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APPLICATION OF ALGEBRAS IN EVALUATING PRODUCTS OF DISTRIBUTIONS

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INTRODUCTION TO THE THEORY OF DISTRIBUTIONS

- Theory of distributions1950s
  - Mathematical meaning of many concepts in the science that were described heuristically
  - Dirac δ - function and its derivatives

- Concepts and their properties were defined heuristically, to be appropriate to the experimental results and to be adequate for analysis and solving the problems that they characterize

  - The operations with those concepts remained mathematically unsupported

$$\delta(x) = \begin{cases} 0, & x \neq 0 \\ \infty & x = 0 \end{cases} \qquad \int_{-\infty}^{\infty} \delta(x)\,dx = 1 \qquad (1)$$

INTRODUCTION TO THE THEORY OF DISTRIBUTIONS

- Necessity for generalizing the concept of function
- Distribution (generalized function)
- Laurent Schwartz, '*Theory of Distributions*' (1950)
  - Many concepts that can not be described with functions can be described applying distributions
  - Concepts that can be described with functions can also be described using distributions

INTRODUCTION TO THE THEORY OF DISTRIBUTIONS

- $\mathcal{D} = \mathcal{D}(\mathbf{R}^n)$ - space of smooth functions with compact support (test functions)
- *Generalized function (distribution) is continuous linear mapping* $f : \mathcal{D} \to \mathbf{C}$

$$f(\varphi) = \langle f, \varphi \rangle \qquad (2)$$

$\mathcal{D}' = \mathcal{D}'(\mathbf{R}^n)$ - the space of distributions with domain $\mathcal{D}$

INTRODUCTION TO THE THEORY OF DISTRIBUTIONS

- f - locally integrable function

$$f(\varphi) = \langle f, \varphi \rangle = \int_{\mathbf{R}^n} f(x)\varphi(x)\,dx \qquad (3)$$

- f - *regular distribution*

- *Singular distributions*
  - Dirac δ distribution:

$$\langle \delta, \varphi \rangle = \varphi(0) \qquad (4)$$

- Differentiation of distributions

$$f \in \mathcal{D}'(\mathbf{R}) \qquad \varphi \in \mathcal{D}(\mathbf{R})$$

- $$\left\langle f^{(n)}(x), \varphi(x) \right\rangle = (-1)^n \left\langle f(x), \varphi^{(n)}(x) \right\rangle \qquad (5)$$

$$f \in \mathcal{D}'(\mathbf{R}^n) \qquad \varphi \in \mathcal{D}(\mathbf{R}^n)$$

$$D^k = \prod_{i=1}^{n} \left(\frac{\partial}{\partial x_i} \right)^{k_i} \qquad k_i \in \mathbf{N_0} \qquad k = \sum_{i=1}^{n} k_i$$

- $$\left\langle D^k f, \varphi \right\rangle = (-1)^k \left\langle f, D^k \varphi \right\rangle \qquad (6)$$

INTRODUCTION TO THE THEORY OF DISTRIBUTIONS

- **A derivative of function, with arbitrary order, will always exist if we consider that function as generalized function (distribution)**

- Two main problems for the theory of distributions:

  - **Product of distributions: two arbitrary distributions can not always be multiplied**

    - the product of distributions is not an associative operation

  - **Differentiation of the product of distributions** (the product of distributions not always satisfy the Leibniz rule)

- The application of distributions in non-linear systems needs products of singular distributions
- Attempts for defining product of distributions that will be generalization of the existing products
- **Regularization method**

$\varphi_n \to \delta(x)$ - delta sequence; f - distribution

$$f_n(x) = (f * \varphi_n)(x) = \langle f(y), \varphi_n(x-y) \rangle \quad (7)$$

- Sequence of smooth functions (f_n); $f_n \to f$
- (f_n) - *regularization of the distribution* f
- $$fg = \lim_{n\to\infty} (f * \varphi_n) \cdot (g * \varphi_n) \quad (8)$$

INTRODUCTION TO THE THEORY OF DISTRIBUTIONS

- with regularization method:
- $$\frac{1}{x}\cdot\delta = -\frac{1}{2}\delta' \qquad (9)$$
- but, $\delta \cdot \delta = \delta^2$ is not defined neither with the regularization method

