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HETEROGENEOUS AGENT (HA) MODELS AND TWO - ASSET HANK MODEL: REVIEW OF SOME COMPUTATIONAL MODELS

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Abstract

These are Heterogeneous Agent HA models with continuum of agents in discrete and continuous time with aggregate uncertainty i.e. these are Krusell-Smith (1997) (1998) types of models. The two asset HANK (Heterogeneous Agent New Keynesian) model uses discrete cosine transform (DCT) technique and sequential equilibrium with recursive individual planning as a sequence of discretized Bellman equations. Some of the models are placed in a two-sector economy as Kiyotaki, Moore (1997) and one sector growth model studied by Huggett (1997). Huggett (1993) is a HACT (heterogeneous agent model in continuous time) model that describes solution for a simple continuous time heterogeneous agent economy. Some of the models are computed with MIT shock which is an unexpected shock that hits an economy at its steady state, leading to a transition path back towards the economy's steady state. Heterogeneous Agent New Keynesian (HANK) models are emerging as leading frameworks to study the impact of monetary and fiscal policy on the macroeconomy. Central idea of this paper is the notion that representative agent models were wrong turn for modern macroeconomics especially for general equilibrium model (some individuals are some are not liquidity constrained) and that central problems of

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macroeconomics cannot arise in representative agent models (debt, bankruptcy, asymmetric information).Results from models prove the importance of distributions of household income, wealth, savings decisions in order HA and HANK models to construct optimal monetary policy as opposed to RANK (Representative Agent New Keynesian) model "lean against the wind " monetary policy response.

Keywords: HANK model, MIT shock, Huggett economy, sequential equilibrium

1. Introduction

Heterogeneity is pervasive in macroeconomic data, for instance households vary in their income, wealth, and consumption while firms vary in productivity, and investment, see Winberry (2018). The heterogeneity of agents is relevant, and it could provide answers for the welfare questions that are crucial in macroeconomics. In a way it is a critique of representative agents' models[3]. Models with heterogenous agents (HA) have become dominant workhorse in macroeconomics since seminal works by: Bewley (1986), Hopenhayn (1992), Huggett (1993), Aiyagari (1994)[4]. Lucas (1987) showed that for standard preferences, aggregate fluctuations have a very small impact on the welfare of a representative consumer. Lucas (1987) estimates that magnitude of the costs of business cycles on total consumption to be remarkably small 0.1%. And this estimation is based on a assumption that of a perfect insurance of idiosyncratic risk. Lucas (1987) instigated growing literature and studies such as Imrohoroğlu (1989) was to "examine whether the magnitude of the costs of business cycles in economies with incomplete insurance differs significantly from the cost estimates found in an environment with perfect insurance". A rapidly growing literature has

[3] Arrow (1951) and Arrow, Debreu (1954), proved that competitive equilibrium in Arrow-Debreu economy is Pareto optimal and discovered class of convex Arrow-Debreu economies for which competitive equilibria always exist. In the case of incomplete, see Geanakoplos (1990)markets this equilibrium may (will) not be efficient see Geanakoplos (1986) or the will be suboptimal constrained.

[4] More complete review of this literature could be read in Heathcote et al.(2009)

