УНИВЕРЗИТЕТ
ГОЦЕ ДЕЛЧЕВ

SIRS+P model of cow mastitis

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1. *Introduction*

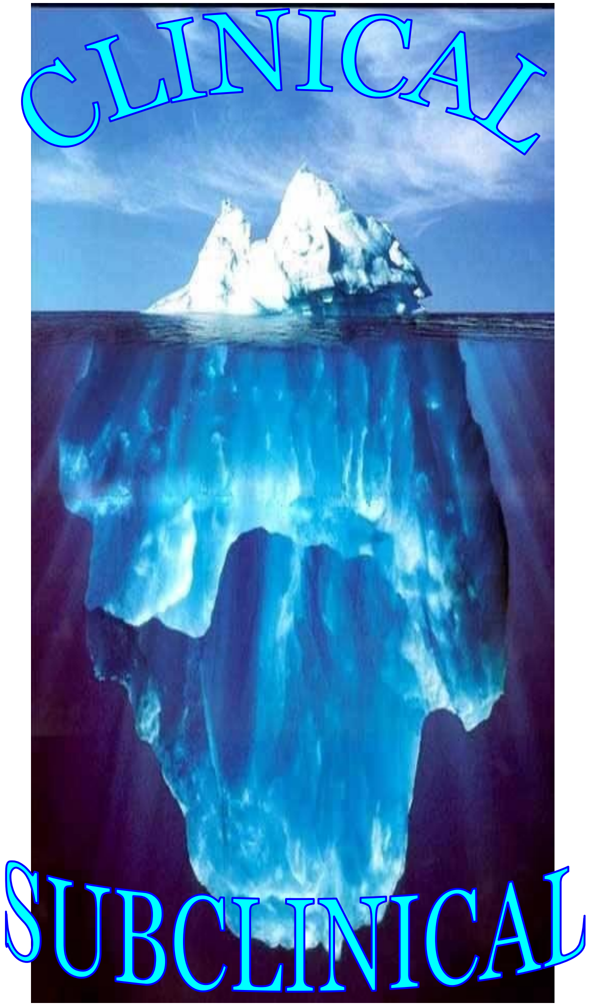
Mastitis remains one of the major disease and biggest epizootiology risk in dairy herds, causing profound economic losses to the entire milk production chain due to decreased milk production and milk quality.

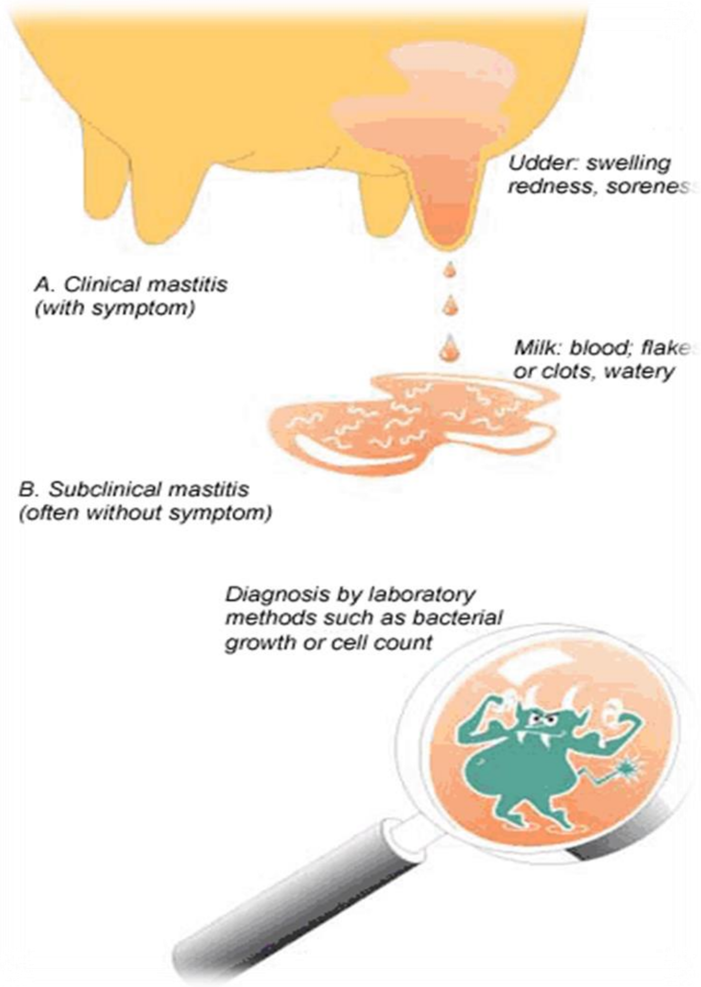
Mastitis is a complex disease which can be defined as an inflammatory reaction of the mammary gland.

Udder health disorders are always related to profound economic losses and have a big influence on dairy cows' welfare and productivity According to the effect on productivity and animal welfare, this disease is far away from the other infective diseases in dairy farms. The success of the management of mastitis in the transition period effectively determines the profitability of the cow during lactation.

Mastitis commonly occurs in cows with high milk production and has a long-lasting effect on milk yield.

