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THIRD ORDER HANKEL DETERMINANT FOR INVERSE FUNCTIONS OF A CLASS OF STARLIKE FUNCTIONS OF ORDER α

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Abstract

The main goal of this paper is to determine an upper bound for the third Hankel determinant for the inverse functions of f, belonging to the class of starlike function of order α $(0 \le \alpha \le \frac{1}{2})$.

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1. Introduction

Let $\mathcal{A}$ is the class of functions f which are analytic in $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ and are normalized such that $f(0) = 0 = f'(0) - 1$, i.e.,

$$f(z) = z + a_2 z^2 + a_3 z^3 + \cdots \tag{1.1}$$

We want to find upper bound of the modulus of the Hankel $H_q(n)(f)$ of a given function f, for $q \geq 1$ and $n \geq 1$ defined with:

$$H_q(n)(f) = \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\ a_{n+1} & a_{n+2} & \cdots & a_{n+q} \\ \vdots & \vdots & & \vdots \\ a_{n+q-1} & a_{n+q} & \cdots & a_{n+2q-2} \end{vmatrix}.$$

The third Hankel determinant is

$$H_3(1)(f) = \begin{vmatrix} 1 & a_2 & a_3 \\ a_2 & a_3 & a_4 \\ a_3 & a_4 & a_5 \end{vmatrix} = a_3(a_2 a_4 - a_3^2) - a_4(a_4 - a_2 a_3) + a_5(a_3 - a_2^2).$$

This research is focused on the class of starlike function of order α. The class of starlike function of order α is defined with

$$S^*(\alpha) = \left\{ f \in \mathcal{A} : Re \frac{z f'(z)}{f(z)} \geq \alpha, z \in \mathbb{D} \right\}.$$

More about this class can be found in Thomas et al. [12]. Research like ours we have from Krishna and Ramreddy in Krishna et al. [4] who obtain an upper bound of the second Hankel determinant $|a_2 a_4 - a_3^2|$ for starlike and convex functions of order α. The bounds of some initial coefficients, the Fekete-Szegö -type inequality and estimation of Hankel determinants of second and third order were discussed in Shi et al. [10]. Upper bound of the Hankel determinant of third order for inverse functions of functions from some classes of univalent functions can find in Obradović et al. [6].

Some of the more significant results for the Hankel determinant of second order for the inverse functions of convex and starlike function can be found in Obradović et al. [7] and for the second Hankel determinant for starlike and convex functions of order alpha in Sim et al. [11]. Sharp bound of third Hankel determinant for inverse coefficients of convex functions can be found in Raza et al. [9] and the sharp bound of the third Hankel determinant $|H_3(1)(f)|$ in Lecko et al., Ahamed et al., and Rath at al., respectively [5], [1] and [8].

For every univalent function in $\mathbb{D}$, there exists inverse at least on the disk with radius 1/4. If the inverse has an expansion

$$f^{-1}(\omega) = \omega + A_2 \omega^2 + A_3 \omega^3 + \cdots , \tag{1.2}$$

then, by using the identity $f(f^{-1}(\omega)) = \omega$, from (1.1) and (1.2) we receive

$$\begin{aligned} A_2 &= -a_2 \\ A_3 &= -a_3 + 2a_2^2 \\ A_4 &= -a_4 + 5a_2a_3 - 5a_2^3 \\ A_5 &= -a_5 + 6a_2a_4 - 21a_2^2a_3 + 3a_3^2 + 14a_2^4. \end{aligned} \tag{1.3}$$

By using the definition of $H_3(1)(f)$ and the relations (1.3), after some calculations, we obtain:

$$\begin{aligned} H_3(1)(f^{-1}) &= A_3(A_2A_4 - A_3^2) - A_4(A_4 - A_2A_3) + A_5(A_3 - A_2^2) \\ &= a_3(a_2a_4 - a_3^2) - a_4(a_4 - a_2a_3) + a_5(a_3 - a_2^2) - (a_3 - a_2^2)^3 \\ &= H_3(1)(f) - (a_3 - a_2^2)^3, \end{aligned}$$

i.e.,

$$H_3(1)(f^{-1}) = H_3(1)(f) - (a_3 - a_2^2)^3. \tag{1.4}$$

2. **Preliminaries**

Let P denote the class of functions p analytic in $\mathbb{D}$, for which $\operatorname{Re}\{p(z)\} > 0$,

$$p(z) = (1 + c_1z + c_2z^2 + c_3z^3 + \cdots) = 1 + \sum_{n=1}^{\infty} c_nz^n, \ \ \forall z \in \mathbb{D}.$$

More about this class can be found in Grenander at al. [2]. For our consideration we need the next lemma which can be found in Kwon et al. [3] and Shi et al. [10].