emerged which studies how this micro heterogeneity shapes[5] our understanding of business cycle fluctuations, see (Auclert (2017), Berger and Vavra (2015), Kaplan, Moll and Violante (2016), and on the firm side: Bachmann, Caballero and Engel (2013), Clementi and Palazzo (2016), Ottonello and Winberry (2017)). Although a neoclassical synthesis dominated quantitative macroeconomics for many decades, heterogeneous agent models were always present and taken seriously as early as the multiple class models of Kalecki (2016) that emphasized heterogeneous marginal propensities to consume and their implications for fiscal policy. Important components of Friedman (1956) were his empirical and theoretical analyses of differences in marginal propensities to consume across classes of consumers who faced stochastic processes of non-financial income with different mixtures of permanent and temporary components, see Sargent (2023). The DSGE (Dynamic stochastic general equilibrium)model proposed by Christiano, Eichenbaum, and Evans (2005) and later estimated by Smets and Wouters (2003) using Bayesian techniques, is currently considered to be a benchmark richly specified DSGE model for a closed economy, see Kolasa et al. (2012).These models may be called Friedmanite DSGE models, since they assume that monetary policy has no effect on real variables such as: output and real interest rate in the long run. But due to sticky prices and wages, monetary policy matters in the short run. The name HANK model was coined by Kaplan et al. (2018). They developed HANK model on the household side with Aiyagari-Huggett-Imrohoroğlu incomplete market model, with one important modification: as in Kaplan, Violante (2014), households can save in two assets, a low-return liquid asset and a high-return illiquid asset that is subject to a transaction cost. The most important lesson we have learned from HANK models is about the transmission mechanism of monetary policy. If we start from canonical representative agent model, there, a cut in the nominal rate induces a rise in consumption expenditures through intertemporal substitution through the aggregate Euler equation. Such rise in expenditures, in turn, leads to an expansion in the demand for labor and, because of nominal rigidities, to an additional round of increase in

[5] Models of heterogeneous agents have become widespread in macroeconomics, at least since Krusell and Smith (1997), Krusell and Smith (1998) developed the first widely applicable algorithm to solve them in an environment of aggregate risk.

expenditures. The size of these indirect general equilibrium effects linked to the Keynesian multiplier are proportional to the magnitude of the aggregate marginal propensity to consume which, in RANK models, is tiny (equal to the discount rate), see Violante (2021)[6]. From the RANK prospect in order to understand the impact of a change in the policy rate on aggregate consumption, all CB needs are two ingredients: expected inflation to convert the nominal rate under control into the real one, and the aggregate intertemporal elasticity of substitution which measures the sensitivity of aggregate consumption to the real rate. From the HANK perspective in order to estimate the aggregate consumption response, one needs a full picture of the joint distribution of marginal propensities to consume, income composition, and the various elements of household balance sheets. In general, households are unequally exposed to aggregate shocks. In HANK models, this heterogeneous sensitivity is a source of amplification of shocks to the extent that income is redistributed from low MPC to high MPC households ((Auclert, 2017); (Bilbiie, 2020), (Patterson, 2021), Slacalek, Tristani, Violante (2020)). The importance of indirect equilibrium channels means that the transmission of monetary policy is crucially mediated by all those mechanisms that contribute to price formation in goods, inputs, credit, housing and financial markets. It is then essential for a central bank to have a deep comprehension of market structure, market frictions as well as of those institutions, see Violante (2020). Household heterogeneity and market incompleteness also alter the strength of their propagation through the macroeconomy[7]. CGE models were also suffering critique for their reliance

[6] Thus, somewhat paradoxically, the channel by which monetary policy affects aggregate output in the standard New Keynesian model differs markedly from the ideas typically associated with John Maynard Keynes i.e. the equilibrium spending multiplier see Keynes (1936). Most undergraduate macroeconomics textbooks argue that the multiplier effect of government purchases is larger than that of transfers, see Keynes (1936),Mankiw (2006). In the standard Keynesian framework, government spending on useless public works has a larger multiplier effect than spending on government transfer payments does, see Ono,Y.(2011).

[7] First, through redistribution channel: exposure to aggregate fluctuations is highest at the**m**extremes of the distribution, second in HANK models, this precautionary saving channel amplifies the negative aggregate shock because the cut in expenditures to build the additional buffer stock of saving piles up onto the initial reduction of aggregate demand, see Acharya and Dogra (2020). And the fiscal policy channel. When the monetary authority cuts

