is $\sigma_i^{DD}(t)$,this controls the overall level of volatility while a is the skew of volatility. DD model convenient form is:

equation 27

$$dS(t) = \sigma_{DD}(aS(t) + (1-a)L)dW(t)$$

The DD model can be seen as an approximation of normal and log-normal model.

equation 28

$$dS(t) \approx \sigma_L\left(S(0) + \sigma_l'\big(S(0)S(t) - S(0)\big)dW(t)\right)$$

And therefore, by setting $\sigma_{DD} := \frac{\sigma_L(S(0))}{S(0)}$,$\beta = \sigma_L'\big(S(0)\big)\frac{S(0)}{\sigma_L(S(0))}$; $L = S(0)$ and by setting $\sigma_{DD} = \sigma_{CEV}S(0)^{\beta-1}, a = \beta, L = S(0)$, shows that displaced diffusion model can be well approximated by a CEV model.

4.2 CEV and DD models

To see how CEV and DD models are related see Marris,D.(1999), and Svoboda,S.(2006).First,we take the following model:

equation 29

$$dS(t) = \big(\eta S(t) + S(0)(1-\eta)\big)\sigma^M dW(t)$$

η, σ^M are constant parameters. The DD model fits here b $a = S(0)\frac{1-\eta}{\eta}$ and $\sigma^{DD} = \eta\sigma^M$,and after taking $\sigma^M = S(0)^{\beta-1}\sigma^{cev}$we can find that :

equation 30

$$\frac{\partial(S(t)^\beta(\sigma^{cev})^2 dt}{\partial S(t)}|_{S(t)=S(0)} = \frac{\partial(S(t)+a)(\sigma^{DD})^2 dt}{\partial S(t)}|_{S(t)=S(0)}$$

As $\beta \approx 1$,DD model approximates CEV model. The following is displaced CEV model:

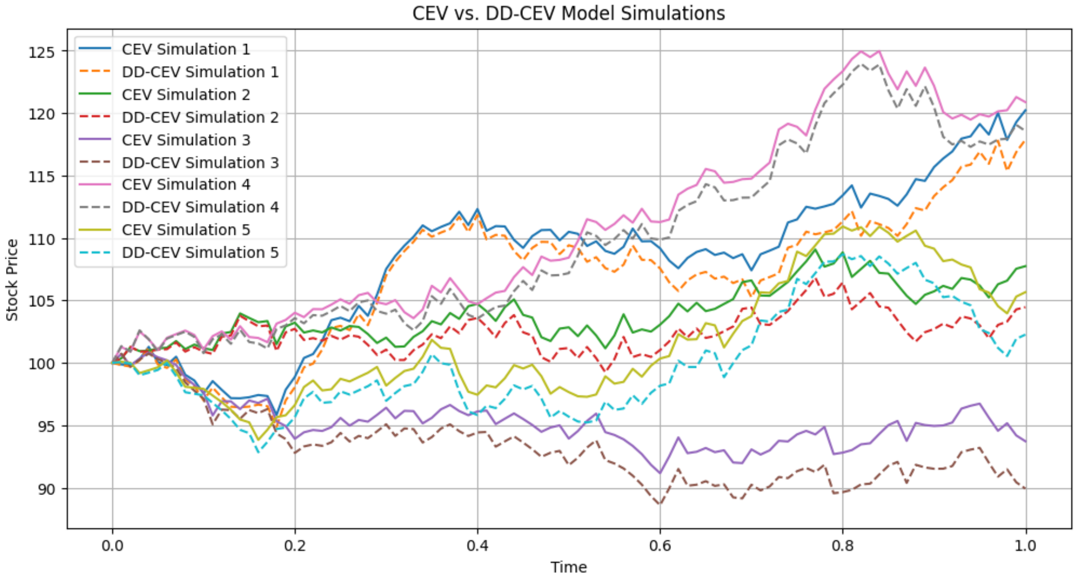
equation 31

$$dS(t) = (S(t)+a)^\beta dW(t)$$
$$S(0) = S_0$$

One the next plot we show comparisons between CEV and DD-CEV model[20].This plot contains 5 separate lines of stock price plotted versus delta time and difference it is shown between forecast of CEV model pricing and DD-CEV model. The larger the time delta the larger is the difference between the CEV and DD-CEV model predictions.

[20] Here the pricing is still possible in closed form (see, for instance, Andersen, L. and Piterbarg, V. (2010).

Figure 6 CEV vs DD-CEV model



Source: authors' own calculations

4.3 CEV approximation

Corollary 1. This is CEV exact formula. Let S_t be a process described in $dS_t = \mu S_t dt + aS_t^{\beta+1} dB_t$ then $\forall t \in [0, T]$,we can express call option fair value C_{ft} see Merino (2020)

equation 32

$$C_{ft} = C_{BS}(\beta - 1)\mathbb{E}_t\left[\int_t^T e^{r(u-t)}\Gamma C_{\widehat{BS}}\left(u, S_u, \sigma_u^{\beta-1}\right)(T-u)\sigma^2 S_t^{2(\beta-1)} du\right] + \frac{(\beta-1)(2\beta-3)}{2}\mathbb{E}_t\left[\int_t^T e^{r(u-t)}\Gamma C_{\widehat{BS}}\left(u, S_u, \sigma_u^{\beta-1}\right)(T-u)\sigma^4 S_t^{4(\beta-1)} du\right] + \frac{(\beta-1)^2}{2}\mathbb{E}_t\left[\int_t^T e^{r(u-t)}\Gamma C_{\widehat{BS}}\left(u, S_u, \sigma_u^{\beta-1}\right)(T-u)^2\sigma^6 S_t^{6(\beta-1)} du\right] + (\beta - 1)\mathbb{E}_t\left[\int_t^T e^{r(u-t)}\Lambda\Gamma C_{\widehat{BS}}\left(u, S_u, \sigma_u^{\beta-1}\right)(T-u)\sigma^4 S_t^{4(\beta-1)} du\right]$$

So, we will write:

equation 33

$$\mathbb{E}\left[e^{r(T-t)} C_{\widehat{BS}}\left(T, S_T, \sigma_u^{\beta-1}\right)\right] = C_{BS}(t, S_t, \sigma S_t^{\beta-1} + (I_{CEV}) + (II_{CEV}) + (III_{CEV})$$

Corollary 2. Let S_t be a process described in $dS_t = \mu S_t dt + aS_t^{\beta+1} dB_t$ then $\forall t \in [0, T]$,we can express call option fair value approximation[21] $C_{ft_{approx}}$ see Merino (2020)

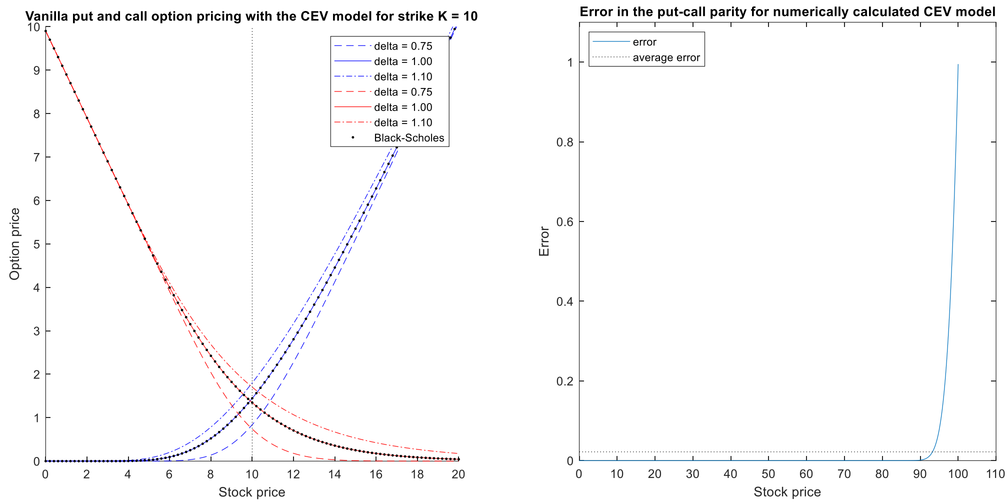
[21] The exact formula can be seen in Cox (1975), Emmanuel, Macbeth (1982),Schroder (1989).

