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CANTOR CONNECTEDNESS OF UNIFORM SPACES

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A **uniformly connected space** (Cantor connected space) is a uniform space X such that *every uniformly continuous function from* X *to a discrete uniform space is* **constant**. A uniform space X is called uniformly disconnected if it is not uniformly connected.

We give an **equivalent definition of Cantor connectedness** based on definition of uniform space by uniform coverings. We prove several properties of Cantor connectedness using the notion of chain.

With this approach, we introduce new concept of connectedness of uniform spaces which is not appeared in the research work.

INTRODUCTION

Familly of coverings $\mathcal{U}$ is a **star refinement** of family $\mathcal{V}$ if for each $U \in \mathcal{U}$, there exists $V \in \mathcal{V}$ such that $stU \subseteq V$.

Notation: $\mathcal{U} <* \ \mathcal{V}$

A Uniform space $X = (X, \underline{\mathcal{U}})$ is a set X with a distinguished family of coverings $\underline{\mathcal{U}} = \{\mathcal{U}_\alpha | \alpha \in I\}$ called **uniform covering** (subset of the set of coverings of X), than form a filter i.e. this family holds the statements:

1. $\{X\}$ is uniform covering for X, i.e. $X \in \underline{\mathcal{U}}$;
2. If $\mathcal{U} <* \ \mathcal{V}$ and $\mathcal{U}$ is uniform covering, than $\mathcal{V}$ is also uniform covering;
3. If $\mathcal{U}$ and $\mathcal{V}$ are uniform coverings, there exists uniform covering $\mathcal{W}$ which is star refinement in $\mathcal{U}$ and $\mathcal{V}$ i.e. $\mathcal{W} <* \ \mathcal{U}$ and $\mathcal{W} <* \ \mathcal{V}$.

Definition 1.1. Let $\mathcal{U}$ be covering of the set X and $x, y \in X$. **Chain in $\mathcal{U}$** which connected $\boldsymbol{x}$ and $\boldsymbol{y}$ (from x to y, from y to x) is a finite sequence of sets $U_1, U_2, \ldots, U_n$ such that $x \in U_1$, $y \in U_n$ и $U_i \cap U_{i+1} \neq \emptyset$ за $i = 1,2,\ldots,n-1$.■

For two points $x, y \in X$ we say that *x* and *y* are **uniformly chainable** if for any uniform covering U of *X*, *x* and *y* are chainable in U.

Now we are able to define connectedness of the space using coverings, as given in [4]:

The topological space *X* is chain connected if any two points in *X* are chainable .

Definition 1.2. Let X be uniform space. X is **uniformly chainable connected** if for each *uniform covering* $\mathcal{U}$ of X and each $x,y \in X$, there exists *chain* in $\mathcal{U}$ which connected x and y.■

Corolary 1.3. X is **uniformly chainable connected,** if for each uniform covering $\mathcal{U}$ of X, each two points $x,y \in X$ are chainable.

Theorem 1.4. Each non empty subset Y from the uniform space X with uniform coverings $\underline{\mathcal{U}}$ is an uniform space Y with uniform coverings $\underline{\mathcal{U}} \cap Y$.

Proof: (Isbel, pg.7)

Every subset from the uniform space will be uniform space too, with respect to uniform coverings.

Let X be a uniform space and $Y \subseteq X$.

Definition 1.5. Uniform subspace Y of the uniform space $X = (X, \underline{\mathcal{U}})$ is the uniform space Y with family consist of uniform coverings $\underline{\mathcal{U}} \cap Y$.

2. UNIFORM CHAIN CONNECTED SET IN A UNIFORM SPACE

Let X be a uniform space and $C \subseteq X$.

Definition 2.1. The set C is **uniform chain connected** in X, if for every uniform covering $\mathcal{U}$ of X and every $x, y \in C$, there exists a chain in $\mathcal{U}$ that connects $\boldsymbol{x}$ and y.

Let X be a uniform space and $C \subseteq Y \subseteq X$.

The first property of a uniform chain connected set, shown in the next theorem, is an implication of uniform chain connectedness from a space to each of its super spaces (X is a super space of C, if C is a subspace of X).

Theorem 2.2. If C is a uniform chain connected in Y, then C is chain connected in X.