Lemma 2.1. *Let $p \in P$. Then, for some $\rho, \sigma, x \in \mathbb{D}$, we have*

$$\begin{aligned} 2c_2 &= c_1^2 + x(4 - c_1^2), \\ 4c_3 &= c_1^3 + 2(4 - c_1^2)c_1x - c_1(4 - c_1^2)x^2 + 2(4 - c_1^2)(1 - |x|^2)\sigma, \\ 8c_4 &= c_1^4 + (4 - c_1^2)x[c_1^2(x^2 - 3x + 3) + 4x] \\ &\quad -4(4 - c_1^2)(1 - |x|^2)[c_1(x - 1)\sigma + \overline{x}\sigma^2 - (1 - |x|^2)\rho]. \end{aligned} \tag{2.1}$$

3. **Main results**

Theorem 3.1. *If $f(z) \in S^*(\alpha)(0 \le \alpha \le \frac{1}{2})$, then*

$$\left|H_3(1)(f^{-1})\right| \le \frac{143}{18}(1 - \alpha)^2.$$

The result is sharp for the function $f \in S^*(\alpha), (0 \le \alpha \le \frac{1}{2})$ given by $f(z) = z + \frac{\sqrt{286}}{6}(1 - \alpha)z^4 + \cdots$.

P r o o f. From $f(z)=z+a_2z^2+a_3z^3+\cdots$ there exists an analytic function $p\in P$ in the unit disk $\mathbb{D}$ with $p(0)=1$ and $Re\{p(z)\}>0$ such that

$$\left\{\frac{zf'(z)-\alpha f(z)}{(1-\alpha)f(z)}\right\}=p(z)\Leftrightarrow zf'(z)-\alpha f(z)=(1-\alpha)f(z)p(z). \tag{3.1}$$

We calculate $f'(z)=1+2a_2z+3a_3z^2+4a_4z^3+\cdots$ and than replacing $f(z), f'(z)$ and $p(z)$ in 3.1 we get

$$[z\{1+2a_2z+3a_3z^2+4a_4z^3+\cdots\}-\alpha\{z+a_2z^2+a_3z^3+a_4z^4+\cdots\}]$$
$$=[(1-\alpha)[\{z+a_2z^2+a_3z^3+a_4z^4+\cdots\}\times\{1+c_1z+c_2z^2+c_3z^3+\cdots\}].$$

Equating the coefficients of powers of z, z^2, z^3 and z^4 after simplifying, we get:

$$\begin{aligned}a_2&=(1-\alpha)c_1\\ a_3&=\frac{(1-\alpha)}{2}(c_2+(1-\alpha)c_1^2)\\ a_4&=\frac{(1-\alpha)}{6}(2c_3+3(1-\alpha)c_1c_2+(1-\alpha)^2c_1^3)\\ a_5&=\frac{(1-\alpha)}{24}(6c_4+6c_1c_3-6c_1c_3\alpha+3(1-\alpha)(c_2+(1-\alpha)c_1^2)c_2\\ &\quad+(1-\alpha)(2c_3+3c_1c_2(1-\alpha)+(1-\alpha)^2c_1^3)c_1).\end{aligned} \tag{3.2}$$

So, from (1.4) and (3.2), we obtain:

$$\begin{aligned}H_3(1)(f^{-1})=\frac{1}{144}(1-\alpha)^2(&17(1-\alpha)^4c_1^6-51(1-\alpha)^3c_1^4c_2\\ &+8(1-\alpha)^2c_1^3c_3+9(1-\alpha)c_1^2\left(5(1-\alpha)c_2^2-2c_4\right)\\ &+24(1-\alpha)c_1c_2c_3-27(1-\alpha)c_2^3+18c_2c_4-16c_3^2),\end{aligned}$$

i.e.

$$\begin{aligned}|H_3(1)(f^{-1})|=|\frac{1}{144}(1-\alpha)^2(&17(1-\alpha)^4c_1^6-51(1-\alpha)^3c_1^4c_2\\ &+8(1-\alpha)^2c_1^3c_3+9(1-\alpha)c_1^2\left(5(1-\alpha)c_2^2-2c_4\right)\\ &+24(1-\alpha)c_1c_2c_3-27(1-\alpha)c_2^3+18c_2c_4-16c_3^2)|.\end{aligned}$$