of "representative agent" and aggregation procedures. If the representative agents' model is estimated with data from heterogenous agents' economy under different policy regimes important parameters vary considerably. For instance, the aggregate labor supply elasticity, which was/is often recognized as a crucial parameter for fiscal policy analysis, depends on cross-sectional distribution of reservation wages ,which distribution is in turn a function of fiscal policy regime, see Auerbach, Kotlikof (1987); and Judd (1987); Prescott (2004),and Chang,Kim,Schorfheide(2013).As per Bernanke (2015), monetary policy is a blunt tool which certainly affects the distribution of income and wealth, although whether its net effect is to increase or reduce inequality is not clear, see Violante (2020). HANK models are useful because they offer a structure to shed light on the interplay between stabilization and redistribution. For instance, in case of positive mark-up shock RANK models propose rise in the nominal rate to cut aggregate demand and tame inflation. An increase in mark-ups reduces the labor share in favor of the owners of capital. A rise in the policy rate which stifles aggregate demand would further hurt workers. HANK model in such a case prescribes opposite i.e. toward a cut in the nominal rate in order to foster the aggregate demand for labor and redistribute income back to workers. This paper will review and solve following HA-DSGE models: Winberry (2018) , Huggett (1997), Huggett (1993), Kiyotaki and Moore (1997), and Two-asset HANK model: Bayer,Luetticke (2020).Last model uses sequential equilibrium by Reiter (2002). These models will be solved in MATLAB or Python programing languages.

the interest rate, borrowers gain. Governments are net borrowers and, as a result, they have extra resources in their budget. The extent of this inflow depends largely on the maturity structure of debt and on how rates at other horizons respond to a change in the short rate ,see Auclert, Rognlie and Straub, (2020). The magnitude of fiscal policy effect depends on the cross-sectional covariance between the change in income and the marginal propensity to consume.

2. Toolbox for Solving and Estimating Heterogeneous Agent Macro Models Winberry (2018)

Firms $j \in [0, 1]$ produce output y_{jt} according to production function:

$$y_{jt} = e^{zt} e^{\varepsilon_{jt}} k_{jt}^{\theta} n_{jt}^{\nu};\, \theta + \nu < 1 \quad (1)$$

Where in previous z_t is an aggregate productivity shock,ε_{jt} is an idiosyncratic productivity shock, k_{jt} is capital,n_{jt} is labor,θ is the elasticity of output with respect to capital, and ν with respect to labor. Aggregate productivity shock is same for all firms and follows AR(1) process:

$$z_{t+1} = \rho_z z_t + \sigma_z \omega_{t+1}^{z}\, ; \omega_{t+1}^{z} \sim \mathcal{N}(0, 1) \quad (2)$$

Gross investment i_{jt} yields:

$$k_{jt+1} = (1-\delta) k_{jt} + i_{jt}\, , \wedge \frac{i_{jt}}{k_{jt}} \notin [-a, a] \quad (3)$$

Where parameter a is around zero investment within which firms do not incur fixed costs but if $\frac{i_{jt}}{k_{jt}} \notin [-a, a]$ the firms must pay fixed adjustment costs[8] ξ_{jt} in units of labor. Households do have utility function:

$$\mathbb{E} \sum_{t=0}^{\infty} \beta^{t} \left[\frac{C_t^{1-\sigma} - 1}{1-\sigma} - \chi \frac{N_t^{1+\alpha}}{1+\alpha}\right] \quad (4)$$

Where β is a discount factor,σ is relative risk aversion parameter,N_t is labor supply, χ is disutility of labor supply, and α is the Frisch elasticity of labor supply[9]. Alternatively :

[8] A common model assumption used by economists to explain this 'lumpy' firm behaviour is that adjustment comes with a fixed cost; a cost that does not depend on the size of the change, but must be paid for any level of adjustment, however small.

$$u\big(c_t(j),h_t(j)\big)=\frac{c_t(j)^{1-\sigma^c}}{1-\sigma^c}\cdot\frac{h_t(j)^{1+\frac{1}{\sigma^L}}}{1+\frac{1}{\sigma^L}} \quad (5)$$