Proof. Let C be uniform chain connected in Y and $\mathcal{U}$ be a covering of X. Then:

$$\mathcal{U}_Y = \mathcal{U} \cap Y = \{U \cap Y | U \in \mathcal{U}\}$$

is a uniform covering of Y. Since C is uniform chain connected in Y, it follows that for every two points $x, y \in C$, there exists a chain $U_1 \cap Y,\ U_2 \cap Y,\ \ldots,\ U_n \cap Y$ of elements of $\mathcal{U}_Y$. Then since $U_i \cap U_{i+1} \neq \emptyset$ for every $i = 1,2,\ldots,n-1$ and $U_i \in \mathcal{U}$ for every $i = 1,2,\ldots,n$, the sequence $U_1,\ U_2,\ \ldots,\ U_n$ is a chain in $\mathcal{U}$ that connects x and y. It follows that C is uniform chain connected in X.■

Remark 2.3. The most important case of the previous theorem is when $Y = C$.■

Remark 2.4. If the set C is a uniform chain connected in X, then each subset of C is a uniform chain connected in X.■

We will give criteria for uniform chain connectedness in a space, using the notion of *infinite star of a coverings* [1].

Let X be a uniform space and $C \subseteq X$.

Let $\mathcal{U}$ be a covering of X and $x \in X$. Then the set $st(x,\mathcal{U})$ is a union of all $U \in \mathcal{U}$ which have nonempty intersection with x. The set:

$$st^n(x,\mathcal{U}) = st\left(st^{n-1}(x,\mathcal{U})\right) \text{ and } st^\infty(x,\mathcal{U}) = \cup_{n=1}^{\infty} st^n(x,\mathcal{U}).$$

Theorem 2.5. The set C is uniform chain connected in X, if and only if for every $x \in C$ and every covering $\mathcal{U}$ of X, $C \subseteq st^\infty(x,\mathcal{U})$.■

Corollary 2.6. The space X is a uniform chain connected in X, if and only if for every $x \in X$ and every covering $\mathcal{U}$ of X,

$$X = st^\infty(x,\mathcal{U}).■$$

3. CHAIN RELATION AND CHAIN COMPONENTS OF UNIFORM CONNECTEDNESS

Let $X = (X, \underline{\mathcal{U}})$ be a uniform space and $x \in C \subseteq X$.

Definition 3.1. A uniform chain component of the point x of C in X, denoted by $V_{CX}(x)$, is the *maximal uniform chain connected subset of* C in X that contains x.■

Proposition 3.2. The set $V_{CX}(x)$ consists of all elements $y \in C$, such that for every covering $\mathcal{U} \in \underline{\mathcal{U}}$ there exists a chain in $\mathcal{U}$ that connects x and y.

If $C = X$, then we use notation $V_X(x)$ or $V(x)$ if we work only with the space X, for $V_{XX}(x)$. Clearly $V_{CX}(x) = C \cap V_X(x)$.

The set C is **uniform chain connected** in X if C **is subset of** $V_X(x)$ for every $x \in C$.

We denote by $U_x(C)$ or $U(C)$, the set that consists of all elements $y \in X$, such that for every uniform covering $\mathcal{U} \in \underline{\mathcal{U}}$ there exists a chain in $\mathcal{U}$ that connects some $x \in C$ and y. This set is a *union of uniform chain components of connectedness*.

If C is a uniform chain connected set in X, since for every $x, y \in C$ the uniform chain components $V(x)$ and $V(y)$ coincide i.e. $V(x) = V(y)$, it follows that $U(C)$ is a uniform chain component and it is denoted by $V(C)$. Clearly $C \subseteq V_x(C)$ and $V(C) = V(x)$ for every $x \in C$.■

Remark 3.3. If the set C is uniform chain connected in X, then each subset of $V(C)$ is uniform chain connected in X.■

Let X be a uniform space and $x, y \in X$.

Definition 3.4. The element x is **uniform chain related** to y in X, (we denote it by $x \sim y$) if for every uniform covering $\mathcal{U}$ of X there exists a chain in $\mathcal{U}$ that connects x and y.

The *uniform chain relation in a uniform space is an* ***equivalence relation*** and it depends on the set X and the family of uniform coverings of X.