Substituting the equation of c_2, c_3 and c_4 from the given Lemma 2.1, in the right side of $|H_3(1)(f^{-1})|$, we have

$$\begin{aligned}|H_3(1)(f^{-1})|=\frac{1}{144}(1-\alpha)^2|&17(1-\alpha)^4c_1^6-\frac{51}{2}(1-\alpha)^3c_1^6\\ &+\frac{51}{2}(1-\alpha)^3c_1^4x(4-c_1^2)+2(1-\alpha)^2c_1^6+4(1-\alpha)^2c_1^4x(4-c_1^2)\\ &-2(1-\alpha)^2c_1^4(4-c_1^2)x^2+4(1-\alpha)^2c_1^3(4-c_1^2)(1-|x|^2)\sigma\\ &+9(1-\alpha)c_1^25(1-\alpha)\frac{1}{4}(c_1^2+x(4-c_1^2))^2-\frac{9}{4}(1-\alpha)c_1^6\\ &-9(1-\alpha)c_1^2(4-c_1^2)xc_1^2(x^2-3x+3)-\frac{9}{4}(1-\alpha)c_1^2(4-c_1^2)x^24\end{aligned}$$

$$+\frac{9}{4}(1-\alpha)c_1^2 4(4-c_1^2)(1-|x|^2)c_1(x-1)\sigma$$
$$-\frac{9}{4}(1-\alpha)c_1^2(-4)(4-c_1^2)(1-|x|^2)\overline{x}\sigma^2$$
$$+\frac{9}{4}(1-\alpha)c_1^2(-4)(4-c_1^2)(1-|x|^2)(1-|\sigma|^2)\rho$$
$$+24(1-\alpha)c_1^6\frac{1}{8}+24(1-\alpha)c_1\frac{1}{2}x(4-c_1^2)\frac{1}{4}c_1^3$$
$$+24(1-\alpha)c_1\frac{1}{2}c_1^3x2(4-c_1^2)+24(1-\alpha)c_1\frac{1}{2}x(4-c_1^2)^2 2c_1x$$
$$-24(1-\alpha)c_1\frac{1}{2}c_1^3x^2(4-c_1^2)-24(1-\alpha)c_1^2\frac{1}{2}x(4-c_1^2)^2x^2$$
$$+24(1-\alpha)c_1^3\frac{1}{2}2(4-c_1^2)(1-|x|^2)\sigma)+24(1-\alpha)c_1\frac{1}{2}x(4-c_1^2)^2 2(1-|x|^2)\sigma)$$
$$-\frac{27}{8}(1-\alpha)c_1^6-\frac{27}{8}(1-\alpha)12xc_1^4+\frac{27}{8}(1-\alpha)3xc_1^6$$
$$-\frac{27}{8}(1-\alpha)c_1^2(x^2(4-c_1^2)^2)-\frac{27}{8}(1-\alpha)x^3(4-c_1^2)^3+9(c_1^2+x(4-c_1^2))c_1^4\frac{1}{8}$$
$$+9(c_1^2+x(4-c_1^2))\frac{1}{8}(4-c_1^2)x[c_1^2(x^2-3x+3)+4x]$$
$$-9(c_1^2+x(4-c_1^2))\frac{1}{2}(4-c_1^2)(1-|x|^2)(c_1(x+1)\sigma+\overline{x}\sigma^2-(1-|\sigma|^2)\rho]$$
$$-c_1^6-4c_1^4(4-c_1^2)x+2c_1^4(4-c_1^2)x^2-4(4-c_1^2)\sigma(1-|x|^2)c_1^3-4c_1^2x^2(4-c_1^2)^2$$
$$+2c_1^2x^3(4-c_1^2)^2-8c_1(4-c_1^2)^2x\sigma(1-|x|^2)+2c_1^2(4-c_1^2)^2x^3-c_1^2(4-c_1^2)^2x^4$$
$$+2c_1(4-c_1^2)^2x^2\sigma(1-|x|^2)+2(4-c_1^2)^2\sigma(1-|x|^2)c_1x^2$$
$$-4(4-c_1^2)^2\sigma^2(1-|x|^2)^2|.$$

