Solving Lorenz System – Part One

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In this work we consider a way solving Lorenz system with series. The coefficients of the series produce a system of difference equations. Solving each equation of this system, a new system of difference equations is produced.

Lorenz system

$$\frac{dx}{dt} = \sigma(y - x)$$
$$\frac{dy}{dt} = x(r - z) - y$$
$$\frac{dz}{dt} = xy - bz$$

depends on three parameters *σ, r, b,* so its solution x[t]=(x[t], y[t],z[t]) depends on these three parameters (σ, r, b).

The solutions of the system are series:

$$x = a_0 + a_1 t + a_2 \frac{t^2}{2!} + \ldots + a_n \frac{t^n}{n!} + \ldots = \sum_{n=0}^{\infty} a_n \frac{t^n}{n!}$$

$$y = b_0 + b_1 t + b_2 \frac{t^2}{2!} + \ldots + b_n \frac{t^n}{n!} + \ldots = \sum_{n=0}^{\infty} b_n \frac{t^n}{n!}$$

$$z = c_0 + c_1 t + c_2 \frac{t^2}{2!} + \ldots + c_n \frac{t^n}{n!} + \ldots = \sum_{n=0}^{\infty} c_n \frac{t^n}{n!}$$

The coefficients a_n, b_n, c_n of the series produce the following system of difference equations:

$$a_n = \sigma\left(b_{n-1} - a_{n-1}\right)$$

$$b_n = (r - c_0)a_{n-1} - b_{n-1} - a_0 c_{n-1} - \sum_{i=1}^{n-2} \binom{n-1}{i} a_i \cdot c_{n-i-1}$$

$$c_n = b_0 a_{n-1} + a_0 b_{n-1} - b c_{n-1} + \sum_{i=1}^{n-2} \binom{n-1}{i} a_i \cdot b_{n-i-1}$$

The system of difference equations has form

$$a_n = a_n\left(\sigma, a_{n-1}, b_{n-1}\right)$$

$$b_n = b_n\left(r, a_0, a_1, \ldots a_{n-1}, b_{n-1}, c_0, c_1, \ldots c_{n-1}, n\right)$$

$$c_n = c_n\left(b, a_0, a_1, \ldots, a_{n-1}, b_0, b_1, \ldots, b_{n-1}, c_{n-1}, n\right)$$

for $n \in N$.

The coefficient a_n will be represented as

$$a_n = \varphi_k^n \cdot a_{n-k} + \psi_k^n \cdot b_{n-k} + \xi_k^n \cdot c_{n-k} +$$

$$+ \sum_{m=0}^{k-1} \sum_{i=k-m}^{n-k-1} \tau_k^n(i,m) a_i \cdot b_{n-i-m-1} + \sum_{m=0}^{k-1} \sum_{i=k-m}^{n-k-1} \pi_k^n(i,m) a_i \cdot c_{n-i-m-1} \qquad (1)$$

for fixed $n \in N$.

In the representation (1) for k=0 , the initial values

$$\varphi_0^n = 1, \psi_0^n = 0, \xi_0^n = 0, \tau_0^n = 0, \pi_0^n = 0.$$

The equation for a_n produce a new system of difference equations.

The system is

$$\varphi_k^n = -\sigma \cdot \varphi_{k-1}^n + \left(r - c_0\right) \cdot \psi_{k-1}^n + b_0 \cdot \xi_{k-1}^n - \sum_{i=1}^{k-1} c_i \cdot \binom{n-k+i}{n-k} \cdot \psi_{k-1-i}^n +$$

$$+ \sum_{i=1}^{k-1} b_i \cdot \binom{n-k+i}{n-k} \cdot \xi_{k-1-i}^n$$

$$\psi_k^n = \sigma \cdot \varphi_{k-1}^n - \psi_{k-1}^n + a_0 \cdot \xi_{k-1}^n + \sum_{i=1}^{k-1} a_i \cdot \binom{n-k+i}{i} \cdot \xi_{k-1-i}^n$$

$$\xi_k^n = -a_0 \cdot \psi_{k-1}^n - b \cdot \xi_{k-1}^n - \sum_{i=1}^{k-1} a_i \cdot \binom{n-k+i}{i} \cdot \psi_{k-1-i}^n$$

$$\tau_k^n = \binom{n-1-m}{i} \xi_m^n$$

$$\pi_k^n = -\binom{n-1-m}{i} \psi_m^n$$

This system of difference equations has form

$$\varphi_k^n = \varphi_k^n\left(\sigma, r, c_0, c_1, \ldots, c_{k-1}, b_0, b_1, \ldots, b_{k-1}, \varphi_{k-1}^n, \psi_{k-1}^n, \ldots, \psi_0^n, \xi_{k-1}^n, \ldots, \xi_0^n, k\right)$$