Remark 3.5. The set C is uniform chain connected in X if and only if for every $x, y \in C$, $x \sim y$.■

Therefore C is not uniform chain connected in X if and only if there exist $x, y \in C$ such that $x \nsim y$.

The uniform chain relation decomposes the space into classes. The classes are uniform chain components.

Let $x, y \in C$. If $y \in V_{CX}(x)$, then $V_{CX}(x) = V_{CX}(y)$. If $V_{CX}(x) \neq V_{CX}(y)$, then $V_{CX}(x) \cap V_{CX}(y) = \emptyset$. As a consequence, the next proposition is valid.

Proposition 3.6. For every $x \in C$, $V_{CX}(x) = C \cap V_{XX}(x)$. Each uniform chain component of X in X contains at most one uniform chain component of C in X.■

Proposition 3.7. *For every* $x \in C$,

$$V_{CC}(x) \subseteq V_{CX}(x) = \bigcup_{y \in V_{CX}(x)} V_{CC}(y) \subseteq V_{XX}(x).■$$

The proposition shows that every uniform chain component of C in X is a union of uniform chain components of C in C and is a subset of uniform chain component of X in X.

Proposition 3.8. The set of all uniform chain connected subsets of C in X consist of all uniform chain components and their subsets.■

4. PROPERTIES OF UNIFORM CHAIN CONNECTED SETS THAT CONSIST UNIFORM CHAIN COMPONENTS

Next we turn to a union of a uniform chain connected sets in a uniform space.

Lemma 4.1. Let $C, D \subseteq X$. If C and D are uniform chain connected in X and $V(C) \cap V(D) \neq \emptyset$, where $V(C)$ and $V(D)$ are uniform chain components of C and D respectively, then the union $V(C) \cup V(D)$ is uniform chain connected in X and:

$$V(C) \cup V(D) = V(C) = V(D).$$

Proof. Let $\mathcal{U}$ be a uniform covering of X and $x, y \in V(C) \cup V(D)$. If $x, y \in V(C)$ or $x, y \in V(D)$, then since $V(C)$ and $V(D)$ are uniform chain connected, there exists a chain in $\mathcal{U}$ that connects x and y. If $x \in V(C)$ and $y \in V(D)$, it follows that firstly there exists $z \in V(C) \cap V(D)$, and secondly that there exist chains in $\mathcal{U}$ that connect x with z, and z with y, from which it follows that there is a chain in $\mathcal{U}$ that connects x and y. So $V(C) \cup V(D)$ is uniform chain connected in X. Since $V(C) = V(x)$ is uniform chain component of some $x \in C$, and $V(C) \cup V(D)$ is a uniform chain connected set that contain x it follows that $V(C) \cup V(D) = V(C)$. Similarly $V(C) \cup V(D) = V(D)$.■

Corollary 4.2. Let $C, D \subseteq X$. If C and D are uniform chain connected in X and $V(C) \cap V(D) \neq \emptyset$, where $V(C)$ and $V(D)$ are uniform chain components of C and D respectively, then the union $C \cup D$ is uniform chain connected in X.

Theorem 4.3. Let $C_i,\ i \in I$ be a family of uniform chain connected subspaces of X. If there exists $i_0 \in I$ such that for every $i \in I$, $V(C_{i_0}) \cap V(C_i) \neq \emptyset$, then the *uniform union* $\bigcup_{i\in I} V(C_i)$ *is a uniform chain connected* in X and $\bigcup_{i\in I} V(C_i) = V(C_i)$ for every $i \in I$.

Proof. Let $\mathcal{U}$ be a uniform covering of X and $C_i,\ i \in I$ be a family of uniform chain connected subspaces of X. Let $x, y \in \bigcup_{i\in I} V(C_i)$, i.e. $x \in V(C_x)$ and $y \in V(C_y)$ for some$x, y \in I$.

Since $V(C_{i_0}) \cap V(C_i) \neq \emptyset$ for every $i \in I$, from the previous lemma, it follows that $V(C_{i_0}) \cup V(C_x)$ is a uniform chain connected in X. Similarly $V(C_{i_0}) \cup V(C_y)$ is a uniform chain connected in X. Then because $C_{i_0} \neq \emptyset$, from the previous lemma it follows that $V(C_{i_0}) \cup V(C_x) \cup V(C_y)$ is a uniform chain connected in X, i.e. for every covering $\mathcal{U}$ of X, there exists a chain in $\mathcal{U}$ that connects x and y. So $\bigcup_{i\in I} V(C_i)$ is a uniform chain connected in X.