For $c_1=c\in[0,2]$, and sorting the expression we obtain

$$H_3(1)(f^{-1})=\frac{1}{144}(1-\alpha)^2(v_1(c,x)+v_2(c,x)\sigma+v_3(c,x)\sigma^2+\psi(c,x,\sigma)\rho),$$

where $|x|\le 1, |\sigma|\le 1, |\rho|\le 1$, and

$$v_1(c,x)=17(1-\alpha)^4c^6-\frac{51}{2}(1-\alpha)^3c^6+2(1-\alpha)^2c^6-c^6$$
$$+\frac{9}{8}c^6+\frac{9}{4}(1-\alpha)^2c^6-\frac{45}{8}(1-\alpha)c^6+3(1-\alpha)c^6$$
$$-\frac{81}{2}(1-\alpha)xc^4+\frac{81}{8}(1-\alpha)xc^6+(4-c^2)[(4-c^2)(\frac{9}{4}(1-\alpha)^2c^2x^2$$
$$+24(1-\alpha)c^2x^2-12(1-\alpha)c^2x^3-\frac{27}{8}(1-\alpha)x^2c^2+\frac{9}{8}x^2c^2(x^2-3x+3)$$
$$+\frac{9}{2}x^3-4c^2x^2+4c^2x^3-c^2x^4+\frac{51}{2}(1-\alpha)^3xc^4+4(1-\alpha)^2c^4x$$
$$-2(1-\alpha)^2c^4x^2+\frac{9}{2}(1-\alpha)^2c^4x-9(1-\alpha)c^4x(x^2-3x+3)-9(1-\alpha)c^2x^2$$

$$+3(1-\alpha)c^4x+24(1-\alpha)c^4x-12(1-\alpha)c^4x^2+\frac{9}{8}c^4x+\frac{9}{8}c^4x(x^2-3x+3)$$
$$+\frac{9}{2}c^2x^2-4c^4x+2c^4x^2],$$
$$v_2(c,x)=(4-c^2)(1-|x|^2)[(4-c^2)(-\frac{9}{2}x(x+1)c-8cx+4cx^2)$$
$$+4(1-\alpha)^2c^3+9(1-\alpha)c^3(x-1)+24(1-\alpha)c^3+24(1-\alpha)cx$$
$$-\frac{9}{2}c^3(x+1)]-4(4-c^2),$$
$$v_3(c,x)=(4-c^2)(1-|x|^2)[9(1-\alpha)c^2\overline{x}-\frac{9}{2}c^2\overline{x}$$
$$-\frac{9}{2}|x|^2(4-c^2)-4(4-c^2)(1-|x|^2)]$$
$$\psi(c,x,\sigma)=(4-c^2)(1-|x|^2)(1-|\sigma|^2)(-9(1-\alpha)c^2+\frac{9}{2}c^2).$$

Setting $|x|=x$, $|\sigma|=y$ and utilizing the assumption $|\rho|\le 1$, we obtain

$$\begin{aligned}|H_3(1)(f^{-1})| &\le \frac{1}{144}(1-\alpha)^2(|v_1(c,x)| \\ &+|v_2(c,x)|y+|v_3(c,x)|y^2+|\psi(c,x,\sigma)|) \\ &\le \frac{1}{144}(1-\alpha)^2Q(c,x,y),\end{aligned} \tag{3.3}$$

where

$$Q(c,x,y)=q_1(c,x)+q_2(c,x)y+q_3(c,x)y^2+q_4(c,x)(1-y^2),$$

and

$$q_1(c,x)=17(1-\alpha)^4c^6+\frac{51}{2}(1-\alpha)^3c^6+\frac{17}{4}(1-\alpha)^2c^6+c^6$$
$$+\frac{1}{8}c^6+\frac{21}{8}(1-\alpha)c^6+\frac{81}{2}(1-\alpha)xc^4+\frac{81}{8}(1-\alpha)xc^6$$
$$+(4-c^2)[(4-c^2)(\frac{9}{4}(1-\alpha)^2c^2x^2$$
$$+\frac{165}{8}(1-\alpha)c^2x^2-12(1-\alpha)c^2x^3+\frac{9}{8}x^2c^2(x^2-3x+3)+\frac{9}{2}x^3$$
$$+4c^2x^2+4c^2x^3+c^2x^4)+\frac{51}{2}(1-\alpha)^3xc^4+\frac{17}{2}(1-\alpha)^2c^4x+2(1-\alpha)^2c^4x^2$$
$$+9(1-\alpha)c^4x(x^2-3x+3)+9(1-\alpha)c^2x^2+27(1-\alpha)c^4x$$
$$+12(1-\alpha)c^4x^2+\frac{23}{8}c^4x+\frac{9}{8}c^4x(x^2-3x+3)+\frac{9}{2}c^2x^2+2c^4x^2]$$
$$q_2(c,x)=(4-c^2)(1-x^2)[(4-c^2)(\frac{9}{2}x(x+1)c+8cx+4cx^2)$$
$$+4(1-\alpha)^2c^3+9(1-\alpha)c^3(x-1)+24(1-\alpha)c^3$$
$$+24(1-\alpha)cx+\frac{9}{2}c^3(x+1)]+4(4-c^2)$$