$$\psi_k^n = \psi_k^n\left(\sigma, a_0, a_1, \ldots, a_{k-1}, \varphi_{k-1}^n, \psi_{k-1}^n, \xi_{k-1}^n, \ldots, \xi_0^n, k\right)$$

$$\xi_k^n = \xi_k^n\left(b, a_0, a_1, \ldots, a_{k-1}, \varphi_{k-1}^n, \psi_{k-1}^n, \ldots, \psi_0^n, \xi_{k-1}^n, k\right)$$

$$\tau_k^n = \tau_k^n\left(a_1, \ldots, a_{n-k-1}, b_1, \ldots, b_{n-k-1}, c_1, \ldots, c_{n-k-1},, \psi_{k-1}^n, \ldots, \psi_0^n, k\right)$$

for fixed $n \in N$ and $k \in N$.

The coefficient b_n will be represented as

$$b_n = \overline{\varphi}_k^n \cdot a_{n-k} + \overline{\psi}_k^n \cdot b_{n-k} + \overline{\xi}_k^n \cdot c_{n-k} +$$

$$+ \sum_{m=0}^{k-1} \sum_{i=k-m}^{n-k-1} \overline{\tau}_k^n (i,m) a_i \cdot b_{n-i-m-1} + \sum_{m=0}^{k-1} \sum_{i=k-tm}^{n-k-1} \overline{\pi}_k^n (i,m) a_i \cdot c_{n-i-m-1} \qquad (2)$$

for fixed $n \in N$.

In the representation (2) for k=0 , the initial values

$$\overline{\varphi}_0^n = 0, \overline{\psi}_0^n = 1, \overline{\xi}_0^n = 0, \overline{\tau}_0^n = 0, \overline{\pi}_0^n = 0.$$

The equation for b_n produce a new system of difference equations.

The system is

$$\bar{\varphi}_k^n = -\sigma \cdot \bar{\varphi}_{k-1}^n + \left(r - c_0\right) \cdot \bar{\psi}_{k-1}^n + b_0 \cdot \bar{\xi}_{k-1}^n - \sum_{i=1}^{k-1} c_i \cdot \binom{n-k+i}{n-k} \cdot \bar{\psi}_{k-1-i}^n +$$

$$+ \sum_{i=1}^{k-1} b_i \cdot \binom{n-k+i}{n-k} \cdot \bar{\xi}_{k-1-i}^n$$

$$\psi_k^n = \sigma \cdot \bar{\varphi}_{k-1}^n - \bar{\psi}_{k-1}^n + a_0 \cdot \bar{\xi}_{k-1}^n + \sum_{i=1}^{k-1} a_i \cdot \binom{n-k+i}{i} \cdot \bar{\xi}_{k-1-i}^n$$

$$\bar{\xi}_k^n = -a_0 \cdot \bar{\psi}_{k-1}^n - b \cdot \bar{\xi}_{k-1}^n - \sum_{i=1}^{k-1} a_i \cdot \binom{n-k+i}{i} \cdot \bar{\psi}_{k-1-i}^n$$

$$\bar{\tau}_k^n = \binom{n-m-1}{i} \bar{\xi}_m^n$$

$$\bar{\pi}_k^n = -\binom{n-m-1}{i} \bar{\psi}_m^n$$

This system of difference equations has form

$$
\begin{aligned}
\bar{\varphi}_k^n &= \bar{\varphi}_k^n\left(\sigma, r, c_0, c_1, \dots, c_{k-1}, b_0, b_1, \dots, b_{k-1}, \bar{\varphi}_{k-1}^n, \bar{\psi}_{k-1}^n, \dots, \bar{\psi}_0^n, \bar{\xi}_{k-1}^n, \dots, \bar{\xi}_0^n, k\right) \\
\bar{\psi}_k^n &= \bar{\psi}_k^n\left(\sigma, a_0, a_1, \dots, a_{k-1}, \bar{\varphi}_{k-1}^n, \bar{\psi}_{k-1}^n, \bar{\xi}_{k-1}^n, \dots, \bar{\xi}_0^n, k\right) \\
\bar{\xi}_k^n &= \bar{\xi}_k^n\left(b, a_0, a_1, \dots, a_{k-1}, \bar{\varphi}_{k-1}^n, \bar{\psi}_{k-1}^n, \dots, \bar{\psi}_0^n, \bar{\xi}_{k-1}^n, k\right) \\
\bar{\tau}_k^n &= \bar{\tau}_k^n\left(a_1, \dots, a_{n-k-1}, b_1, \dots, b_{n-k-1}, c_1, \dots, c_{n-k-1},, \bar{\psi}_{k-1}^n, \dots, \bar{\psi}_0^n, k\right)
\end{aligned}
$$

for fixed $n \in N$ and $k \in N$.