COLOMBEAU ALGEBRA

- Overcoming the problem with product of distributions

- Construction of algebra *A* with properties:

1) Contains the space of distributions $\mathcal{D}'(\mathbf{R}^n)$ and $f(x) \equiv 1$ is neutral element in *A*
2) There exist linear differential operators $\partial_i : A \to A$ which satisfy the Leibnitz's rule
3) ∂_i generalizes the derivation on the space of distributions
4) The product in *A* generalizes the product of continuous functions

- **Schwartz's impossibility result**
  - the functions $f(x) = x$ и $g(x) = |x|$ are considered
- derivative of their classical product:

$$\partial\left(x|x|\right) = 2|x| \qquad (10)$$

$$\partial^2\left(x|x|\right) = 2\partial\left(|x|\right) \qquad (11)$$

- Derivative of their product in *A*

$$\partial\left(x\cdot|x|\right) = |x| + x\cdot\partial\left(|x|\right) \qquad (12)$$

$$\partial^2\left(x\cdot|x|\right) = 2\partial\left(|x|\right) + x\cdot\partial^2\left(|x|\right) \qquad (13)$$

$$\partial^2\left(x\cdot|x|\right) = 2\partial\left(|x|\right) + 2x\cdot\delta \qquad (14)$$

COLOMBEAU ALGEBRA

- Schwartz's impossibility result
  - From (11) and (14) it follows: $x \cdot \delta = 0$ (15)

- *Theorem*: In *A*, if $x \cdot a = 0$ then $a = 0$.

- From (15) $\Rightarrow$ $\delta = 0$

COLOMBEAU ALGEBRA

- New theory of generalized functions, more general then the theory of distributions
- Jean-Francois Colombeau
  - *New generalized function and multiplication of distributions* (1984)
  - *Elementary introduction to new generalized functions* (1985)

- **Colombeau algebra**
  - The product in the algebra generalizes classical product of C^{∞} - functions

CONSTRUCTION OF THE COLOMBEAU ALGEBRA

- $\mathbf{N_0} = \mathbf{N} \cup \{0\}$ - non negative integers
- $\mathcal{D}(\mathbf{R}^n)$ - the space of C^∞ - functions $\varphi : \mathbf{R}^n \to \mathbf{C}$ with compact support

- for $j \in \mathbf{N_0}$ and $q \in \mathbf{N_0}$ the next sets are defined:

$$A_0(\mathbf{R}^n) = \left\{ \varphi(x) \in \mathcal{D}(\mathbf{R}^n) \middle| \int_{\mathbf{R}^n} \varphi(x)\,dx = 1 \right\}$$

$$A_q(\mathbf{R}^n) = \left\{ \varphi(x) \in D(\mathbf{R}^n) \middle| \int_{\mathbf{R}^n} \varphi(x)\,dx = 1, \int_{\mathbf{R}^n} x^j \varphi(x)\,dx = 0; 1 \le |j| \le q \right\}$$

$q \ge 1$ $\quad x = (x_1, x_2, \ldots, x_n) \in \mathbf{R}^n$ $\quad j = (j_1, j_2, \ldots, j_n) \in \mathbf{N}^n$

$|j| = j_1 + j_2 + \ldots + j_n$ $\quad x^j = (x_1)^{j_1} (x_2)^{j_2} \cdots (x_n)^{j_n}$

CONSTRUCTION OF THE COLOMBEAU ALGEBRA

- $A_0 \supset A_1 \supset A_2 \supset A_3 \ldots$

- *Theorem:* The sets A_q are non empty sets.

  Proof: J.F.Colombeau, *Elementary Introduction to New Generalized Functions* (1985)

- For $\varphi \in A_q\left(\mathbf{R}^n\right)$ and $\varepsilon > 0$ we denote:

$$\varphi_\varepsilon\left(x\right) = \frac{1}{\varepsilon^n}\varphi\left(\frac{x}{\varepsilon}\right) \qquad (16)$$

$$\check{\varphi}(x) = \varphi\left(-x\right) \qquad (17)$$

CONSTRUCTION OF THE COLOMBEAU ALGEBRA

- $\mathcal{E}\left(\mathbf{R}^n\right)$ - an algebra of functions $f(\varphi, x) : A_0\left(\mathbf{R}^n\right) \times \mathbf{R}^n \to \mathbf{C}$ that are infinitely differentiable regarding the second variable x (with fixed test function φ)
- The space $C^\infty\left(\mathbf{R}^n\right)$ is subalgebra of $\mathcal{E}\left(\mathbf{R}^n\right)$ (those functions that don't depend of φ)
- Embedding of the distributions in $\mathcal{E}\left(\mathbf{R}^n\right)$ is such that embedding of $C^\infty\left(\mathbf{R}^n\right)$ - functions is identity