equation 34

$$C_{ft_{approx}} = C_{BS}\left(T, S_T, \sigma S_t^{\beta-1}\right) + \frac{1}{2}(\beta-1)r\sigma^2 S_t^{2(\beta-1)}\Gamma C_{\widehat{BS}}\left(t, S_t, \sigma S_t^{\beta-1}\right)(T-t)^2$$
$$+\frac{1}{4}(\beta-1)(2\beta-3)\sigma^4 S_t^{4(\beta-1)}\Gamma C_{\widehat{BS}}\left(t, S_t, \sigma S_t^{\beta-1}\right)(T-t)^2$$
$$+\frac{1}{6}(\beta-1)^2\sigma^6 S_t^{6(\beta-1)}\Gamma^2 C_{\widehat{BS}}\left(t, S_t, \sigma S_t^{\beta-1}\right)(T-t)^3$$
$$+\frac{1}{2}(\beta-1)\sigma^4 S_t^{4(\beta-1)}\Lambda\Gamma C_{\widehat{BS}}\left(t, S_t, \sigma S_t^{\beta-1}\right)(T-t)^2 + \epsilon_t$$

Where ϵ_t is error term, now we have that $\epsilon_t \leq (\beta-1)^2\Pi(t,T,r,\sigma,\beta)$ and Π is increasing function on every parameter. Proof is derived in Merino (2020). Next plot, shows put and call parity in Vanila option pricing with the CEV model (with different deltas(0.75,1,1.1) and their comparison with Black-Scholes result, and the error in put call parity for numerically calculated CEV model.

Figure 7 Vanila put and call option pricing for strike $K = 10$ and comparison with B-S results and error in put-call parity for numerically calculated CEV



Source: authors' own calculations based on code available at: https://github.com/fhqvst/cev/tree/master/figures

Parameters for estimation of this model were: $T = \frac{1}{2}; X = 100; n = 100; m = 500; K = 10;$ $g = @(x)\ x - K;\ r = 0.02;\ deltas = \{0.75, 1, 1.1\};\ \sigma = 0.5;$

5. Crank-Nicolson and other finite difference methods (FDMs) with Galerkin Method of Weighted Residuals (GMWR)

The Crank-Nicolson finite difference method represents an average of the implicit method and the explicit method. For Crank-Nicolson see (Crank,Nicolson (1947)).For the implicit method forward and backward difference approximations are: $\frac{\partial f}{\partial S} = \frac{f_{i,j+1} - f_{i,j}}{\Delta S}; \frac{\partial f}{\partial S} = \frac{f_{i,j+1} - f_{i,j-1}}{\Delta S}$, where $f_{i,j}$ is the value of option at the (i,j) point, and ΔS is the change in stock-price. For the explicit method previous notation is given as: $\frac{\partial f}{\partial S} = \frac{f_{i+1,j+1} - f_{i+1,j+1}}{2\Delta S}; \frac{\partial^2 f}{\partial S^2} = \frac{f_{i+1,j+1} + f_{i+1,j-1} - 2f_{i,j-1}}{\Delta S^2}$, see Hull (2012). Finite difference approximation at implicit method at (i,j) is given as: $\frac{\partial^2 f}{\partial S^2} = \frac{f_{i,j+1} + f_{i,j-1} - 2f_{i,j}}{\Delta S^2}$; since $S = j\Delta S$ we can rearrange and knowing previous like:

equation 35

$$\frac{f_{i+1,j}-f_{i,j}}{\Delta t}+(r-q)j\Delta S\frac{f_{i.j+1}+f_{i,j-1}}{2\Delta S}+\frac{1}{2}\sigma^2 j^2\Delta S^2\frac{f_{i,j+1}+f_{i,j-1}-2f_{i,j}}{\Delta S^2}=rf_{i,j}$$
$$f_{i+1,j}=a_j f_{i+1,j-1}+b_j f_{i+1,j}+c_j f_{i,j+1}$$

Where :

equation 36

$$a_j=\frac{1}{2}(r-q)j\Delta t-\frac{1}{2}\sigma^2 j^2\Delta t$$
$$b_j=1+\sigma^2 j^2\Delta t+r\Delta t$$
$$c_j=-\frac{1}{2}(r-q)j\Delta t-\frac{1}{2}\sigma^2 j^2\Delta t$$

or in explicit difference method the difference equation is given as:

equation 37

$$\frac{f_{i+1,j}-f_{i,j}}{\Delta t}+(r-q)j\Delta S\frac{f_{i+1.j+1}+f_{i+1,j-1}}{2\Delta S}+\frac{1}{2}\sigma^2 j^2\Delta S^2\frac{f_{i+1,j+1}+f_{i+1,j-1}-2f_{i+1,j}}{\Delta S^2}=rf_{i,j}$$
$$f_{i,j}=a^* f_{i+1,j-1}+b_j^* f_{i+1,j}+c_j^* f_{i+1,j+1}$$

Where q is dividend yield, r is risk free interest rate. Or in previous:

equation 38

$$\begin{cases} a^*=\frac{1}{1+r\Delta t}\left(-\frac{1}{2}(r-q)j\Delta t+\frac{1}{2}\sigma^2 j^2\Delta t\right) \\ b_j^*=\frac{1}{1+r\Delta t}(1-\sigma^2 j^2\Delta t) \\ c_j^*=\frac{1}{1+r\Delta t}\left(\frac{1}{2}(r-q)j\Delta t+\frac{1}{2}\sigma^2 j^2\Delta t\right) \end{cases}$$

Finite difference methods were first applied by Schwartz, E. S. (1977). This equation $f_{i+1,j}=a_j f_{i+1,j-1}+b_j f_{i+1,j}+c_j f_{i,j+1}$ (implicit difference method) and this equation $f_{i,j}=a^* f_{i+1,j-1}+b_j^* f_{i+1,j}+c_j^* f_{i+1,j+1}$, the Crank-Nicolson method averages these two equations:

equation 39

$$f_{i,j}+f_{i-1,j}=a_j f_{i-1,j-1}+b_j f_{i-1,j}+c_j f_{i-1,j+1}+a_j^* f_{i,j-1}+b_j^* f_{i,j}+c_j^* f_{i,j+1}$$
$$g_{i,j}=f_{i,j}-a_j^* f_{i,j-1}-b_j^* f_{i,j}-c_j^* f_{i,j+1}$$
$$g_{i,j}=a_j f_{i-1,j-1}+b_j f_{i-1,j}+c_j f_{i-1,j+1}-f_{i-1,j}$$

Crank-Nicolson method is similar in implementation to finite difference method, but his advantage is in faster convergence[22]. The goal is to discretize Black-Scholes-Merton equation: $\frac{\partial f}{\partial t}+rS\frac{\partial f}{\partial S}+$

[22] In matrix form Crank-Nicolson method is: $CF_{i-1}=DF_i+K_{i-1}+K_i, i=1,..n$ and the equation was:$-\bar{a}f_{i-1,j-1}+\left(1-\bar{b}_j\right)f_{i-1,j}-\bar{c}f_{i-1,j+1}=\bar{a}f_{i,j-1}+\left(1-\bar{b}_j\right)f_{i,j}+\bar{c}f_{i,j+1}$.Previous equation is only stable if: $\| C^{-1}D \|_\infty \leq 1$ this is Crank-Nicolson Finite Difference Stability Condition. Previous shows the infinity norm of the product of the matrices $C^{-1}D$. Heuristically, if the infinity norm of$C^{-1}D$ is less than 1 then successive values of F_i in $\| C^{-1}D \|_\infty \leq 1$ get smaller and smaller, and hence the algorithm converges, or is stable. In previous $\bar{a}_j=\frac{\delta t}{4}(\sigma^2 j^2-rj); \bar{b}_j=-\frac{\delta t}{2}(\sigma^2 j^2+r); \bar{c}_j=\frac{\delta t}{4}(\sigma^2 j^2+rj)$.