Where σ^c is the risk aversion, and σ^L is the Frisch elasticity of labor supply. $u(.)$ represents the utility increasing from consumption $c_t(j)$,and decreasing from hours worked $h_t(j)$. Welfare is the sum fo current and expected utilities:$w_t(j)=\sum_{\tau=o}^{+\infty}\beta^\tau u\big(c_{t+\tau}(j),h_{t+\tau}(j)\big)$.Following Khan, A.,Thomas, J. K. (2008), implications of household optimizations are incorporated into the firm's optimization problem by approximating the transformed value function:

$$\hat{v}(\varepsilon,k;s)=\lambda(s)max_n\{e^z e^\epsilon k^\theta n^v-w(s)n\}+E_\xi[max\{v^a(\varepsilon,k;s)-\xi\lambda(s)w(s),v^n(\varepsilon,k,s\}] \quad (6)$$

Where s is the state vector and $\lambda(s)=C(s)^{-\sigma}$ is the marginal utility of consumption in equilibrium and also:

$$\begin{aligned} v^a(\varepsilon,k,s) &= \max_{k'\in\mathbb{R}} -\lambda(s)(k'-(1-\delta)k)+\beta E[\hat{v}(\varepsilon',k';s'(z';s)|\varepsilon] \\ v^n(\varepsilon,k,s) &= \max_{k'\in[(1-\delta-a)k,(1-\delta+a)k]} -\lambda(s)(k'-(1-\delta)k)+\beta E[\hat{v}(\varepsilon',k';s'(z';s)|\varepsilon,k,s] \end{aligned} \quad (7)$$

unconstrained capital choice is $k^a(\varepsilon,k,s)$ and constrained is $k^n(\varepsilon,k,s)$,firms will pay fixed costs if $v^a(\varepsilon,k,s)-\xi\lambda(s)w(s)\geq v^n(\varepsilon,k,s)$,there is a unique threshold between these two options:

$$\tilde{\xi}(\varepsilon,k,s)=\frac{v^a(\varepsilon,k,s)-v^n(\varepsilon,k,s)}{\lambda(s)w(s)} \quad (8)$$

Where $\hat{\xi}(\varepsilon,k,s)$ is the threshold with a bounded support :$\hat{\xi}(\varepsilon,k,s)=\min\big[\max\{0,\tilde{\xi}(\varepsilon,k,s),\bar{\xi}\}\big]$.As for the equilibrium:

Definition 1:A recursive competitive equilibrium for the model is a set $\hat{v}(\varepsilon,k,s),n(\varepsilon,k,s)$,$k^a(\varepsilon,k,s);\hat{\xi}(\varepsilon,k,s),\lambda(s),w(s);s'(z';s)=\big(z';\mu'(z\mu)\big)$ such that :

i. Firm optimization takes $\lambda(s),w(s);s'(z';s)$ as given $\hat{v}(\varepsilon,k,s),n(\varepsilon,k,s),k^a(\varepsilon,k,s),k^n(\varepsilon,k,s);\ \hat{\xi}(\varepsilon,k,s)$ to solve optimization

[9] The Frisch elasticity measures the relative change of working hours to a one-percent increase in real wage, given the marginal utility of wealth λ .In the steady-state benchmark model is given as: $\frac{dh/h}{dw/w}=\frac{1-h}{h}\left(\frac{1-\eta}{\eta}\theta-1\right)^{-1}$

problem
$\widehat{v}(\varepsilon,k;s)=\lambda(s)max_n\{e^z e^\epsilon k^\theta n^v-w(s)n\}+E_\xi[max\{v^a(\varepsilon,k;s)-\xi\lambda(s)w(s),v^n(\varepsilon,k,s\}]$
and $\tilde{\xi}(\varepsilon,k,s)=\frac{v^a(\varepsilon,k,s)-v^n(\varepsilon,k,s)}{\lambda(s)w(s)}$.