The coefficient c_n will be represented as

$$c_n = \overline{\overline{\varphi}}_k^n \cdot a_{n-k} + \overline{\overline{\psi}}_k^n \cdot b_{n-k} + \overline{\overline{\xi}}_k^n \cdot c_{n-k} + \qquad (3)$$

$$+ \sum_{m=0}^{k-1} \sum_{i=k-m}^{n-k-1} \overline{\overline{\tau}}_k^n (i,m) a_i b_{n-i-m-1} + \sum_{m=0}^{k-1} \sum_{i=k-m}^{n-k-1} \overline{\overline{\pi}}_k^n (i,m) a_i c_{n-i-m-1}$$

for fixed $n \in N$.

In the representation (3) for k=0 , the initial values

$$\overline{\overline{\varphi}}_0^n = 0, \overline{\overline{\psi}}_0^n = 0, \overline{\overline{\xi}}_0^n = 1, \overline{\overline{\tau}}_0^n = 0, \overline{\overline{\pi}}_0^n = 0.$$

The equation for c_n produce a new system of difference equations.

The system is

$$\bar{\bar{\varphi}}_k^n = -\sigma\cdot\bar{\bar{\varphi}}_{k-1}^n + \left(r - c_0\right)\cdot\bar{\bar{\psi}}_{k-1}^n + b_0\cdot\bar{\bar{\xi}}_{k-1}^n - \sum_{i=1}^{k-1} c_i\cdot\binom{n-k+i}{n-k}\cdot\bar{\bar{\psi}}_{k-1-i}^n +$$

$$+\sum_{i=1}^{k-1} b_i\cdot\binom{n-k+i}{n-k}\cdot\bar{\bar{\xi}}_{k-1-i}^n$$

$$\bar{\bar{\psi}}_k^n = \sigma\cdot\bar{\bar{\varphi}}_{k-1}^n - \bar{\bar{\psi}}_{k-1}^n + a_0\cdot\bar{\bar{\xi}}_{k-1}^n + \sum_{i=1}^{k-1} a_i\cdot\binom{n-k+i}{i}\cdot\bar{\bar{\xi}}_{k-1-i}^n$$

$$\bar{\bar{\xi}}_k^n = -a_0\cdot\bar{\bar{\psi}}_{k-1}^n - b\cdot\bar{\bar{\xi}}_{k-1}^n - \sum_{i=1}^{k-1} a_i\cdot\binom{n-k+i}{i}\cdot\bar{\bar{\psi}}_{k-1-i}^n$$

$$\bar{\bar{\tau}}_k^n = \binom{n-m-1}{i}\bar{\bar{\xi}}_m^n$$

$$\bar{\bar{\pi}}_k^n = -\binom{n-m-1}{i}\bar{\bar{\psi}}_m^n$$

This system of difference equations has form

$$\overline{\overline{\varphi}}_k^n = \overline{\overline{\varphi}}_k^n\left(\sigma, r, c_0, c_1, \ldots, c_{k-1}, b_0, b_1, \ldots, b_{k-1}, \overline{\overline{\varphi}}_{k-1}^n, \overline{\overline{\psi}}_{k-1}^n, \ldots, \overline{\overline{\psi}}_0^n, \overline{\overline{\xi}}_{k-1}^n, \ldots, \overline{\overline{\xi}}_0^n, k\right)$$

$$\overline{\overline{\psi}}_k^n = \overline{\overline{\psi}}_k^n\left(\sigma, a_0, a_1, \ldots, a_{k-1}, \overline{\overline{\varphi}}_{k-1}^n, \overline{\overline{\psi}}_{k-1}^n, \overline{\overline{\xi}}_{k-1}^n, \ldots, \overline{\overline{\xi}}_0^n, k\right)$$

$$\overline{\overline{\xi}}_k^n = \overline{\overline{\xi}}_k^n\left(b, a_0, a_1, \ldots, a_{k-1}, \overline{\overline{\varphi}}_{k-1}^n, \overline{\overline{\psi}}_{k-1}^n, \ldots, \overline{\overline{\psi}}_0^n, \overline{\overline{\xi}}_{k-1}^n, k\right)$$

$$\overline{\overline{\tau}}_k^n = \overline{\overline{\tau}}_k^n\left(a_1, \ldots, a_{n-k-1}, b_1, \ldots, b_{n-k-1}, c_1, \ldots, c_{n-k-1}, , \overline{\overline{\psi}}_{k-1}^n, \ldots, \overline{\overline{\psi}}_0^n, k\right)$$

for fixed $n \in N$ and $k \in N.$

Conclusion

The systems of difference equations for coefficients a_n, b_n, c_n have identical forms, but theirs initial values are different.