$\frac{1}{2}\sigma^2 S^2 \frac{\partial^2 f}{\partial S^2} = rf$, now we use central approximation for $\partial f/\partial t$ and central approximation for $\partial f/\partial S$ and standard approximation for$\partial^2 f/\partial S^2$:

equation 40

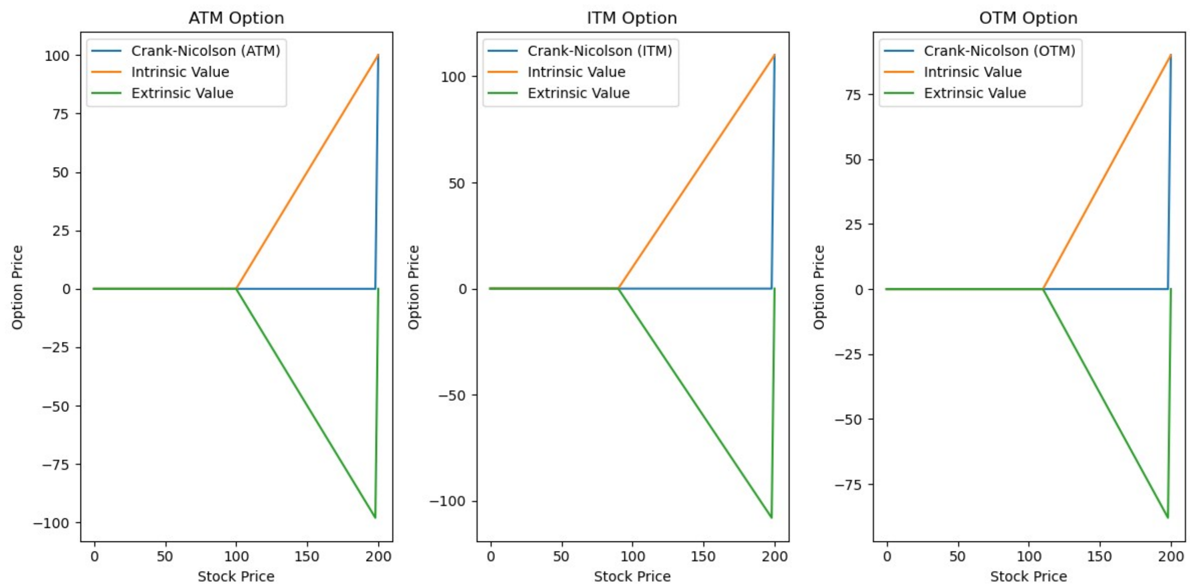
$$\frac{\partial f_{i-\frac{1}{2},j}}{\partial t} = \frac{f_{i,j}-f_{i-1,j}}{\delta t} + \mathcal{O}(\delta t^2)$$

$$\frac{\partial f_{i-\frac{1}{2},j}}{\partial S} = \frac{1}{2}\left[\frac{\partial f_{i-1,j}}{\partial S} + \frac{\partial f_{i,j}}{\partial S}\right] = \frac{1}{2}\left[\frac{f_{i-1,j+1}+f_{i-1,j-1}}{2\delta S} + \frac{f_{i,j+1}+f_{i,j-1}}{2\delta S}\right] + \mathcal{O}(\delta S^2)$$

$$\frac{\partial f_{i-\frac{1}{2},j}}{\partial S^2} = \frac{1}{2}\left[\frac{\partial^2 f_{i-1,j}}{\partial S^2} + \frac{\partial^2 f_{i,j}}{\partial S^2}\right] = \frac{1}{2}\left[\frac{f_{i-1,j+1}-2f_{i-1,j}+f_{i-1,j-1}}{\partial S^2} + \frac{f_{i,j+1}-2f_{i,j}+f_{i,j-1}}{\partial S^2}\right] + +\mathcal{O}(\delta S^2)$$

Hence the Crank-Nicolson method converges at the rates of $\mathcal{O}(\delta t^2)$ and $\mathcal{O}(\delta S^2)$. This is a faster rate of convergence than either the explicit method, or the implicit method. Next we will show comparisons between Crank-Nicolson vs Implicit FD model vs explicit FD model for ATM,ITM,OTM options[23].

Figure 8 ATM (At-The-Money), ITM (In-The-Money), and OTM (Out-Of-The-Money) option pricing comparisons Crank-Nicolson vs intrinsic and extrinsic value



Source: Author's own calculations

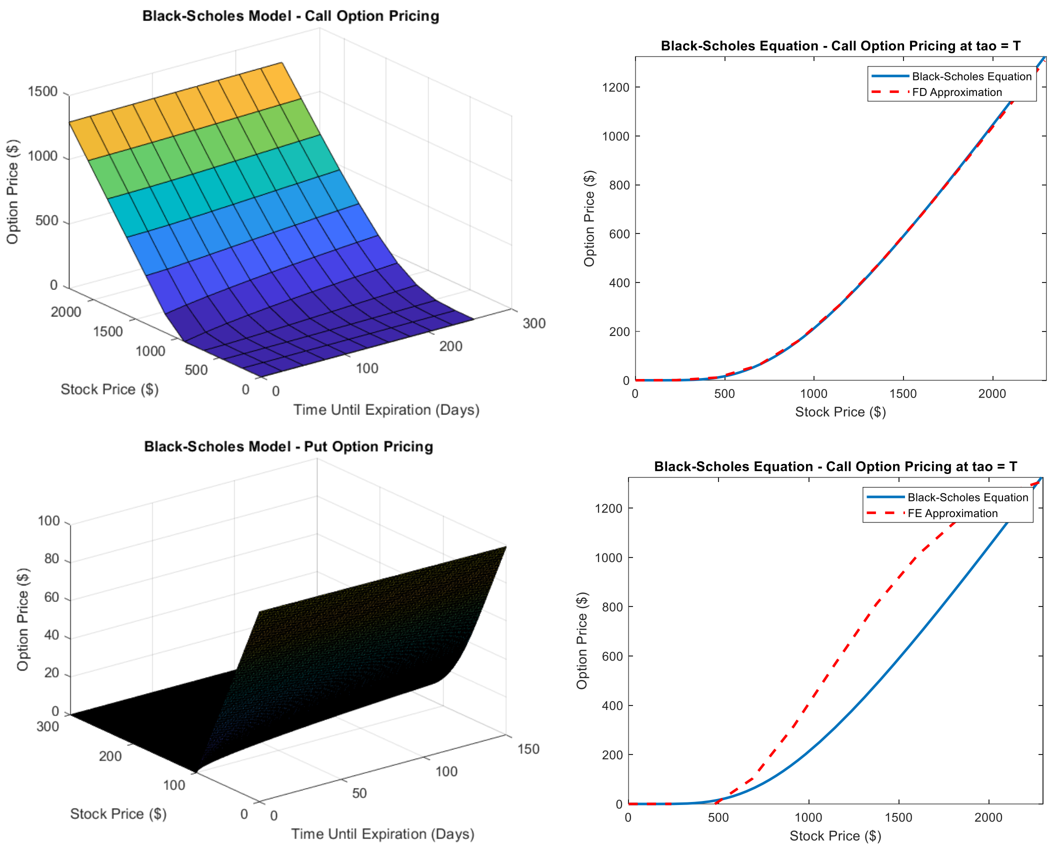
In this case Crank-Nicolson model provides higher intrinsic value[24] for the ATM case and ITM and OTM case for the price after the initial stock price, but with diminishing returns, intrinsic minus

[23]In an ATM option, the difference between the strike price and the current market price is minimal. For call options, stock price is above the agreed upon strike price. For put options, stock price is below the agreed upon strike price. For call options, an ITM option has a strike price below the current market price. For put options, it has a strike price above the current market price.If the strike price is higher than the underlying stock price, the option is out-of-the-money (OTM), OTM options typically do not have intrinsic value and rely on extrinsic value (time value and volatility) for any potential profit. For call options, stock price is below the agreed upon strike price. For put options, stock price is above the agreed upon strike price.

[24] Intrinsic value is the price difference between the current stock price and the strike price. An option's time value or extrinsic value of an option is the amount of premium above its intrinsic value.

extrinsic value is zero at the last price. Next, we will compare Finite difference method for option pricing with Black-Scholes equation for put and call options and finite element (FE) approximations.