Since $V(C_i) = V(x_i)$ is a uniform chain component of some $x_i \in C_i$ for every $i \in I$, and $\bigcup_{i\in I} V(C_i)$ is a uniform chain connected set that contain x it follows that:

$$\bigcup_{i\in I} V(C_i) = V(C_i) \text{ for every } i \in I.\blacksquare$$

Corollary 4.4. Let $C_i,\ i \in I$ be a family of uniform chain connected subspaces of X. If there exists $i_0 \in I$ such that for every $i \in I$, $C_{i_0} \cap C_i \neq \emptyset$, then the union $\bigcup_{i\in I} C_i$ is a uniform chain connected in X.

Corollary 4.5. If every two points x and y of $C \subseteq X$ are in a uniform chain connected set C_{xy} in X, then C is uniform chain connected in X.■

5. UNIFORMLY CONTINUOUS FUNCTIONS

Let X and Y be uniform spaces and $x, y \in X$.

Definition 5.1. A **uniformly continuous function** $f: X \to Y$ is defined as one where the <u>*inverse images of uniform covers are again uniform covers.*</u>

The next theorem shows that the ***uniform chain connectedness is invariant with respect to uniformly continuous function.***

Theorem 5.2. If C is uniform chain connected in X and $f: X \to Y$ is a uniform continuous function, then $f(C)$ is uniform chain connected in $f(X)$.

Proof. The function $f: X \to Y$ is uniform continuous if and only if $f: X \to f(X)$ is uniform continuous.

Let $f(x), f(y) \in f(C)$ and $\mathcal{V}$ be a uniform covering of $f(X)$. Then $\mathcal{U} = f^{-1}(\mathcal{V})$ is a uniform covering of X. Since C is uniform chain connected in X, there exists a chain $f^{-1}(V_1), f^{-1}(V_2), \ldots, f^{-1}(V_n)$ in $\mathcal{U}$ that connects x and y. Since:

$$f^{-1}(V_i) \cap f^{-1}(V_{i+1}) \neq \emptyset,\ i = 1,2,\ldots,n-1,$$

it follows that:

$$\emptyset \neq f\left(f^{-1}(V_i)\right) \cap f\left(f^{-1}(V_{i+1})\right) = V_i \cap V_{i+1},\ i = 1,2,\ldots,n;$$

i.e. $V_1, V_2, \ldots, V_n$ is a chain in $\mathcal{V}$ that connects $f(x)$ and $f(y)$.■

And…our motivation and history from ISBELL book (AMS, 1964)

Uniform topology and uniform continuity. The *uniform topology* of a uniform space X is defined as follows. A subset N of X is a neighborhood of a point x of N if for some uniform covering $\mathscr{U}$, N contains $\mathrm{St}(x, \mathscr{U})$. N is open if it is a neighborhood of each of its points.

We wish to prove

11. Theorem. *Every uniform space is a completely regular Hausdorff space in the uniform topology.*

The construction required for this proof can be made to yield another important theorem; so we put it off for a while. Observe here that the uniform space X is at least a T_1 space; for X is an open set (obvious), any union of open sets is open (obvious), the intersection of any two open sets is open (easy exercise), and a point is closed (proof follows). For any point x, for any point $y \neq x$, by definition there must be a uniform covering $\mathscr{U}$, no element of which contains both x and y. Then $\mathrm{St}(y, \mathscr{U}) \subset X - \{x\}$. Thus $X - \{x\}$ is a neighborhood of each of its points, and the point x is closed.

covering $\mathscr{U}$ of X such that, for each element U of $\mathscr{U}$, $f(U)$ is contained in some element of $\mathscr{V}$. We can restate this in terms of the convenient notion of the *inverse image* $f^{-1}(\mathscr{V})$ of a covering $\mathscr{V}$, which is the set of all $f^{-1}(V)$, $V \in \mathscr{V}$; f is uniformly continuous if and only if the inverse image of every uniform covering is uniform. (This is equivalent to the definition since a covering having a uniform refinement is uniform.)

12. *Every uniformly continuous function is continuous.*

THANK YOU FOR YOUR ATTENTION

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