CONSTRUCTION OF THE COLOMBEAU ALGEBRA

- $\mathcal{E}_M\left[\mathbf{R}^n\right]$ - subalgebra of $\mathcal{E}\left(\mathbf{R}^n\right)$ with elements such that for every compact subset K from $\mathbf{R}^n$ and every $p \in \mathbf{N_0}$ there exists $q \in \mathbf{N}$ such that for arbitrary $\varphi \in A_q\left(\mathbf{R}^n\right)$ there exist $c > 0,\ \eta > 0$ and the relation holds:

$$\sup_{x \in K}\left|\partial^p f\left(\varphi_\varepsilon, x\right)\right| \le c\varepsilon^{-q} \qquad (18)$$

for $0 < \varepsilon < \eta$.

  - Functions in $\mathcal{E}\left(\mathbf{R}^n\right)$ which derivatives on compact sets are bounded with negative powers of ε

CONSTRUCTION OF THE COLOMBEAU ALGEBRA

$f \in C\left(\mathbf{R}^n\right)$

- With the mapping:

$$F(\varphi, x) = \int_{\mathbf{R}^n} f(y)\varphi(y-x)\,dy = \int_{\mathbf{R}^n} f(x+t)\varphi(t)\,dt \qquad (19)$$

the space $C\left(\mathbf{R}^n\right)$ is embedded in $\mathcal{E}_M\left[\mathbf{R}^n\right]$.

- $C^\infty\left(\mathbf{R}^n\right)$ is contained in $\mathcal{E}_M\left[\mathbf{R}^n\right]$ in a way that $f(x)$ are those functions $f(\varphi, x)$ that don’t depend of φ .

- $C^\infty\left(\mathbf{R}^n\right) \subset C\left(\mathbf{R}^n\right)$ are embedded in $\mathcal{E}_M\left[\mathbf{R}^n\right]$ with (19).

$$f(x) \neq \int_{R^n} f(x+\varepsilon t)\varphi(t)\,dt \qquad (20)$$

- Ideal such that the difference in (20) will vanish

CONSTRUCTION OF THE COLOMBEAU ALGEBRA

- $\mathcal{I}\left[\mathbf{R}^n\right]$ is an ideal in $\mathcal{E}_M\left[\mathbf{R}^n\right]$ consisting of functions $f(\varphi, x)$ such that for every compact subset K of $\mathbf{R}^n$ and each $p \in \mathbf{N}_0$ there exist $q \in \mathbf{N}$ such that for any $r \geq q$ and each $\varphi \in A_r\left(\mathbf{R}^n\right)$ there exist $c > 0,\ \eta > 0$ and it holds:

$$\sup_{x \in K}\left|\partial^p f\left(\varphi_\varepsilon, x\right)\right| \leq c\varepsilon^{r-q} \qquad (21)$$

For $0 < \varepsilon < \eta$.

  - The elements of $\mathcal{I}\left[\mathbf{R}^n\right]$ are called *null functions*

CONSTRUCTION OF THE COLOMBEAU ALGEBRA

- ***Generalized functions in Colombeau theory are elements of*** *quotient algebra*

$$\mathcal{G} \equiv \mathcal{G}\left(\mathbf{R}^n\right) = \frac{\mathcal{E}_M\left[\mathbf{R}^n\right]}{\mathcal{I}\left[\mathbf{R}^n\right]} \qquad (22)$$

- In $\mathcal{E}_M\left[\mathbf{R}^n\right]$ the equivalence relation is defined ' $\sim$ ' :

$$F_1 \sim F_2 \Leftrightarrow F_1 - F_2 \in \mathcal{I}\left[\mathbf{R}^n\right] \qquad (23)$$