Figure 9



Source :Author's own calculations based on a MATLAB code available at: https://github.com/EricJXShi/Black-Scholes-FEM

Previous models use Classical Black-Scholes and transformed B-S model:

equation 41

$$\frac{\partial V}{\partial t} - rS\frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} - rV = 0$$
$$\frac{\partial F}{\partial \tau} = \frac{1}{2}\sigma^2 \frac{\partial^2 F}{\partial x^2}$$

Now we define $\tau = T - t$ and PDE becomes: $-\frac{\partial V}{\partial t} - rS\frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} - rV = 0$.Boundary conditions for put and call options are :

equation 42

$$V(0,\tau) = 0, V(\infty,\tau) = S - Ke^{-r\tau}, V(S,0) = \max(S - K, 0) \text{ .... Boundary conditions for call options}$$
$$V(0,\tau) = Ke^{-r\tau}, V(\infty,\tau) = 0, V(S,0) = \max(K - S, 0) \text{ .... Boundary conditions for put options}$$

Theoretical solution for Call option price is: $Call_{option_{price}} = sN(d_1) - Ke^{-rt}N(d_2)$ where $d_1 = \frac{\ln\left(\frac{S}{K}\right)+\left(r+\frac{\sigma^2}{2}\right)}{\sigma\sqrt{\tau}}$ and $d_2 = \frac{\ln\left(\frac{S}{K}\right)+\left(r-\frac{\sigma^2}{2}\right)\tau}{\sigma\sqrt{\tau}}$.Now Applying Galerkin Method of Weighted Residuals (GMWR) to $-\frac{\partial \mathrm{V}}{\partial \mathrm{t}} - rS\frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} - rV = 0$, results in :

equation 43

$$\int_{S_1^e}^{S_2^e} \left(-\frac{\partial \mathrm{V}}{\partial \mathrm{t}} - rS\frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} - rV\right) N_i \, dS = 0$$

S_1^e, S_2^e are limits of integration. The residual equation is:

$$R = \frac{\partial \mathrm{V}}{\partial \mathrm{t}} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} - rS\frac{\partial V}{\partial S} - rV$$

Where V can be approximated $\sim$ by the following:

equation 44

$$\tilde{V}(S,\tau) = \sum_{i=1}^{2} N_i(S) \cdot V_i(\tau)$$

This equation $\int_{S_1^e}^{S_2^e} \left(-\frac{\partial \mathrm{V}}{\partial \mathrm{t}} - rS\frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} - rV\right) N_i \, dS = 0$ transforms into:

equation 45

$$[K] \cdot \frac{\partial V_j}{\partial \tau} + [C] \cdot V_j = [\alpha]$$

Now applying Crank-Nicolson result to previous equation:

equation 46

$$[K] \cdot \frac{V_j^{\tau+\Delta\tau} - V_j^{\tau}}{\Delta\tau} = \theta\left([\alpha] - [C] \cdot V_j^{\tau+\Delta\tau}\right) + (1-\theta) \cdot [\alpha] - [C] \cdot V_j^{\tau})$$

Where $\theta = \frac{1}{2}$ and α is a byproduct from using integration by parts and was canceled in the derivation.The matrices in previous equation are defined as:

equation 47

$$[K] = \frac{\Delta S}{2}\int_{-1}^{1} N_j N_i d\eta, N_j is\ a\ column\ vector\ of\ the\ element\ of\ shape\ f-nctions.$$

$$[C] = \frac{\Delta S}{2}((\sigma^2 - r)\int_{-1}^{1} S\frac{\partial N_j}{\partial S} N_i d\eta + \frac{1}{2}\sigma^2 \int_{-1}^{1} S^2 \frac{\partial N_j}{\partial S}\frac{\partial N_i}{\partial S} d\eta + r\int_{-1}^{1} N_j N_i d\eta)$$

$$[\alpha] = \frac{1}{2}\sigma^2 S^2 N_i \left(\frac{\partial \tilde{V}}{\partial S}\right)|_{S_1}^{S_2}$$

For Galerkin method see ,

D Galerkin Method of Weighted Residuals (GMWR) to the stock price dimension and CrankNicolson to the time dimension

6.Greeks

Let $V(0, S_0)$ denotes fair price at $\tau = 0$ European call option with strike price E and time to maturity T , then the Black-Scholes valuation formula is given as[25]:

equation 48

$$V(0,S_0) = S_0\mathcal{N}\left(\frac{\log\left(\frac{S_0}{E}\right)+\left(r+\frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}}\right) - Ee^{-rT}\mathcal{N}\left(\frac{\log\left(\frac{S_0}{E}\right)+\left(r-\frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}}\right)$$
$$= S_0\mathcal{N}(d_1) - Ee^{-rT}\mathcal{N}(d_2)$$

Where : $d_1 = \frac{\log\left(\frac{S_0}{E}\right)+\left(r+\frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}}$; $d_2 = \frac{\log\left(\frac{S_0}{E}\right)+\left(r-\frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}} = d_1 - \sigma\sqrt{T}$. Each Greek letter measures a different dimension to the risk in an option position and the aim of a trader is to manage the Greeks so that all risks are acceptable, see Hull (2012).Risk free rate is r and stock volatility is σ. The partial derivatives of $V = V(0, S_0)$ with respect to these variables is extremely important in practice. In the next table we will present definitions for Greeks in Black-Scholes model :

Table 1 Greeks and their definitions in Black-Scholes model

Greek	Definition in Black -Scholes model
Delta Δ	$\frac{\partial V}{\partial S_0} = \mathcal{N}(d_+)$
Gamma Γ	$\frac{\partial^2 V}{\partial S_0^2} = \frac{\mathcal{N}'(d_+)}{S_0\sigma\sqrt{T}}$
Theta Θ	$-\frac{\partial V}{\partial T} = -\frac{S_0\mathcal{N}'(d_+)\sigma}{2\sqrt{T}} - rKe^{-rT}\mathcal{N}(d_-)$
Vega ν	$\frac{\partial V}{\partial \sigma} = S_0\sqrt{T}\mathcal{N}'(d_+)$
Rho ρ	$\frac{\partial V}{\partial T} = KTe^{-rT}\mathcal{N}(d_-)$

Source: textbook definitions of Greeks in Black-Scholes model

Delta measures sensitivity to a small change in the price of the underlying asset. The delta of a European option is therefore sensitive to: the time to expiry (t),the volatility of the underlying (σ)the moneyness ($\frac{S}{K}$). Gamma measures the change of rate of delta. A short position in option is negative gamma. In this case, the trader will need to sell stocks if the stock price goes down and buy stocks if the stock price goes up to be delta hedged (sell low – buy high). While Rho measures sensitivity to the applicable risk-free interest rate. As the delta, it is positive for calls and negative for puts. Theta measures the sensitivity to the passage of time. The financial definition of Theta Θ is: $-\frac{\partial V}{\partial T}$ and with this definition, if you are "long an option, then you are short theta. "And Vega measures the sensitivity to volatility. The need to understand Vega only became important after trading options became as liquid as it is today. The formulas produced so far for Delta, Theta, Gamma, Vega, and Rho have been for a European option on a non-dividend-paying stock. Next table shows how this change when the stock pays a continuous dividend yield at rate q.