$$q_3(c,x) = (4-c^2)(1-x^2)[9(1-\alpha)c^2x + \frac{9}{2}c^2x + \frac{9}{2}x^2(4-c^2)$$
$$+4(4-c^2)(1-x^2)]$$
$$q_4(c,x) = (4-c^2)(1-x^2)(9(1-\alpha)c^2 + \frac{9}{2}c^2).$$

Next, to find the upper bound of $|H_3(1)(f^{-1})|$, we have to maximize $Q(c,x,y)$ in the closed cuboid $\Omega : [0,2]\times[0,1]\times[0,1]$. So, we need to consider the following cases:

1) In the interior of Ω:

$$\frac{\partial Q(c,x,y)}{\partial y} = 0 \text{ imples that } q_2(c,x) + 2y(q_3(c,x) - q_4(c,x)) = 0,$$

so

$$y = -\frac{q_2(c,x)}{2(q_3(c,x) - q_4(c,x))} < 0$$

for $c \in (0,2)$ and $x \in (0,1)$. Hence, we deduce that Q has no critical point in the interior of Ω.

2) On the edges of Ω: For $x = 1, y = 0$, we get

$$\begin{aligned} Q(c,1,0) &= q_1(c,1) + q_4(c,1) \\ &= 17(1-\alpha)^4c^6 + \frac{51}{2}(1-\alpha)^3c^6 + \frac{17}{4}(1-\alpha)^2c^6 \\ &+\frac{21}{8}(1-\alpha)c^6 + \frac{81}{2}(1-\alpha)c^4 + \frac{81}{8}(1-\alpha)c^6 + \frac{9}{8}c^6 \\ &+(4-c^2)[(4-c^2)(\frac{9}{4}(1-\alpha)^2c^2 \\ &+\frac{165}{8}(1-\alpha)c^2 - 12(1-\alpha)c^2 + \frac{9}{8}c^2 + 9c^2 + \frac{9}{2}) \\ &+\frac{51}{2}(1-\alpha)^3c^4 + \frac{17}{2}(1-\alpha)^2c^4 + 2(1-\alpha)^2c^4 \\ &+9(1-\alpha)c^4 + 9(1-\alpha)c^2 + 27(1-\alpha)c^4 \\ &+12(1-\alpha)c^4 + \frac{23}{8}c^4 + \frac{25}{8}c^4 + \frac{9}{2}c^2] \leq 1144, \end{aligned} \tag{3.4}$$

for $0 < c < 2$ and $0 < \alpha < \frac{1}{2}$. For $x = 1, y = 1$, we get

$$Q(c,1,1) = q_1(c,1) + q_2(c,1) + q_3(c,1) \leq 1144,$$

for $0 < c < 2$.

2.1) On $x = 0, y = 1$, we get

$$Q(c,0,1) = q_1(c,0) + q_2(c,0) + q_3(c,0)$$

$$
\begin{aligned}
= 17(1-\alpha)^4c^6 + \frac{51}{2}(1-\alpha)^3c^6 + \frac{17}{4}(1-\alpha)^2c^6 \\
+c^6 + \frac{1}{8}c^6 + \frac{21}{8}(1-\alpha)c^6 \\
+(4-c^2)[4(1-\alpha)^2c^3 \\
-9(1-\alpha)c^3 + 24(1-\alpha)c^3 + \frac{9}{2}c^3 + 4 + 4(4-c^2)] \leq 496
\end{aligned} \tag{3.5}
$$

for $0 < c < 2$ and $0 < \alpha < \frac{1}{2}$.

2.2) For $c = 0, y = 0$,

$$Q(0,x,0) = q_1(0,x) + q_4(0,x) = 72x^3 \leq 72$$

which is equivalent to $x^3 \leq 1$, which is true.

2.3) If $c = 0, y = 1$ we can see

$$Q(0,x,1) = q_1(0,x) + q_2(0,x) + q_3(0,x)$$

$$= 72x^3 + 16 + 4(1-x^2) + 18x^2 + 16(1-x^2) = 72x^3 - 2x^2 + 36 \leq 106$$

which is equivalent to $72x^3 - 2x^2 - 70 \leq 0$ which is true.