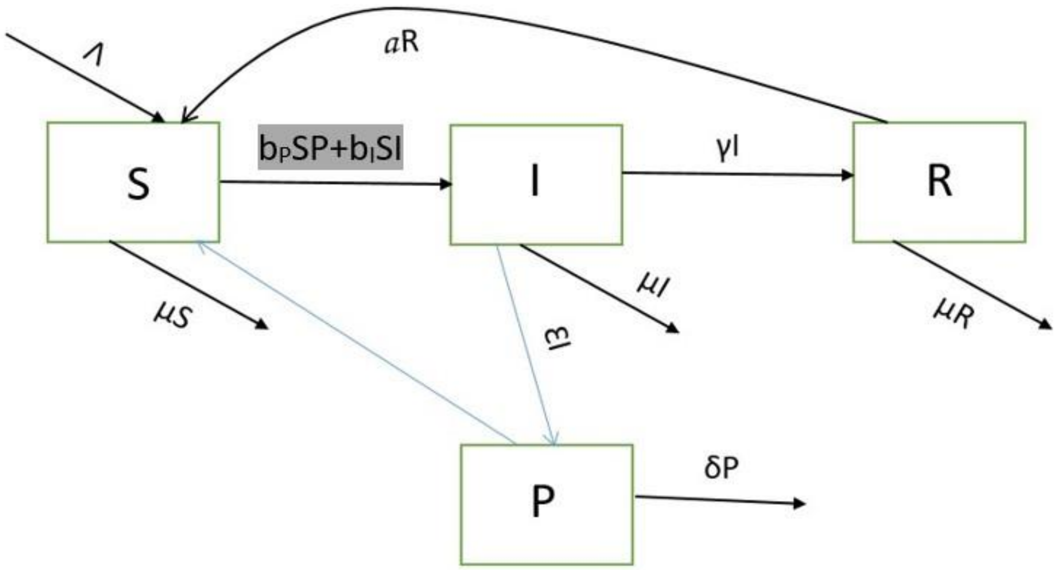
- Contagious and environmental;
- Clinical and subclinical;
- Peracute, acute, subacute and chronical;
- *mastitis catarrhalis, mastitis parenchymatosa, mastitis interstitialis;*





The model is dynamic, and the population of cows on the farm is divided into three groups: susceptible S(t), infected I(t), and recovered R(t) at the time t, so that the cow population $N=S+I+R$. Since the disease can also arise from bacteria present in the environment where the cows live, the influence of the environment is taken into account by P(t) - measuring the concentration of bacterium in the environment. On the other hand, cows do not develop immunity to this disease, so it can recur multiple times, which implies that this aspect must be considered in constructing the model.

The parameters	Meaning
Λ	a recruited rate
b_P	a transmission rate of disease to cows from the bacterium from the environment
b_I	a transmission rate of disease from cow to cow
γ	recovery rate
ε	a shedding rate of the bacterium by infectious cows
δ	a removal rate of the bacterium
μ	a removal rate of cows for some reason (coming out the cow from lactation or taking out the cow from the farm by reason)
α	disease recurrence rate



The differential equations of model are

$$\frac{dS}{dt} = \Lambda - b_P SP - b_I SI - \mu S + \alpha R$$
$$\frac{dI}{dt} = b_P SP + b_I SI - (\mu + \gamma) I$$
$$\frac{dR}{dt} = \gamma I - (\mu + \alpha) R$$
$$\frac{dP}{dt} = \varepsilon I - \delta P$$

with the initial values $S(0) = S_0 > 0, I(0) = I_0 > 0, R(0) = R_0 > 0, P(0) = P_0 > 0.$

The bounded region of the model is given by Theorem 1:

Theorem 1. Assume the system of differential Equation holds, then the feasible solution set for initial conditions remains in the bounded region

$$\Omega = \{(S, I, R, P) \,|\, 0 < N \leq \frac{\Lambda}{\mu}\}$$

The stability of the model is given by Theorem 2:

Theorem 2. Let $\left(S, I, R, P\right) \in \mathbb{R}^4,\ S(t) \geq S_0, I(t) \geq I_0, R(t) \geq R_0, P(t) \geq P_0$ then the solution set $\left\{\left(S(t), I(t), R(t), P(t)\right)\right\}$ of the model is positive for all $t \geq 0.$

The equilibrium points for the model are found when every equation of the system is equalized to zero.

$$\begin{aligned} &\Lambda - b_P SP - b_I SI - \mu S + \alpha R = 0 \\ &b_P SP + b_I SI - (\mu + \gamma) I = 0 \\ &\gamma I - (\mu + \alpha) R = 0 \\ &\varepsilon I - \delta P = 0 \end{aligned}$$

The equilibrium points are the disease-free equilibrium point (DFE) and the endemic equilibrium point (EE).