ii. Household optimization $\lambda(s)=C(s)^{-\sigma}$; $C(s)=\int\left[e^z e^\varepsilon k^\theta n(\varepsilon,k,s)^v+(1-\delta)k-\left(\frac{\hat{\xi}(\varepsilon,k,s)}{\bar{\xi}}\right)k^a(\varepsilon,k,s)-\left(1-\frac{\hat{\xi}(\varepsilon,k,s)}{\bar{\xi}}\right)k^n(\varepsilon,k,s)\right]d\mu(\varepsilon,k)$
and $\int\left(n(\varepsilon,k,s)+\frac{\hat{\xi}(\varepsilon,k,s)^2}{2\bar{\xi}}\right)d\mu(\varepsilon,k)=\left(\frac{w(s)\lambda(s)}{\chi}\right)^{\frac{1}{\alpha}}$

iii. Law of motion for all feasible sets $\triangle_\varepsilon;\triangle_k$ is
$\mu'(z,\mu)(\triangle_\varepsilon\times\triangle_k)=\int\int p(\rho_\varepsilon\varepsilon+\sigma_\varepsilon\omega^\varepsilon\in\triangle_\varepsilon)d\omega^\varepsilon\times\left[\frac{\hat{\xi}(\varepsilon,k,s)}{\bar{\xi}}\right]\{k^a(\varepsilon,k,s)\in\triangle_k\}+\left(1-\frac{\hat{\xi}(\varepsilon,k,s)}{\bar{\xi}}\right)\{k^n(\varepsilon,k,s)\in\triangle_k\}d\mu(\varepsilon,k)$

iv. Law of motion for aggregate shocks is $z'=\rho_z z+\omega_z';\omega_z'\sim\mathcal{N}(0,\sigma_z)$

The PDF of the distribution of firms is given as:

$$g(\varepsilon,k)\cong g_0 exp\left\{g_1^1(\varepsilon-m_1^1)+g_1^2(k-m_1^2)+\sum_{i=2}^{n_g}\sum_{j=0}^{i}g_i^j\left[(\varepsilon-m_1^1)^{i-j}(k-m_1^2)-m_i^j\right]\right\} \tag{9}$$

Previous is following Algan et al. (2008) and n_g indexes the degree of approximation and $g_0,g_1^1,g_2^2;\{g_i^j\}_{i,j=2,0}^{n_g,i}$ are parameters and $m_1^1,m_1^2;\{m_i^j\}_{i,j=2,0}^{n_g,i}$ are centralized moments of distribution. And the moments[10] are implied by the parameters :

[10]Normalization is done by
$g(\varepsilon,k)\cong g_0 exp\left\{g_1^1(\varepsilon-m_1^1)+g_1^2(k-m_1^2)+\sum_{i=2}^{n_g}\sum_{j=0}^{i}g_i^j\left[(\varepsilon-m_1^1)^{i-j}(k-m_1^2)-m_i^j\right]\right\}=1$

$$m_1^1 = \int\int \varepsilon g(\varepsilon, k) d\varepsilon dk$$
$$m_1^2 = \int\int k g(\varepsilon, k) d\varepsilon dk \quad (10)$$
$$m_i^j = \int\int (i-j)^{i-j} (k - m_1^2)^j g(\varepsilon, k) d\varepsilon dk \; ; i = 2, \dots, n_g, j = 0, \dots, i$$

Firms value function is given as:

$$v(\varepsilon, k, z, m) \cong \sum_{i=1}^{n_\varepsilon} \sum_{j=1}^{n_k} \theta_{ij}(z, m) T_i(\varepsilon) T_j(k) \quad (11)$$

Where $T_i(\varepsilon), T_j(k)$ are Chebyshev polynominals. Chebyshev nodes can be computed as:$x_t = cos\left[\frac{\pi(t-1)}{T}\right], t = 1, \dots T$.The points $\{v_t\}_t^T = 1$ are found via transformation like this: $v_t = \frac{\bar{b}+\omega_L+(\bar{b}-\omega_L)x_t}{2}$. Chebyshev polynomials can be defined recursively as $T_0(x) = 1, T_1(x) = x, T_{n+1}(x) = 2xT_n(x) + T_{n-1}(x)$. The coefficients of these polynomials for a function $f(x)$ can be obtained by the following integral:$a_n = \frac{2}{\pi}\int_{-1}^{1} \frac{f(x)T_n(x)}{(1-x^2)^{\frac{1}{2}}} dx$.Hubbard, Kirkegaard and Paarsch imposed 5 in/equality constraints on the equilibrium bid functions[11], that are approximated by the Chebyshev polynomials of order K (Hubbard, Kirkegaard et al. 2013).Bellman equation for this problem is given as:

$$\hat{v}(\varepsilon, k_j; z, m) = \lambda(z,m) max_n\{e^z e^\epsilon k^\theta n^v - w(z,m)n\} + \lambda(z,m)(1-\delta)k + \left(\frac{\hat{\xi}(\varepsilon_i, k_j, z, m)}{\bar{\xi}}\right)$$

$$\begin{pmatrix} -\lambda(z,m)k^a(\varepsilon_i, k_j, z, m) - w(z,m)\frac{\hat{\xi}(\varepsilon_i, k_j, z, m)}{2} \\ +\beta E_{(z'|z)}\left[\int \hat{v}\left(\rho_\varepsilon \varepsilon_i + \sigma_\varepsilon \omega_\varepsilon' k^a(\varepsilon_i, k_j, z, m); z', m'(z,m)\right) p(\omega_\varepsilon') d\omega_\varepsilon'\right] \end{pmatrix}$$

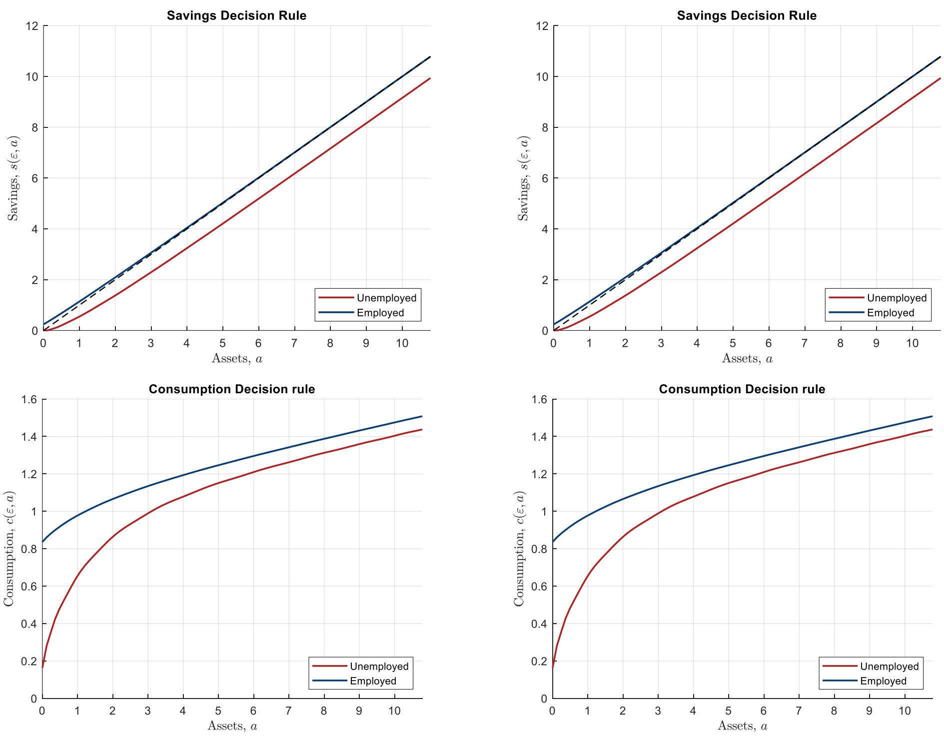
$$\left(1 - \frac{\hat{\xi}(\varepsilon_i, k_j, z, m)}{\bar{\xi}}\right) \begin{pmatrix} -\lambda(z,m)k^n(\varepsilon_i, k_j, z, m \\ +\beta E_{(z'|z)}\left[\int \hat{v}\left(\rho_\varepsilon \varepsilon_i + \sigma_\varepsilon \omega_\varepsilon' k^n(\varepsilon_i, k_j, z, m); z', m'(z,m)\right) p(\omega_\varepsilon') d\omega_\varepsilon'\right] \end{pmatrix}$$

(12)

[11] 1. $\varphi_n(v) = \omega_L$,2. $\varphi_n(\bar{b}) = \omega_H$ 3. $\sum_{m\neq n}(\bar{b} - \bar{v}) f_m(\bar{b}) \varphi_m'(v) = 1$,4. $\varphi(\omega_L) = \frac{N-1}{N}$,5. $\varphi_n(v_j - 1) \leq \varphi_n(v_j)$,for some uniform array $j = 2, \dots, J$.