2.4) If $c = 0, x = 0$ we can see that

$$Q(0,0,y) = q_2(0,0)y + q_3(0,0)y^2 = 16y + 16y^2 \leq 32, 0 < y < 1.$$

2.5) If $c = 0, x = 1$ we can see that

$$Q(0,1,y) = q_1(0,1) + q_2(0,1)y + q_3(0,1)y^2 + q_4(0,1)(1-y^2)$$

$$= 72x^3 + 16y \leq 88, 0 < y < 1.$$

2.6) If $c = 2, x = 0$ we can see that

$$
\begin{aligned}
Q(2,0,y) &= q_1(2,0) + q_2(2,0)y + q_3(2,0)y^2 \\
&= 17(1-\alpha)^4c^6 + \frac{51}{2}(1-\alpha)^3c^6 + \frac{17}{4}(1-\alpha)^2c^6 \\
&\quad + \frac{1}{8}c^6 + \frac{21}{8}(1-\alpha)c^6 \leq 496,
\end{aligned}
$$

$$0 < y < 1, 0 < \alpha < \frac{1}{2}.$$

2.7) If $c = 2, x = 1$ we can see that

$$Q(2,1,y) = q_1(2,1) + q_2(2,1)y + q_3(2,1)y^2$$

$$= 17(1-\alpha)^4c^6 + \frac{51}{2}(1-\alpha)^3c^6 + \frac{17}{4}(1-\alpha)^2c^6$$

$$+c^6 + \frac{1}{8}c^6 + \frac{21}{8}(1-\alpha)c^6 + \frac{81}{2}(1-\alpha)c^4 + \frac{81}{8}(1-\alpha)c^6 \leq 1128,$$

$$0 < y < 1, 0 < \alpha < \frac{1}{2}.$$

2.8) If $c = 2, y = 0$ we can see that

$$Q(2,x,0) = q_1(2,x)$$
$$= 17(1-\alpha)^4c^6 + \frac{51}{2}(1-\alpha)^3c^6 + \frac{17}{4}(1-\alpha)^2c^6$$
$$+c^6 + \frac{1}{8}c^6 + \frac{21}{8}(1-\alpha)c^6 + \frac{81}{2}(1-\alpha)xc^4 + \frac{81}{8}(1-\alpha)xc^6 \leq 1128,$$
$$0 < x < 1, 0 < \alpha < \frac{1}{2}.$$

We also have

$$Q(2,x,1) = q_1(2,x) = Q(2,1,y),$$
$$Q(2,0,0) = Q(2,0,y).$$

3) On the faces of Ω:

3.1) On $x = 0$ we have

$$Q(c,0,y) = 17(1-\alpha)^4c^6 + \frac{51}{2}(1-\alpha)^3c^6 + \frac{17}{4}(1-\alpha)^2c^6$$
$$+c^6 + \frac{1}{8}c^6 + \frac{21}{8}(1-\alpha)c^6 + 4(1-\alpha)^2c^3(4-c^2)y - 9(1-\alpha)c^3(4-c^2)$$
$$+24(1-\alpha)c^3(4-c^2)y + \frac{9}{2}c^3(4-c^2)y + 4(4-c^2)y + (4-c^2)4(4-c^2)y^2$$
$$+(4-c^2)(9((1-\alpha)c^2 + \frac{9}{2}c^2)(1-y^2)$$
$$= 17(1-\alpha)^4c^6 + \frac{51}{2}(1-\alpha)^3c^6 + \frac{17}{4}(1-\alpha)^2c^6$$
$$+c^6+\frac{1}{8}c^6+\frac{21}{8}(1-\alpha)c^6+(4-c^2)(4(1-\alpha)^2c^3y-9(1-\alpha)c^3+24(1-\alpha)c^3y+\frac{9}{2}c^3y$$
$$+4y + 4y^2 + (1-y^2)9(1-\alpha)c^2 + \frac{9}{2}c^2(1-y^2)) \leq 32,$$
$$0 < y < 1, 0 < \alpha < \frac{1}{2}.$$
$$\frac{\partial Q(c,0,y)}{\partial y} = (4-c^2)(4(1-\alpha)^2c^3 + 24(1-\alpha)c^3$$
$$+\frac{9}{2}c^3 + 4 + 8y(4-c^2) - 18y(1-\alpha)c^2 - 9yc^2) = 0$$
$$\frac{\partial Q(c,0,y)}{\partial c} = 102(1-\alpha)^4c^5 + 153(1-\alpha)^3c^5$$
$$+\frac{51}{2}(1-\alpha)^2c^5 + 6c^5 + \frac{3}{4}c^5 + \frac{63}{4}(1-\alpha)c^5$$
$$+(4-c^2)(4(1-\alpha)^2c^3 + 24(1-\alpha)c^3 + \frac{9}{2}c^3 + 4 + 8y - 18y(1-\alpha)c^2 - 9c^2y = 0.$$