The disease-free equilibrium point (DFE) is

$$X^0 = (S^0, I^0, R^0, P^0) = (\frac{\Lambda}{\mu}, 0, 0, 0).$$

The basic reproduction number for the model is obtained by using the next-generation matrix method and it has a form:

$$\Re_0 = \frac{\Lambda}{\mu} \cdot \frac{b_I \delta + b_P \varepsilon}{\delta(\gamma + \mu)}$$

The endemic equilibrium point (EE) is

$$X^* = (S^*, I^*, R^*, P^*) = \left(\frac{\Lambda}{\mu \Re_0}, \frac{\Lambda(\mu+\alpha)}{(\mu+\alpha)(\mu+\gamma)-\alpha\gamma}\cdot\left(1-\frac{1}{\Re_0}\right), \frac{\Lambda\gamma}{(\mu+\alpha)(\mu+\gamma)-\alpha\gamma}\cdot\left(1-\frac{1}{\Re_0}\right), \frac{\Lambda\varepsilon(\mu+\alpha)}{\delta[(\mu+\alpha)(\mu+\gamma)-\alpha\gamma]}\cdot\left(1-\frac{1}{\Re_0}\right)\right)$$

About the local stability of equilibria, two theorems are proved:

Theorem 3. The point DFE X^0 is locally stable for $\Re_0 < 1$, locally unstable for $\Re_0 > 1$.

Theorem 4. The point EE X^* is locally stable for $\Re_0 > 1$, locally unstable for $\Re_0 < 1$.

Remark. In proving those two theorems, the Routh–Hurwitz stability criterion is used as a necessary and sufficient condition for the stability of a dynamical system.

Also, about the global stability of equilibria, two theorems are proved.

Theorem 5. The point DFE $X^0 = (\hat{X}, 0, 0)$ is globally stable in Ω for $\Re_0 < 1$, if the conditions of Castillo-Chavez are satisfied.

Explanation. The model will be written in this form:

$$\frac{dX}{dt} = F(X, Y)$$

$$\frac{dY}{dt} = G(X, Y), \quad G(X, 0) = 0$$

where $X = (S, R) \in R^2, \; Y = (I, P) \in R^2$ are uninfected and infected system state $F(X, Y)$ and $G(X, Y)$ with $I = 0, P = 0.$

The equilibrium of the reduced system $\frac{dX}{dt} = F(X, 0)$ is $\hat{X} = (\frac{\Lambda}{\mu}, 0).$

The conditions of Castillo-Chavez:

$H_1 : For \; \frac{dX}{dt} = F(\hat{X}, 0), \hat{X}$ is globally stable;

$$H_2 : G(X, Y) = D_Y \, G(\hat{X}, 0) Y - \hat{G}(X, Y), \; \hat{G}(X, Y) \geq 0 \; for \; all \; (X, Y) \in \Omega.$$

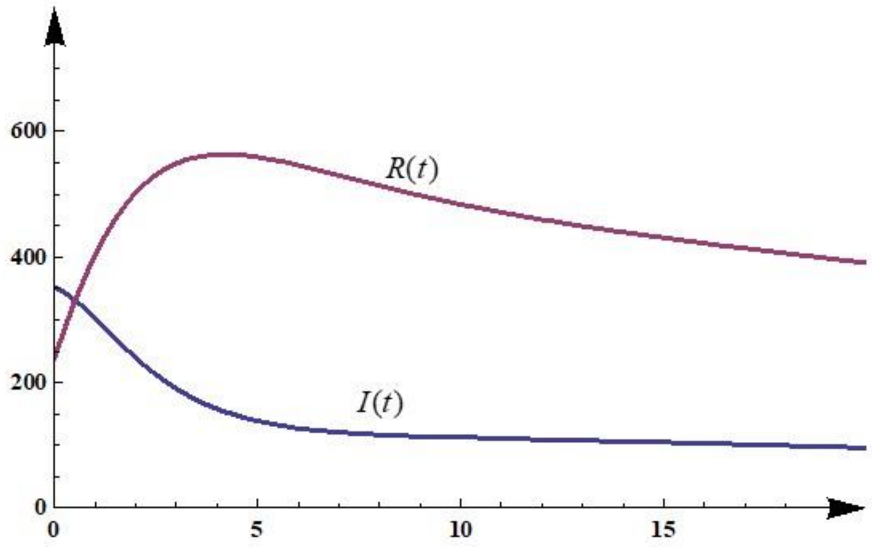
Theorem 6. The point EE X^* is globally stable in Ω for $\Re_0 > 1$.

Explanation. The Lyapunov function V(t) is constructed to prove this theorem. According to LaSalle, the point EE is globally stable in Ω if $\frac{dV}{dt} \leq 0$.

2. Case Study

For our farm:

1. The total population of the cows in the farm is N=1031 cows, with the initial values S(0)=439, I(0)=352, R(0)=240;
2. The parameters are ⍰=0.1863, bp= 0.0003041, bI= 0.0008964, γ= 0.6824, ε= 0.031746, δ= 0.105263;
3. If the percentage of cows entering the farm is the same as those leaving the farm, then the basic reproductive number will be $\Re_0$= 0.867198<1.
4. So with these parameters, we cannot have an epidemic, and the disease on the farm can be controlled. This means that mastitis cannot cause significant damage to the farmers.



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