[25] In this section $\mathcal{N}(x) := \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} \exp\left(\frac{y^2}{2}\right) dy$ and $d_\pm = \frac{1}{\sigma\sqrt{T}}\left[\log\left(\frac{S(0)}{K}\right) + rT \pm \frac{\sigma^2 T}{2}\right]$

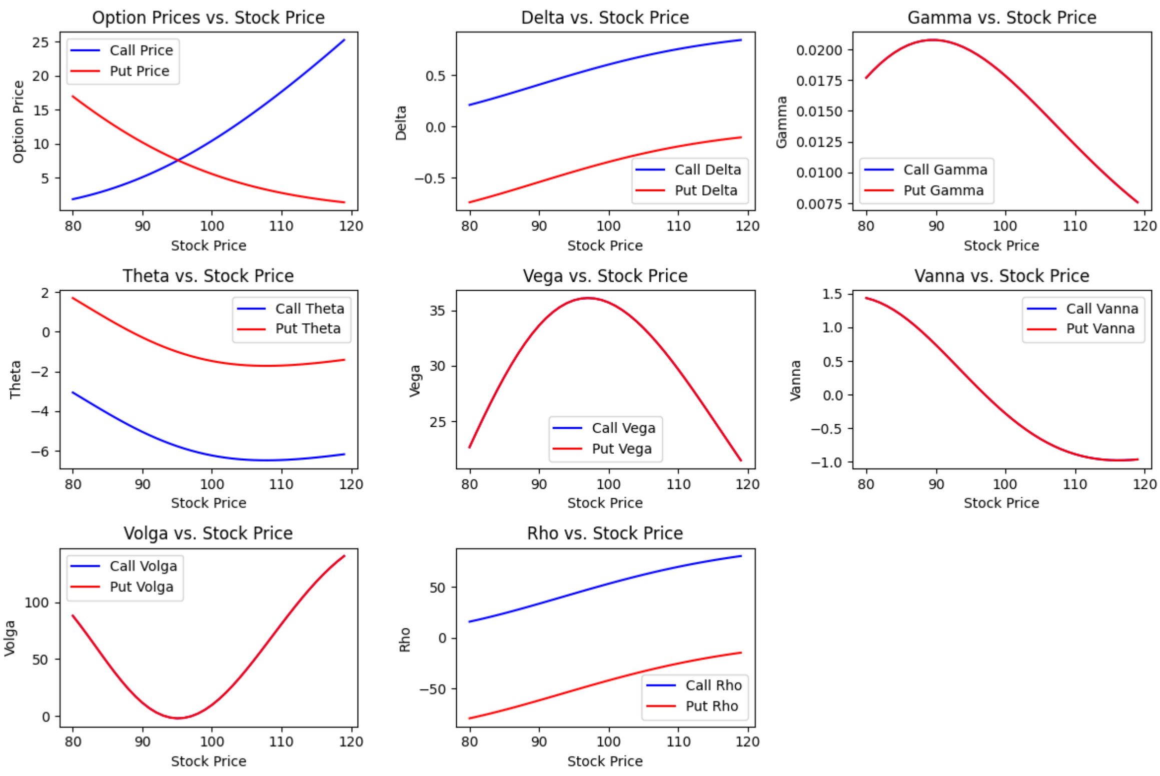
Table 2Greek letters for European options on an asset that provides a yield at rate q

Greek letter	Call option	Put option
Delta Δ	$e^{-qT}(N(d_1)$	$e^{-qT}[(N(d_1)-1]$
Gamma Γ	$\frac{N'(d_1)e^{-qT}}{S_0\sqrt{T}}$	$\frac{N'(d_1)e^{-qT}}{S_0\sqrt{T}}$
Theta Θ	$-\frac{S_0N'(d_1)\sigma e^{-qT}}{2\sqrt{T}} + qS_0N(d_1)e^{qT} - rKe^{-rT}N(d_2)$	$-\frac{S_0N'(d_1)\sigma e^{-qT}}{2\sqrt{T}} + qS_0N(-d_1)e^{qT} - rKe^{-rT}N(-d_2)$
Vega ν	$S_0\sqrt{T}N'(d_1)e^{-qT}$	$S_0\sqrt{T}N'(d_1)e^{-qT}$
Rho ρ	$KTe^{-rT}N(d_2)$	$-KTe^{rT}N(-d_2)$

Source: textbook definitions of Greeks in Black-Scholes model, see Hull (2012)

Where $d_1 = \frac{\ln\left(\frac{S_0}{K}\right)+\left(r+\frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}$ and $d_2 = \frac{\ln\left(\frac{S_0}{K}\right)+\left(r-\frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}} = d_1 - \sigma\sqrt{T}$. And: $c = S_0N(d_1) - Ke^{-rT}N(d_2)$, $p = Ke^{-rT}N(-d_2) - S_0N(-d_1)$. Next, graphically are presented Greeks versus stock price.

Figure 10 Greeks versus stock price



Source : Authors' own calculations

6.1 Relationship between Delta, Theta, and Gamma

The price of a single derivative dependent on a non-dividend-paying stock must satisfy the differential equation $dS = \mu Sdt + \sigma Sdz$.

Previous is stock price process. Suppose now that f is the price of a call option or other derivative contingent on S. The variable f must be some function of S and t.Hence from the assumptions : $dS = \mu S dt + \sigma S dz$ and it follows from Itô's lemma (see Itô (1951)) ,and Kiyosi (1944)[26] that (derivation includes expansion in Taylor series),if Π is a twice differentiable scalar function $V(t, P)$

equation 49

$$d\Pi = \frac{\partial \Pi}{\partial t}dt + \frac{\partial V}{\partial t}(\mu_t dt + \sigma_t dS_t) + \frac{1}{2}\frac{\partial^2 V}{\partial \Pi^2}(\mu_t^2 dt^2 + 2\mu_t\sigma_t dt dS_t + \sigma_t^2\ dS^2 t\ ) + \cdots$$

$$d\Pi = \left(\frac{\partial V}{\partial t} + \mu\frac{S\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 V}{\partial S^2} + \delta\mu S\right) + \left(\sigma S\frac{\partial V}{\partial S} + \delta\sigma\right)dz$$

Where in previous $= V(S,t)$ option price at time t,and $w(P,t)$ is the value of option, δ-stocks value of portfolio $\Pi = V + \delta S$ change in portfolio is given as: $d\Pi = dV + \delta VS$.

Lemma 1. Itô's lemma : Let $z(u)$ be a Wienner process [27] and then:

equation 50

$$V_t - V_0 = \int_0^t f_x(z(u),u)dz(u) - \int_0^t f_\tau(z(u),u)du + \frac{1}{2}\int_0^t f_{x,x}(z(u),u)du$$

Where $V_t = f(z(t),\tau)l\ 0 \le \tau \equiv T - t \le T, f \in C^{2,1}\big((0,\infty) \times [0,T]\big)$

Theorem 2 . Itô's lemma : Now let $f(t,x)$ be a smooth function of two variables,and let X_t be a stochastic process satisfying $dX_t = \mu_t dt + \sigma_t dB_t$ where B_t is Brownian motion, .Then we have:

equation 51

$$df(t,X_t) = \left(\frac{\partial f}{\partial t} + \mu_t\frac{\partial f}{\partial x} + \frac{1}{2}\sigma_t^2\frac{\partial^2 f}{\partial x^2}\right)dt + \frac{\partial f}{\partial x}dB_t$$

Proof. Now we have

equation 52

$$\begin{aligned} df(t,X_t) &= \frac{\partial f}{\partial t}dt + \frac{\partial f}{\partial x}dX_t + \frac{1}{2}\frac{\partial^2 f}{\partial X^2}(dX_t)^2 \\ &= \left(\frac{\partial f}{\partial t} + \mu_t\frac{\partial f}{\partial x} + \frac{1}{2}\sigma_t^2\frac{\partial^2 f}{\partial X^2}\right)dt + \sigma_t\frac{\partial f}{\partial x}dB_t + \cdots dt dB_T + \cdots (dt)^2 \ldots\ldots \nexists\ dt dB_T \\ &+ \cdots (dt)^2 \blacksquare \end{aligned}$$

Now for the discrete versions $\Delta S = \mu S\Delta t + \sigma S\Delta z$ and:

equation 53

$$\Delta f = \left(\frac{\partial f}{\partial S}\mu S + \frac{\partial f}{\partial t} + \frac{1}{2}\frac{\partial^2 f}{\partial S^2}\sigma^2 S^2\right)\Delta t + \frac{\partial f}{\partial S}\sigma S\Delta z$$

---

[26] In mathematics, Itô's lemma is an identity used in Itô calculus to find the differential of a time-dependent function of a stochastic process.

[27] Wienner process is a continuous-time stochastic process $W(t)$ for $t >= 0$ with $W(0) = 0$ and such that the increment $W(t) - W(s)$ is Gaussian with mean 0 and variance $t - s$ for any $0 <= s < t$, and increments for nonoverlapping time intervals are independent. Brownian motion (i.e., random walk with random step sizes) is the most common example of a Wiener process.