3.2) On $c = 0$ we have

$$Q(0, x, y) = 72x^3 + 16y + 4(1 - x^2)(18x^2 + 16(1 - x^2))y^2$$
$$= 72x^3 + 16y + 8(1 - x^2)(8 + x^2)y^2 \leq 72x^3 + 16 + 8(1 - x^2)(8 + x^2)$$
$$= 72x^3 + 16 + 64 + 8x^2 - 64x^2 - 8x^4$$
$$= 80 + 72x^3 - 8x^4 - 56x^2 \leq 88,$$

which is equivalent to

$$72x^3 - 8x^4 - 56x^2 \leq 8, 0 < x < 1,$$

which is true.

3.3) On $c = 2$ we have

$$Q(2, x, y) = q_1(2, x) = 17(1 - \alpha)^4c^6 + \frac{51}{2}(1 - \alpha)^3c^6$$
$$+\frac{17}{4}(1 - \alpha)^2c^6 + c^6 + \frac{1}{8}c^6 + \frac{21}{8}(1 - \alpha)c^6$$
$$+\frac{81}{2}(1 - \alpha)xc^4 + \frac{81}{8}(1 - \alpha)xc^6 = 17(1 - \alpha)^4 64$$
$$+\frac{51}{2}(1 - \alpha)^3 64 + \frac{17}{4}(1 - \alpha)^2 64 + 64 + \frac{1}{8}64$$
$$+\frac{21}{8}(1 - \alpha)64 + \frac{81}{2}(1 - \alpha)x16 + \frac{81}{8}(1 - \alpha)x64 \leq 1144$$
$$0 < \alpha < \frac{1}{2}, 0 < x, y < 1.$$

3.4) On $x = 1$ we have

$$Q(c, 1, 1) = q_1(c, 1) + q_2(c, 1) + q_3(c, 1) \leq 1144$$

for $0 < c < 2, 0 < \alpha < \frac{1}{2}$.

3.5) On $y = 0$ we have

$$Q(c, x, 0) = q_1(c, x) + q_4(c, x) = 17(1 - \alpha)^4c^6 + \frac{51}{2}(1 - \alpha)^3c^6$$
$$+\frac{17}{4}(1 - \alpha)^2c^6 + c^6 + \frac{1}{8}c^6 + \frac{21}{8}(1 - \alpha)c^6$$
$$+\frac{81}{2}(1 - \alpha)xc^4 + \frac{81}{8}(1 - \alpha)xc^6$$
$$+(4 - c^2)[(4 - c^2)(\frac{9}{4}(1 - \alpha)^2c^2x^2$$
$$+\frac{165}{8}(1 - \alpha)c^2x^2 - 12(1 - \alpha)c^2x^3 + \frac{9}{8}x^2c^2(x^2 - 3x + 3) + \frac{9}{2}x^3$$
$$+4c^2x^2 + 4c^2x^3 + c^2x^4) + \frac{51}{2}(1 - \alpha)^3xc^4 + \frac{17}{2}(1 - \alpha)^2c^4x + 2(1 - \alpha)^2c^4x^2$$
$$+9(1 - \alpha)c^4x(x^2 - 3x + 3) + 9(1 - \alpha)c^2x^2 + 27(1 - \alpha)c^4x$$

$$+12(1-\alpha)c^4x^2+\frac{23}{8}c^4x+\frac{9}{8}c^4x(x^2-3x+3)+\frac{9}{2}c^2x^2+2c^4x^2]$$
$$+(4-c^2)(1-x^2)(9(1-\alpha)c^2+\frac{9}{2}c^2).$$

