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SOLUTION TO THE CATENARY PROBLEM BY APPLYING THE FUNDAMENTAL LEMMA OF VARIATIONAL CALCULUS

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Abstract. In this paper, we will prove a theorem for a functional where we prove that the necessary condition for the extreme of a functional is that the variation of a functional is equal to zero and we will give an example of its application to the catenary problem.

1. Introduction

Many problems in mathematics are naturally formulated in terms of identifying a function that minimizes some quantity of interest. A natural example from geometry is the seemingly simple question: which is the shortest length path between two points in R^n? While everyone knows that such a path is the straight-line segment connecting the two points, proving that such is the case is more subtle than such a simple question would suggest. A more complex example of a question in the same vein is to ask: given some open set Ω, and some boundary conditions, can we identify a surface defined on this set, satisfying the boundary conditions, that has the minimum possible area?

The setting of the calculus of variations is over functionals on general normed vector spaces, specifically vector spaces of functions, the methods of results of the calculus of variations are remarkably simple and powerful and bear a great deal of resemblance to the machinery of finite-dimensional real analysis.

The Euler-Lagrange equations are a very useful result in variational analysis, since many naturally occurring problems in mathematics, physics and other domains of application can be formulated in terms of minimizing or maximizing an integral on a given domain.

2. Results and discussion

We will explore for the extreme of the functional

$$v[y(x)] = \int_{x_0}^{x_1} F(x, y(x), y'(x))\, dx \,, \tag{2.1}$$

with the limit points of the allowable set of curves: $y(x_0) = y_0$ and $y(x_1) = y_1$ [1]. We will consider that the function $F(x, y, y')$ is three times differentiable. We know that the necessary condition for the extreme that is the variation in the functional is equal to zero. We will now show how the main theorem is applied to the given functional (2.1).

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Let us assume that the extreme reached on two times differentiable curve $y = y(x)$ (required only the existence of a derived from the first line of residue curves, otherwise, it may be that of the curve on which the extreme is reached, there is a second derived). We are taking some close to $y = y(x)$ limit curves $y = \overline{y}(x)$ and include curves $y = y(x)$ and $y = \overline{y}(x)$ to the family curves with one parameter

$$y(x,\alpha) = y(x) + \alpha(\overline{y}(x) - y(x)) \ .$$

When $\alpha = 0$, we get the curve $y = y(x)$; when $\alpha = 1$, we get $y = \overline{y}(x)$.

As we already know, the difference $\overline{y}(x) - y(x)$ is called the variation of the function $y(x)$ and it means with the δy [2].

The variation δy in variational problems plays a role analogous to the role of the increase Δx of an independent variable x in problems for the study of the extreme of the function $f(x)$. The variation of the function $\delta y = \overline{y}(x) - y(x)$ is a function of the x.

This function can be differentiated once or several times, as $(\delta y)' = \overline{y}'(x) - y'(x) = \delta y'$ it is generated of the variance that is equal to the variance of the generated, and similarly

$$(\delta y)'' = \overline{y}''(x) - y''(x) = \delta y'',$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$(\delta y)^{(k)} = \overline{y}^{(k)}(x) - y^{(k)}(x) = \delta y^{(k)}.$$

And so, we analyse the family $y = y(x,\alpha)$, where $y(x,\alpha) = y(x) + \alpha\delta y$, containing the $\alpha = 0$ curves, of which it reaches an extreme, and in some $\alpha = 1$ close tolerances and curves that are called curves of comparison.

If we look at the values of the functional (2.1), from the family of curves $y = y(x,\alpha)$, only it turns into a function of α :

$$v[y(x,\alpha)] = \varphi(\alpha),$$

As in the case where we consider $v[y(x,\alpha)]$ to be a functional that depends on a parameter, the value of the parameter α determines the curve of the family $y = y(x,\alpha)$, which determines the value of the functional. $v[y(x,\alpha)]$.

Theorem 2.1.