Now substituting $\Delta S = \mu S \Delta t + \sigma S \Delta z$ and $\Delta f = \left(\frac{\partial f}{\partial S}\mu S + \frac{\partial f}{\partial t} + \frac{1}{2}\frac{\partial^2 f}{\partial S^2}\sigma^2 S^2\right)\Delta t + \frac{\partial f}{\partial S}\sigma S \Delta z$ into change of portfolio equation $\Delta\Pi = -\Delta f + \frac{\partial f}{\partial S}\Delta S$ it comes from portfolio value: $\Pi = -f + \frac{\partial f}{\partial S} S$ we get :

equation 54

$$\Delta\Pi = \left(-\frac{\partial f}{\partial t} - \frac{1}{2}\frac{\partial^2 f}{\partial S^2}\sigma^2 S^2\right)\Delta t$$

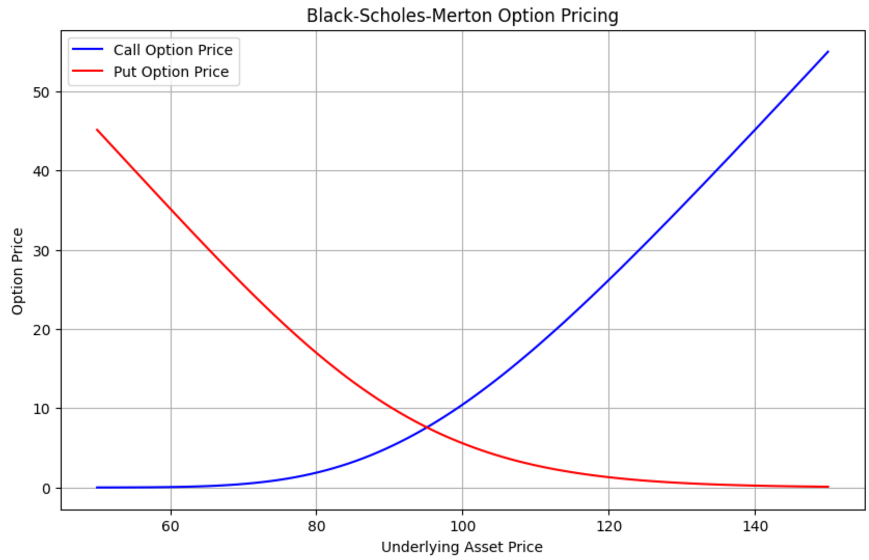
Since $\nexists z$ $\Delta\Pi$ is riskless for time Δt .So this follows $\Delta\Pi = r\Pi\Delta t$,and substituting Π and $\Delta\Pi$ into $\Delta\Pi = r\Pi\Delta t$ we get :

equation 55

$$\left(\frac{\partial f}{\partial t} + \frac{1}{2}\frac{\partial^2 f}{\partial S^2}\sigma^2 S^2\right)\Delta t = r\left(f - \frac{\partial f}{\partial S} S\right)\Delta t \Rightarrow \frac{\partial f}{\partial t} + rS\frac{\partial f}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 f}{\partial S^2} = rf$$

This is Black-Merton-Scholes equation.

Figure 11 Black-Scholes-Merton option pricing



So now:

equation 56

$$\frac{\partial\Pi}{\partial t} + rS\frac{\partial\Pi}{\partial S} + \frac{1}{2}\sigma^2 S^2\frac{\partial^2\Pi}{\partial S^2} = r\Pi$$

Since :

equation 57

$$\Theta = \frac{\partial\Pi}{\partial t}; \Delta = \frac{\partial\Pi}{\partial S}; \Gamma = r\Pi$$

We have that :

$$\Theta + rS\Delta + \frac{1}{2}\sigma^2 S^2 \Gamma = r\Pi$$

For a delta neutral portfolio, $\Delta = 0$ we have that:$\Theta + \frac{1}{2}\sigma^2 S^2 \Gamma = r\Pi$,see Hull,J.(2012). Next, we will compare stock-option prices for different option pricing methods and actual volatility vs implied volatility for different option pricing methods.

7.Stock-price vs Option price and Implied volatility vs Actual volatility for different option pricing methods

In this section we will graphically depict stock-price vs option price for different option pricing methods and actual volatility vs Implied volatility for different option pricing methods.

Figure 12 Stock-price vs option price for different option pricing methods: Black-Scholes,Black-Scholes-Mertton,Bachelier,CEV,GPL,GP-STATE-SPACE,SABR,martingale option pricing, Monté Carlo option pricing

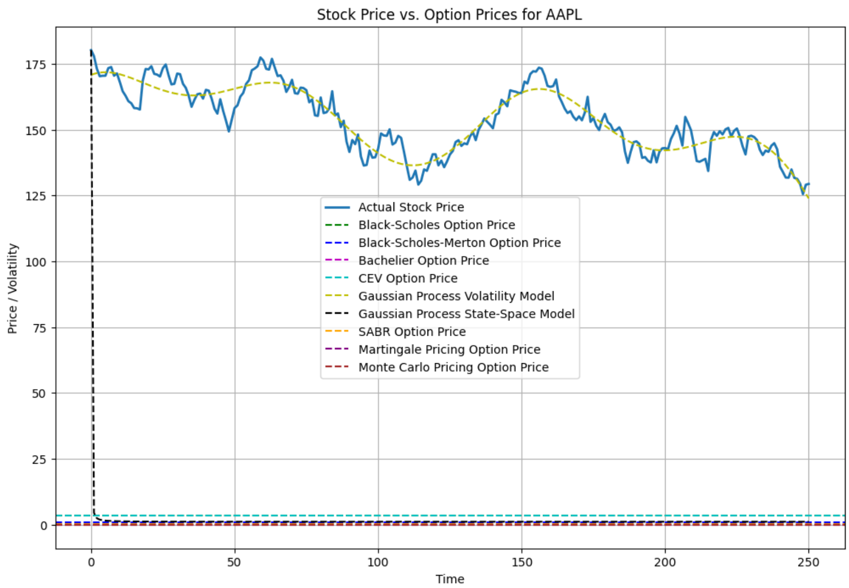
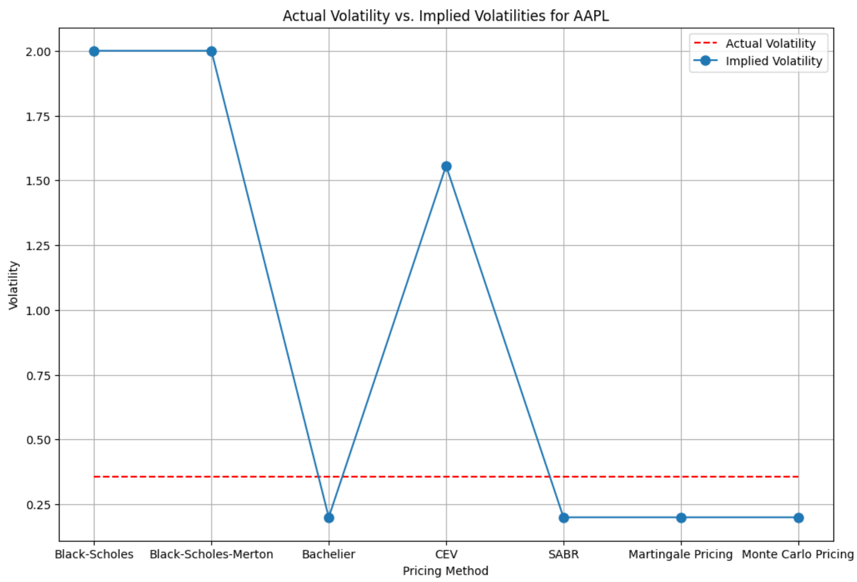


Figure 13 actual volatility vs Implied volatility for different option pricing methods: Black-Scholes,Black-Scholes-Mertton,Bachelier,CEV,GPL, SABR,martingale option pricing, Monté Carlo option pricing