- ***The generalized functions in Colombeau theory are an equivalence classes of smooth functions***
- New generalized functions (Colombeau generalized functions)

EMBEDDING OF DISTRIBUTIONS IN COLOMBEAU ALGEBRA

- $f \in C^{\infty}(\mathbf{R}^n) \qquad f \to f(\varphi, x) \in \mathcal{G}(\mathbf{R}^n)$

$$f(\varphi, x) = f(x) \qquad (24)$$

- $f \in C(\mathbf{R}^n) \qquad f \to f(\varphi, x) \in \mathcal{G}(\mathbf{R}^n)$

$$f(\varphi, x) = \int_{\mathbf{R}^n} f(y)\varphi(y-x)\,dy = \int_{\mathbf{R}^n} f(x+y)\varphi(y)\,dy \qquad (25)$$

- $f \in \mathcal{D}'(\mathbf{R}^{\mathbf{n}}) \qquad f \to f(\varphi, x) \in \mathcal{G}(\mathbf{R}^n)$

$$f(\varphi, x) = \left(f * \overset{\vee}{\varphi}\right)(x) = \left\langle f(y), \varphi(y-x)\right\rangle = \int_{\mathbf{R}^n} f(y)\varphi(y-x)\,dy \qquad (26)$$

ASSOCIATION IN COLOMBEAU ALGEBRA

- The space of smooth functions $C^{\infty}(\mathbf{R}^n)$ is subalgebra of the Colombeau algebra $\mathcal{G}(\mathbf{R}^n)$
- The space of continuous functions $C(\mathbf{R}^n)$ and the space of distributions $\mathcal{D}'(\mathbf{R}^n)$ are not subalgebras of Colombeau algebra $\mathcal{G}(\mathbf{R}^n)$
  - If f, g are two continuous functions (or distributions which classical product exists), the embedding of their classical product, fg, and the product of their embeddings $f \cdot g$ in $\mathcal{G}$ may not coincide
  - This difference of the products has been overcome introducing the concept of „association“ in $\mathcal{G}$

ASSOCIATION IN COLOMBEAU ALGEBRA

- Generalized functions $F, G \in \mathcal{G}\left(\mathbf{R}^n\right)$ are said to be ***associated*** ($F \approx G$) if for each representatives $f\left(\varphi_\varepsilon, x\right)$ and $g\left(\varphi_\varepsilon, x\right)$ and each $\psi\left(x\right) \in \mathcal{D}\left(\mathbf{R}^n\right)$, there exists $q \in \mathbf{N_0}$ such that for any $\varphi\left(x\right) \in A_q\left(\mathbf{R}^n\right)$ holds:

$$\lim_{\varepsilon \to 0_+} \int_{\mathbf{R}^n} \left| f\left(\varphi_\varepsilon, x\right) - g\left(\varphi_\varepsilon, x\right) \right| \psi\left(x\right) dx = 0 \tag{27}$$

- Generalized function $F \in \mathcal{G}$ is associated with the distribution $u \in \mathcal{D}'$ ($F \approx u$) if for each representative of that generalized function $f\left(\varphi_\varepsilon, x\right)$ and each $\psi\left(x\right) \in \mathcal{D}\left(\mathbf{R}^n\right)$, there exist $q \in \mathbf{N_0}$ such that for any $\varphi\left(x\right) \in A_q\left(\mathbf{R}^n\right)$ holds:

$$\lim_{\varepsilon \to 0_+} \int_{\mathbf{R}^n} f\left(\varphi_\varepsilon, x\right) \psi\left(x\right) dx = \left\langle u, \psi \right\rangle \tag{28}$$

ASSOCIATION IN COLOMBEAU ALGEBRA

- Previous definitions are independent of the representatives chosen

- The distribution associated, if it exists, is unique

- To an element of Colombeau algebra, with this process of association, is associated element in $\mathcal{D}'$, which allows us to consider obtained results in the sense of distribution.
  - Not any element in Colombeau algebra has an associated distribution!

ASSOCIATION IN COLOMBEAU ALGEBRA

- *Theorem:* If $f, g \in C(\mathbf{R}^n)$ are two continuous functions, their product $f \cdot g$ in $\mathcal{G}(\mathbf{R})$ is associated with their classical product fg in $C(\mathbf{R}^n)$.