If the functional $v(y) = \int_{x_0}^{x_1} F(x, y, y')dx$ *has a local extreme in* y, *the necessary condition for the extreme of the functional is*

$$\int_{x_0}^{x_1} [F_y - \frac{d}{dx} F_{y'}]\delta y \, dx = 0, \tag{2.2}$$

Proof of theorem 2.1.
We analyse the function $\varphi(\alpha)$. It reaches its extreme at $\alpha=0$, and when $\alpha=0$, we get $y=y(x)$, and the functional, in assumption, reaches the extreme compared with any permissible curve, and in particular, in terms of families of near curves $y=y(x,\alpha)$ [2].
The necessary condition for the extreme of the function $\varphi(\alpha)$ at $\alpha=0$, as is known, is that its derivative is equal to zero at $\alpha=0$, i.e.

$$\varphi'(0)=0\ .$$

Since

$$\varphi(\alpha)=\int_{x_0}^{x_1} F(x,y(x,\alpha),y_x'(x,\alpha))\,dx,$$

It

$$\varphi'(\alpha)=\int_{x_0}^{x_1}\left[F_y'\frac{\partial}{\partial\alpha}y(x,\alpha)+F_{y'}'\frac{\partial}{\partial\alpha}y'(x,\alpha))\right]dx,$$

Where

$$F_y'=\frac{\partial}{\partial y}F(x,y(x,\alpha),y'(x,\alpha)),$$

$$F_{y'}'=\frac{\partial}{\partial y'}F(x,y(x,\alpha),y'(x,\alpha)),$$

$$\frac{\partial}{\partial\alpha}y(x,\alpha)=\frac{\partial}{\partial\alpha}[y(x)+\alpha\delta y]=\delta y$$

$$\frac{\partial}{\partial\alpha}y'(x,\alpha)=\frac{\partial}{\partial\alpha}[y'(x)+\alpha\delta y']=\delta y',$$

And we get

$$\varphi'(\alpha)=\int_{x_0}^{x_1}\left[F_y(x,y(x,\alpha),y'(x,\alpha))\delta y+F_{y'}(x,y(x,\alpha),y'(x,\alpha))\delta y'\right]dx,$$

$$\varphi'(0)=\int_{x_0}^{x_1}\left[F_y(x,y(x),y'(x))\delta y+F_{y'}(x,y(x),y'(x))\delta y'\right]dx\quad (npu\ \alpha=0).$$

As we know, $\varphi'(0)$ is called the variation of the functional and means δv.
The necessary condition for the extreme of the functional is that its variation is equal to zero

$$\delta v=0\ .$$

For the functional (2.1), this condition has a type of

$$\int_{x_0}^{x_1} [F_y '\delta y + F_{y'}^{'}\delta y'] \, dx = 0 \qquad (2.3)$$

Integrate the equation (2.3) in parts, whereas $\delta y' = (\delta y)'$, we get

$$\delta v = [F_{y'}^{'}\delta y']\Big|_{x_0}^{x_1} + \int_{x_0}^{x_1} [F_y ' - \frac{d}{dx} F_{y'}^{'}]\delta y \, dx =$$

$$= \int_{x_0}^{x_1} F_y^{'}\delta y \, dx + F_{y'}^{'}(x_1, y(x_1,\alpha), y'(x_1,\alpha))\delta y(x_1) - F_{y'}^{'}(x_0, y(x_0,\alpha), y'(x_0,\alpha))\delta y(x_0) =$$

$$= \int_{x_0}^{x_1} F_y^{'}\delta y \, dx + F_{y'}^{'}(x_1, y(x_1,\alpha), y'(x_1,\alpha))(\bar{y}(x_1) - y(x_1))$$

$$- F_{y'}^{'}(x_0, y(x_0,\alpha), y'(x_0,\alpha))(\bar{y}(x_0) - y(x_0)) - \int_{x_0}^{x_1} (\delta y) dF_{y'}^{'} =$$

$$= \int_{x_0}^{x_1} F_y^{'}\delta y \, dx + F_{y'}^{'}(x_1, y(x_1,\alpha), y'(x_1,\alpha))(0)$$

$$- F_{y'}^{'}(x_0, y(x_0,\alpha), y'(x_0,\alpha))(0) - \int_{x_0}^{x_1} (\delta y) \frac{d}{dx} F_{y'}^{'}$$

Since all of the possible (permissible) curves in the given problem pass through fixed limit points, we get

$$\delta v = \int_{x_0}^{x_1} [F_y ' - \frac{d}{dx} F_{y'}^{'}]\delta y \, dx \ .$$

□

Note 2.1.