The highest volatility was depicted by GP-volatility model, and it is somehow following the actual stock price/volatility. While in some models, namely CEV methods of option pricing volatility might be close to infinite since price/volatility result is near zero. As we defined it implied volatility is the market's forecast of a likely movement in a security's price, Black-Scholes, and Black-Scholes -Merton model exert implied volatility above actual volatility, also CEV model shows similar pattern of implied volatility, while SABR and Martingale option pricing with Bachelier show implied volatility below actual volatility. Hagan, P.S., Kumar, D., Lesniewski, A.S., Woodward, D.E. (2002). SDEs of the model are given as:

equation 58

$$\begin{cases} dS_t = \sigma_t S_t^{\beta} dW_t \\ d\sigma_t = \sigma_t v dZ_t \\ S(0) = S_0 \\ \sigma(0) = \sigma_0 \\ \langle DW_t, dZ_t \rangle - \rho dt \end{cases}$$

Here S_0 is the spot asset price and σ_0 is the spot value of volatility. The other model parameters are the CEV parameter (constant elasticity of variance[28]) β, the volatility of volatility ν and the correlation ρ between the Brownian motions W and Z driving the asset and the volatility dynamics. The original SABR pricing formulae is given as:
equation 59

$$\sigma_{SABR}(K,T) \approx \frac{\sigma_0}{(SK)^{\frac{1-\beta}{2}}\left(1+\frac{(1-\beta)^2}{24}\log^2\left(\frac{S}{K}\right)+\frac{(1-\beta)^4}{1920}\log^4\left(\frac{F}{K}\right)+..\right)}\frac{z}{x(z)}\left(1+\left(\frac{(1-\beta)^2\sigma_0^2}{24(SK)^{1-\beta}}+\frac{\rho\beta\nu\sigma_0}{4(SK)^{\frac{1-\beta}{2}}}+\nu^2\frac{2-3\rho^2}{24}\right)T+\cdots\right)$$

Where $z = \frac{\frac{\nu}{\alpha}(fK)^{\frac{1-\beta}{2}}logf}{K}$; and $x(z) = \log\left\{\frac{\sqrt{1-2\rho z+z^2}+z-\rho}{1-\rho}\right\}$ for the special case of ATM (at the money) options :
equation 60

$$\sigma_{ATM} = \sigma_B(f,f) = \frac{\alpha}{f(1-\beta)}\left\{1+\left[\frac{(1-\beta)^2}{24}+\frac{\alpha^2}{f^{2-2\beta}}+\frac{1}{4}\frac{\rho\beta\alpha v}{f(1-\beta)}+\frac{2-3\sigma^2}{24}v^2\right]t_{ex}+.\right\}$$

Martingales[29] are out of the reach for this paper, but we will define $E[Z|\mathcal{F}_n]$ where $\mathcal{F}_n$ is σ-algebra, which denotes the conditional expectation of Z given all the information that is available to us on the nth stage. The symbol $\mathcal{F}_n$ denotes subsets of S or collection of all events $\mathcal{A}$ [30]. Martingale is a zero drift process: $d\theta = \sigma dZ$ see Ross (1976).And $E(\theta_t) = \theta_0$ .Now if f, g are prices of securities traded we define: $\phi = f/g$ where g is numeraire.

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[28] Constant elasticity of variance (CEV)model is a stochastic volatility model that attempts to capture stochastic volatility and the leverage effect. The standard CEV model : $dS_t = \mu S_t dt + \sigma_{cev}\, S_t^{\beta} dW_t$, $S(0) = S_0$.This model is due: Schroder, M. (1989) and Andersen, L. ,Andreasen, J.(2000), see Kienitz, Wetterau (2012)

[29] A sequence of random variables $X_0, X_1$with finite means such that conditional expectations of X_{n+1} given $X_0, X_1, X_2, \dots, X_n$ are equal to X_n i.e. $\langle X_{n+1}|X_0, .., X_n\rangle = X_n$

[30] This $\mathcal{F}_n$ is a σ-algebra, which means that any finite or countable union of elements of $\mathcal{F}_n$ is again in $\mathcal{F}_n$, and that the complement of a set in $\mathcal{F}_n$ is again in $\mathcal{F}_n$.

Theorem 3: The equivalent martingale measure result shows that, when there are no arbitrage opportunities, ϕ is a martingale for some choice of the market price of risk.

Proof: suppose that σ_f, σ_g are volatilities of prices f, g.Derivative price is $df = \mu f dt + \sigma f dZ$ where $\mu = r + \lambda\sigma$ so that $df = (r + \lambda\sigma) f dt + \sigma f dZ$ or in a world where market price of risk is σ_g we have:

equation 61

$$df = (r + \lambda\sigma_g\sigma_f) f dt + \sigma_f f dZ$$
$$dg = (r + \sigma_g^2) g dt + \sigma_g g dZ$$

Using Itô's lemma we get:

equation 62

$$dlnf = \left(r + \sigma_g\sigma_f - \frac{\sigma^2 f}{2}\right) dt + \sigma_f dZ$$
$$dlng = \left(r + \frac{\sigma_g^2}{2}\right) dt + \sigma_g dZ$$

And :$d\left(\ln\frac{f}{g}\right) = -\frac{(\sigma_f - \sigma_g)^2}{2} dt + (\sigma_f - \sigma_g) dZ$. By Itô's lemma $d\left(\frac{f}{g}\right) = (\sigma_f - \sigma_g)\frac{f}{g} dZ$ This shows that $\frac{f}{g}$ is a martingale and proves the equivalent martingale measure result[31] ■.

For this model see more in Hull (2012). Let ω_L^a is additive martingale adjustment variable. This adjustment is chosen such that the discounted and dividend d adjusted price process $-\exp(-(r-d)t)S$ is a martingale, and this is given by $\omega_L^a(t) = \mathbb{E}[L(T)]$.Here $L(t)$ is a **Lèvy process**- L let be is an infinite divisible random variable $\forall t \in [0, \infty]$.$L$ can be written as the sum of a diffusion, a continuous Martingale and a pure jump process; i.e:

equation 63

$$L_t = at + \sigma B_t + \int_{|x|<1} x d\widetilde{N}_\tau + \int_{|x|\geq 1} x dN_\tau(\cdot, dx), \forall t \geq 0$$

In previous expression $a \in \Re$, B_t is the standard Brownian motion, N is defined to be the Poisson random measure of the Lèvy process. Lèvy -Khintchine formula: from the previous property it can be shown that for $\forall\tau \geq 0$ one has that :

equation 64

$$E\left|e^{inL_t}\right| = e^\wedge(-\tau\psi(u)$$
$$\psi(u) = -iau + \frac{\sigma^2}{2}u^2 + \int_{|x|\geq 1}(1 - e^{iux}) dv(x) + \int_{|x|<1}(1 + e^{iux} + iux) dv(x)$$

$a \in \Re; \sigma \in [0, \infty); v > 0$ borel measure and σ is Lèvy measure. More so $v(\cdot) = E[N_1(\cdot, A)]$,see Applebaum (2004). So additive martingale adjustment for Black-Scholes model is :

equation 65

$$\omega_{BS}^a(t) = -\frac{\sigma^2}{2} t$$

[31] In a risk neutral world $\frac{f_0}{g_0} = E_g\left(\frac{f_T}{g_T}\right)$ where E_g denotes the expected value in a world that is forward risk neutral with respect to g.