- *Theorem:* If $f \in C^\infty\left(\mathbf{R}^n\right)$ and $T \in \mathcal{D}'\left(\mathbf{R}^n\right)$, the product $f \cdot T$ in $\mathcal{G}\left(\mathbf{R}^n\right)$ is associated with the classical product $f\,T$ in $\mathcal{D}'\left(\mathbf{R}^n\right)$

- *Theorem*: If S and T are two distributions in $\mathcal{D}'\left(\mathbf{R}^n\right)$ and their classical product ST in $\mathcal{D}'\left(\mathbf{R}^n\right)$ exists, then the product of these two distributions $S \cdot T$ in $\mathcal{G}\left(\mathbf{R}^n\right)$ is associated with their classical product ST .

ASSOCIATION IN COLOMBEAU ALGEBRA

- ✓ Two distributions embedded in Colombeau algebra are new (Colombeau) generalized functions
- ✓ Product of two distributions in $\mathcal{G}$ is in general new (Colombeau) generalized function (for which there may not exist associated distribution)
- ✓ *If for the product of two distributions in* $\mathcal{G}$ *there exists an associated distribution, we say that there exists* ***the Colombeau product of those two distributions***
- ✓ **If the classical product of two distributions exists, then their Colombeau product also exists and is the same with the first one**

RESULTS ON PRODUCTS OF DISTRIBUTIONS IN COLOMBEAU ALGEBRA

- $$\ln|x|\cdot\delta^{(s-1)}(x)\approx\frac{-1}{s}\delta^{(s-1)}(x) \qquad s=1,2,\ldots \qquad \textbf{(29)}$$
- $$x_+^{-k}\cdot\delta^{(p)}(x)\approx\frac{(-1)^k k\cdot p!}{(p+k+1)!}\delta^{(k+p)}(x) \qquad k=1,2,\ldots \qquad p=0,1,2,\ldots$$
- $$x_+^{-r-1/2}\cdot x_-^{-k-1/2}\approx\frac{(-1)^{r+k}\pi}{2(r+k)!}\delta^{(r+k)}(x) \qquad r=0,1,2,\ldots \qquad k=0,1,2,\ldots$$

RESULTS ON PRODUCTS OF DISTRIBUTIONS IN COLOMBEAU ALGEBRA

- $$x_+^{-r-1/2} \cdot x_-^{k-1/2} \approx C_{r,k}\delta^{(r-k)}(x) \qquad r = 0,1,2,\dots \qquad k = 0,1,2,\dots \qquad r \geq k$$

$$C_{r,k} = \frac{(-1)^r (2k-1)!!k!r!\pi}{2(4k-1)!!(2r-1)!!(r-k)!(r+k)!}\sum_{q=0}^{2k}(-1)^q \binom{2k}{q}\binom{r-k}{k-q}(2(r+q)-1)!!(2(k-q)-1)!!$$

$$x_-^{-k} \cdot \delta^{(p)}(x) \approx \frac{k \cdot p!}{(p+k+1)!}\delta^{(k+p)}(x) \qquad k = 1,2,\dots \qquad p = 0,1,2,\dots$$

NEW RESULTS ON PRODUCTS OF DISTRIBUTIONS IN COLOMBEAU ALGEBRA

- $$\ln^2|x|\cdot\delta^{(s-1)}(x)\approx\frac{2}{s^2}\delta^{(s-1)}(x) \qquad s=1,2,\ldots \qquad \textbf{(30)}$$

- $$\ln^3|x|\cdot\delta^{(s-1)}(x)\approx\frac{-3!}{s^3}\delta^{(s-1)}(x) \qquad s=1,2,\ldots \qquad \textbf{(31)}$$

- $$\ln^r|x|\cdot\delta^{(s-1)}(x)\approx\frac{(-1)^r r!}{s^r}\delta^{(s-1)}(x) \qquad s=1,2,\ldots \qquad r=0,1,2,\ldots \qquad \textbf{(32)}$$

- $$f(x)\cdot\delta^{(r)}(x)\approx\sum_{i=0}^{r}\binom{r}{i}(-1)^{r-i}f^{(r-i)}(0)\delta^{(i)}(x) \qquad r=0,1,2,\ldots \qquad \textbf{(33)}$$

THANK YOU FOR YOUR ATTENTION