The first multiplier $F_y ' - \frac{d}{dx} F_{y'}^{'}$ of the curve $y = y(x)$ reaches the extreme of the continuous function, and the second multiplier δy, random for the choice of the curve in comparison $y = \bar{y}(x)$, is an arbitrary function having passed only certain general conditions, namely: the function δy in the border points $x = x_0$, and $x = x_1$ is equal to zero, continuous and differentiable once or several times, δy or δy and $\delta y'$ are small in absolute value [1].

To simplify the obtained necessary condition (2.2), we will use the following lemma:

Fundamental lemma of the variational calculus.

If for any continuous function $\eta(x)$ *is true*

$$\int_{x_0}^{x_1} \Phi(x)\,\eta(x)\,dx = 0,$$

Where the function $\Phi(x)$ *is continuous in the interval* $[x_0, x_1]$,

$$\Phi(x) \equiv 0$$

in this interval [3].

Proof of the fundamental lemma of variational calculus.

We accept that, in the point $x = \bar{x}$, resting in the interval (x_0, x_1) , $\Phi(x) \neq 0$, is a contradiction.
Indeed, from the continuity of the function $\Phi(x)$, it follows that if $\Phi(\bar{x}) \neq 0$, $\Phi(x)$, it does not change the characters in the vicinity of $\bar{x}$ ($x_0 \leq x \leq x_1$).
We choose the function $\eta(x)$ which also retains the mark in that vicinity and is equal to zero outside of this vicinity. We get

$$\int_{x_0}^{x_1} \Phi(x)\eta(x)\,dx = \int_{\bar{x}_0}^{\bar{x}_1} \Phi(x)\eta(x)\,dx \neq 0,$$

Since product $\Phi(x)\eta(x)$ retains its mark in the interval $x_0 \leq x \leq x_1$ and is equal to zero in the same interval.
And so, we come to a contradiction, therefore $\Phi(x) \equiv 0$.

□

Note 2.2.
Adoption of the lemma and its proof remain unchanged if the function $\eta(x)$ requires the following restrictions:

$$\eta(x_0) = \eta(x_1) = 0,$$

$\eta(x)$ There is a continuous derived to line n ,

$$\left|\eta^{(s)}(x)\right| < \varepsilon, \quad (s = 0,1,\ldots,q; q \leq n) \ .$$

The function $\eta(x)$ can be selected, e.g.:

$$\eta(x) = \begin{cases} k(x-\bar{x}_0)^{2n}(x-\bar{x}_1)^{2n}, & x \in [\bar{x}_0, \bar{x}_1] \\ 0 & x \in [x_0, x_1] \setminus [\bar{x}_0, \bar{x}_1] \end{cases} ,$$

where n is a positive number, k is a constant.

Apparently, the function $\eta(x)$ satisfies the above conditions: it is a continuous, there is a continuous derived to line $2n-1$, in the points x_0 and x_1 it is equal to zero, and by reducing the factor by k, we can do $\left|\eta^{(s)}(x)\right| < \varepsilon$ for the $\forall x \in [x_0, x_1]$.

Now we will apply the fundamental lemma of variational calculus to simplify the above necessary condition for the extreme (2.2) of the functional (2.1).

Consequence 2.1.

If the functional $v(y) = \int\limits_{x_0}^{x_1} F(x, y, y')dx$ *reaches extreme of the curve* $y = y(x)$ *, and* F_y' *and* $\frac{d}{dx}F_{y'}'$ *are continuous, then* $y = y(x)$ *is a solution to the differential equation (equation of Euler)*

$$F_y - \frac{d}{dx}F_{y'} = 0,$$

Or in an expanded form

$$F_y - F_{xy'} - F_{yy'}y' - F_{y'y'}y'' = 0 \text{ [4].}$$

Proof of consequence 2.1.