Monte-Carlo are simulation methods. Now, let's consider problem of computing expectation:$\theta = E[f(X)], X \sim f(X)$.Monte Carlo simmulation (MC) approach specifies generating N independent draws from the distribution$f(X), X_1, \dots, X_n$ and approx.:

equation 66

$$E[f(X)] \approx \hat{\theta}_N \equiv \frac{1}{N}\sum_{i=1}^{N} f(X_i)$$

By the law of large numbers, the approximation $\hat{\theta}_N$ converges to the true value as $N \to \infty$ .Monte Carlo estimates $\hat{\theta}_N$ is unbiased: $E[\hat{\theta}_N] = \theta$ by the Central limit theorem we have :

equation 67

$$\sqrt{N}\frac{\hat{\theta}_N - \theta}{\sigma} \Rightarrow \mathcal{N}(0,1), \sigma^2 = Var[f(X)]$$

Boyle (1977) is the first researcher to introduce Monte Carlo simulation into finance. Monte Carlo (MC) simulation is the primary method for pricing complex financial derivatives, such as contracts whose payoff depends on several correlated assets or on the entire sample path of an asset price, see Q,Jia (2009). The option price μ is written as an integral that represents the mathematical expectation of the discounted payoff under a so-called risk-neutral probability measure.

equation 68

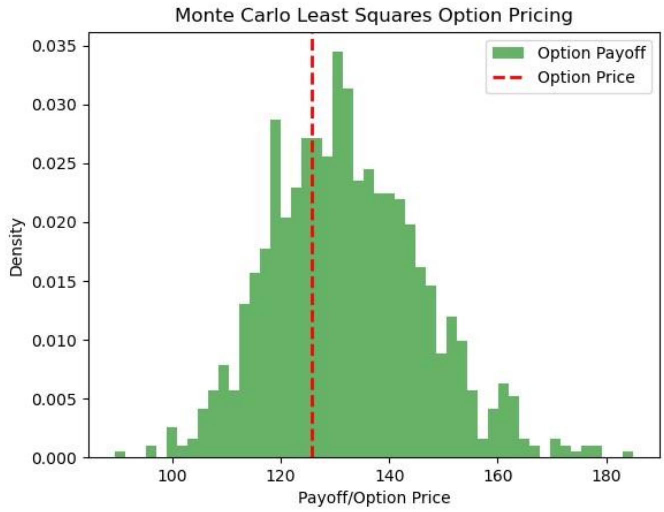
$$\mu = \mu(f) = \int_0^1 \dots \int_0^1 f(u_0, \dots, u_{t-1}) du_0 \dots du_{t-1} = \int_{(0,1)^t} f(u) du = E[f(U)]$$

$u(S(t))$ is the option payout function. Feynman-Kac formula connects the solutions of a specific class of partial differential equations to an expectation which establishes the mathematical link between the PDE formulation of the diffusion problems we encounter in finance and Monte Carlo simulations. *Feynman-Kac formula*- Suppose $\exists \mathcal{P}(t,x)$ that satisfies :$\frac{\partial \mathcal{P}}{\partial t} + f(t,x)\frac{\partial \mathcal{P}}{\partial x} + \frac{1}{2}\rho^2(t,x)\frac{\partial^2 \mathcal{P}}{\partial x^2} - R(x)\mathcal{P} + h(t,x) = 0$ s.t $\mathcal{P}(t,x) = \psi(x)$. Then $\exists \widetilde{W}(t)$ and a measure Q where solution is given as $\mathcal{P}(t,x) = E_Q[\int_t^T \mathcal{V}(t,u)h(u,x(u))du + \mathcal{V}(t,T)\psi(x(t))|\mathcal{F}_t]; t < T \;\; dx(t) = f(t,x(t))dt + \rho(t,x(t))d\widetilde{W}(t); \mathcal{V}(t,u) = \exp\left(-\int_t^u R(x(s)ds\right)$ given that $\int_t^T E_Q\left[\left(\rho(s,x(s))\frac{\partial \mathcal{P}}{\partial x}(s,x(s))\right]^2 \middle| \mathcal{F}_t\right]$. Longstaff and Schwartz (2001),have proposed LSM Least squares Monte Carlo method.Here we consider price time zero price $V_\tau(h)$ depending on a payoff and stopping time τ. We have: $V_\tau(h) = esupp_{\tau stoppingtime} V\tau(h)$.

Theorem 3. Doob's Optional Stopping Theorem :Suppose that X_0 is a known constant, that $X_0, X_1, X_2, \dots$ is a martingale, and that T is a bounded stopping time. Then $E[X_T] = X_0$.If $(X, \mathfrak{F})$ is a martingale and τ is stopping time,and if τ is finite i.e. $\mathbb{P}(\tau < \infty) = 1$ and $\mathbb{E}[|X_\tau|] < \infty$ and $\lim_{n\to\infty} \mathbb{E}[X_n \mathbb{I}_{\{\tau > n\}}] = 0$,then we have that the martingale property is preserved under random stopping.

Proof: It can be see that $X_\tau = X_{\tau \wedge n} + (X_\tau - X_n) \cdot \mathbb{I}_{\{\tau > n\}}$,since $\tau \wedge n$ is bounded stopping time,we know that martingale property is preserved hence, $\mathbb{E}[X_\tau] = \mathbb{E}[X_0] + \mathbb{E}[X_\tau \cdot \mathbb{I}_{\{\tau > n\}}] - \mathbb{E}[X_n \cdot \mathbb{I}_{\{\tau > n\}}]$.Here we can see $\lim_{n\to\infty} \mathbb{E}[X_n \mathbb{I}_{\{\tau > n\}}] = 0$ and $\mathbb{E}[X_\tau \cdot \mathbb{I}_{\{\tau > n\}}] = \sum_{k=n+1}^{\infty} \mathbb{E}[X_\tau \cdot \mathbb{I}_{\{\tau = k\}}]$ so that we know that limit $\lim_{n\to\infty} \sum_{k=n+1}^{\infty} \mathbb{E}[X_\tau \cdot \mathbb{I}_{\{\tau = k\}}] = 0$,we have that $\mathbb{E}[X_\tau] = \mathbb{E}[X_0]$ as $n \to \infty$ ∎For more on martingales see Grimmett, G. R.; Stirzaker, D. R. (2001).

Figure 14 Monte Carlo least squares option pricing



Source: Authors' own calculation

8.Conclusion

This paper reviewed LVM models, GP-LVM, CEV model, DD model, Crank-Nicolson, and other finite difference methods, Greeks, SABR, martingales, and LSMC option pricing. The effects of changing the volatility on paths generated by Bachelier, Black-Scholes were estimated, and the results proved similar patterns in the scenario of σ low and σ high compared to the base scenario for both models. GP-Vol, GP-SSM, GARCH model vs simulated volatility graph favored GP-Vol, GP-SSM over GARCH model when contrasting their forecast vs simulated volatilities. Vanila options put and call pricing with different deltas (0.75,1,1.1) showed that results are most similar to the Black-Scholes when delta=1. Crank-Nicolson model provides higher intrinsic value for the ATM case and ITM and OTM case for the price after the initial stock price, but with diminishing returns, intrinsic minus extrinsic value is zero at the last price. Comparison of Finite difference method (FD) for option pricing with Black-Scholes equation for put and call options and finite element (FE) approximations proved that FD is identical to B-S approximation, while there is a gap between Black-Scholes call option pricing at $\tau = T$,B-S approximates higher strike prices compared to FE method. When we compared Greeks we observed that there is no put and call parity in delta Δ (low put and call prices are reported for low values of delta) for negative delta. For Theta (Theta indicates the amount an option's price would decrease as the time to expiration decreases, all else equal) call and put price increase when θ is lowered (Theta, usually expressed as a negative number for long positions, indicates how much the option's value will decline every day up to maturity), and for rho (represents the rate of change between an option's value and a 1% change in the interest rate) is negative for long puts but puts and call increase with ρ ,for Gamma (measures the rate of change of the Delta of the option with respect to a move in the underlying asset) higher spot price is associated with lower gamma Γ , in the case of Vega ν spot price exerts concave relationship when compared to the measure of implied volatility (Vega).In Vanna call options have positive vanna while put have negative vanna , negative vanna reading indicates that as volatility increases, the portfolio delta becomes more negative and spot price reaches maximum, in the case of volga (a second-order derivative indicating the change in vega with respect to change in volatility) association between Volga and stock price is convex. The conclusion in this paper is that Gaussian process Local volatility model (GP-LVM) forecast proved to be closest to the actual stock price when compared with B-S,Bachelier,CEV,GP-Vol,GP-SSM,SABR,M-C,and martingale option pricing. While implied volatility was lower than actual volatility in SABR,martingale, and M-C option pricing methods. And implied volatility was higher than implied volatility in Black-Scholes, Black-Scholes-Merton model, and Bachelier.

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