Then

$$\frac{\partial Q(c,x,0)}{\partial x}=\frac{81}{2}(1-\alpha)c^4+\frac{81}{8}(1-\alpha)c^6$$
$$+(4-c^2)[(4-c^2)(\frac{9}{2}(1-\alpha)^2c^2x+\frac{165}{4}(1-\alpha)c^2x-36(1-\alpha)c^2x^2$$
$$+\frac{9}{8}c^2(4x^3-9x^2+6x)+\frac{27}{2}x^2$$
$$+8c^2x+12c^2x^2+4c^2x^3)+\frac{51}{2}(1-\alpha)^3c^4+\frac{17}{2}(1-\alpha)^2c^4+4(1-\alpha)^2c^4x$$
$$+9(1-\alpha)c^4(3x^2-6x+3)+18(1-\alpha)c^2x+27(1-\alpha)c^4$$
$$+24(1-\alpha)c^4x+\frac{23}{8}c^4+\frac{9}{8}c^4(3x^2-6x+3)+9c^2x+4c^4x]$$
$$+(4-c^2)(-2x)(9(1-\alpha)c^2+\frac{9}{2}c^2)=0$$

Also, if we calculate

$$\frac{\partial Q(c,x,0)}{\partial c}=0,$$

we get system of equations that has no critical point on $(0,2)\times(0,1)$, and

$$\max Q(c,x,0)\leq 80$$

for $c=0, x=0$.

3.6) On $y=1$ we have

$$Q(c,x,1)=q_1(c,x)+q_2(c,x)+q_3(c,x)$$
$$\frac{\partial Q(c,x,1)}{\partial x}=\frac{81}{2}(1-\alpha)c^4+\frac{81}{8}(1-\alpha)c^6$$
$$+(4-c^2)[(4-c^2)(\frac{9}{2}(1-\alpha)^2c^2x$$
$$+\frac{165}{4}(1-\alpha)c^2x-36(1-\alpha)c^2x^2+\frac{9}{8}c^2(4x^3-9x^2+6x)+\frac{27}{2}x^2$$
$$+8c^2x+12c^2x^2+4c^2x^3)$$
$$+\frac{51}{2}(1-\alpha)^3c^4+\frac{17}{2}(1-\alpha)^2c^4+4(1-\alpha)^2c^4x$$
$$+9(1-\alpha)c^4(3x^2-6x+3)+18(1-\alpha)c^2x+27(1-\alpha)c^4$$
$$+24(1-\alpha)c^4x+\frac{23}{8}c^4+\frac{9}{8}c^4(3x^2-6x+3)+9c^2x+4c^4x]$$
$$+(4-c^2)(-2x)[(4-c^2)(\frac{9}{2}x(x+1)c+8cx+4cx^2)$$
$$+4(1-\alpha)^2c^3+9(1-\alpha)c^3(x-1)+24(1-\alpha)c^3$$

$$+24(1-\alpha)c^2x+\frac{9}{2}c^3(x+1)]$$

$$+(4-c^2)(1-x^2)[(4-c^2)(\frac{9}{2}(2x+1)c+8c+8cx)$$

$$+9(1-\alpha)c^3+24(1-\alpha)c^2+\frac{9}{2}c^3]$$

$$+(4-c^2)(-2x)[9(1-\alpha)c^2x+\frac{9}{2}c^2x+\frac{9}{2}x^2(4-c^2)$$

$$+4(4-c^2)(1-x^2)]+(4-c^2)(1-x^2)[9(1-\alpha)c^2+\frac{9}{2}c^2+9x(4-c^2)$$

$$+4(4-c^2)(-2x)]=0.$$

If we calculate

$$\frac{\partial Q(c,x,1)}{\partial c}=0,$$

we will got a system which has no solutions on $(0,2)\times(0,1)$. So $Q(c,x,1)$ has no critical point on $(0,2)\times(0,1)$ and hance,

$$\max Q(c,x,1)=80$$

for $c=0, x=0$.

4) On the vertices of Ω:

4.1) We note that

$$Q(0,0,1)=80,$$

$$Q(0,0,0)=80.$$

Summerizing 1) to 4), we get

$$\max_{(c,x,y)\in\Omega}\{Q(c,x,y)\}\le 1144.$$

Consequently, from inequality (3.3) we have that

$$\left|H_3(1)(f^{-1})\right|\le\frac{143}{18}(1-\alpha)^2.$$

□

In the end, we will show that the upper bound is sharp. We consider the function f obtained by choosing $a_2=a_3=a_5=0$ and $a_4=\frac{\sqrt{286}}{6}(1-\alpha)$. So, it follows from $H_3(1)(f)$ that

$$\left|H_3(1)(f^{-1})\right|=\frac{143}{18}(1-\alpha)^2,$$

which prove that the bound is sharp.

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