The proof of consequence 2.1 follows immediately from the fundamental lemma of variational calculus.

This equation is called the equation of Euler (1744 year). The integral curve $y = y(x, C_1, C_2)$ equation of Euler is called extreme.

To find a curve, that is the reached extreme of functional (2.1) we integrate the equation of Euler and spell out random constants, satisfying the general solution of this equation, of the conditions of borders $y(x_0) = y_0, y(x_1) = y_1$.

Only if they are satisfied with these conditions, the extreme of functional can be reached.

However, in order to determine whether they are really extreme (maximum or minimum), the sufficient conditions for the extreme must be studied.

To recall, that border problem

$$F_y' - \frac{d}{dx}F_{y'}' = 0, \quad y(x_0) = y_0, \quad y(x_1) = y_1 ,$$

not always has a solution, and if there is a solution, then this may not be sole.

It should be taken into account that in many variational problems the existence of solutions is evident, from physical or geometrical sense of the problem, and in the solution of the equations of Euler satisfying the border conditions, only a single extreme may be the solution of the given problem.

Catenary problem

Consider a string with uniform density μ of length l suspended from two points of equal height and at distance D. By the formula for the arc length,

$$l = \int_S dS = \int_{S_1}^{S_2} \sqrt{1 + y'^2}\, dx,$$

where S is the path of the string, and S_1 and S_2 are the boundary conditions [5].

The curve has to minimize its potential energy

$$U = \int_S g\mu y \,.\, dS = \int_{S_1}^{S_2} g\mu y . \sqrt{1 + y'^2}\, dx,$$

and is subject to the constraint

$$\int_{S_1}^{S_2} \sqrt{1 + y'^2}\, dx = l,$$

where g is the force of gravity.

Because the independent variable x does not appear in the integrand, the Beltrami identity may be used to express the path of the string as a separable first order differential equation

$$L - \frac{\partial L}{\partial y'} = \mu g y \sqrt{1 + y'^2} + \lambda \sqrt{1 + y'^2} - \left[\mu g y \frac{y'^2}{\sqrt{1+y'^2}} + \lambda \frac{y'^2}{\sqrt{1+y'^2}} \right] = C,$$

where λ is the Lagrange multiplier [6],[7].

It is possible to simplify the differential equation as such:

$$\frac{g\rho y - \lambda}{\sqrt{1+y'^2}} = C.$$

Solving this equation gives the hyperbolic cosine, where C_0 is a second constant obtained from integration

$$y = \frac{C}{\mu g} cosh\left[\frac{\mu g}{C}(x + C_0)\right] - \frac{\lambda}{\mu g}.$$

The three unknowns C, C_0, and λ can be solved for using the constraints for the string's endpoints and arc length L, though a closed-form solution is often very difficult to obtain.

3. Conclusion

It should be taken into account that in many variational problems the existence of solutions is evident, from physical or geometrical sense of the problem, and in the solution of the equations of Euler satisfying the border conditions, only a single extreme may be the solution of the given problem.

References

[1] *Assoc. Dr. B. Златанов* - Metric spaces;
[2] *L.Э.Эльсгольц* - ДИФФЕРЕНЦИАЛЬНЫЕ equations AND ВАРИАЦИОННОЕ ИСЧИСЛЕНИЕ;
[3] *Russak I.B.* - MA4311, 1996, Calculus of variations;
[4] *Brunt B.* - Springer, 2004, The calculus of variations (ISBN 0387402470);
[5] *Byerly W.* - Cambridge, 1917, Introduction to the calculus of variations;
[6] *Dr. Hosein Naderpour* – 2017, CALCULUS OF VARIATIONS;
[7] *PROF. ARNOLD ARTHURS* – 2021, CALCULUS OF VARIATIONS.

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