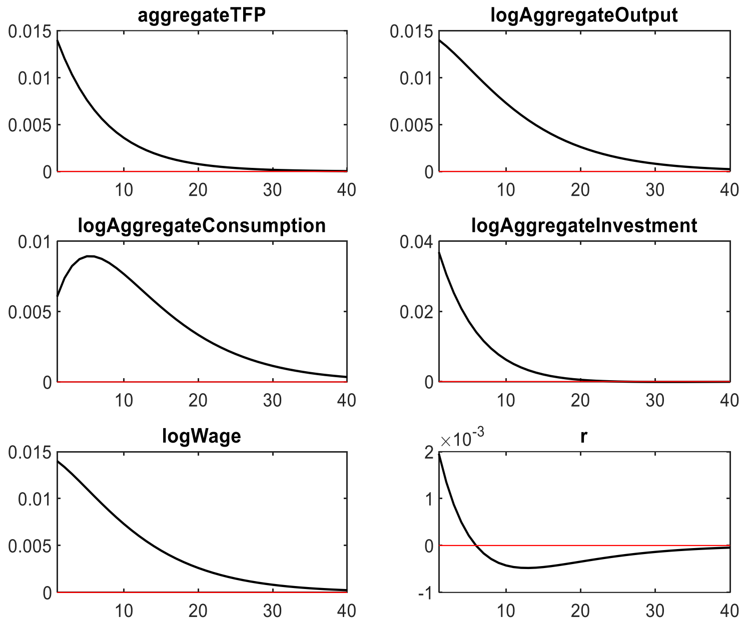
Parametrization follows A.,Thomas, J. K. (2008) and :$\beta = 0.96$ (discount factor), $\sigma = 1$ (utility curvature),$\alpha = \lim \alpha \to 0$ (inverse Frisch), χ(labor disutility) $\chi = \frac{1}{3}N$, $\upsilon = 0.64$ (labor share),$\theta = 0.256$ (capital share),$\delta = 0.085$ (capital depreciation), $\rho_z = 0.859$(Aggregate TFP AR(1)),$\sigma_z = 0.859$(Aggregate TFP AR(1)),$\bar{\xi} = 0.083$(fixed cost),$a = 0.011$(no fixed cost region), $\rho_\varepsilon = 0.859$(Idiosyncratic TFP AR(1)), $\sigma_\varepsilon = 0.859$(Idiosyncratic TFP AR(1)).Next follows computation results of previous model.

Figure 1. Computes and analyzes steady state with no aggregate shocks.



Source: Authors own calculations based on a code available at:
https://github.com/JohannesPfeifer/winberryAlgorithmCodes

Figure 2 first order approximation of aggregate dynamics



Source: Authors own calculations based on a code available at: https://github.com/JohannesPfeifer/winberryAlgorithmCodes

3. G(Global)DSGE: A Toolbox for Solving DSGE Models with Global Methods: Steady States and Transition Paths in Heterogeneous Agent Models as per Huggett (1997)

This model draws on a seminal work by Huggett (1997). This model is included in GDSGE toolbox that solves non-linear Dynamic Stochastic General Equilibrium (DSGE) models with a global method based on the Simultaneous Transition and Policy Function Iteration (STPFI) algorithm introduced in Cao, Luo, and Nie (2023).Decision problem is characterized by and Euler equation:

$$u'(c_t) = \beta \mathbb{E}_t[(1 + r_{t+1})u'(c_{t+1})] + \lambda_t \quad (13)$$

Where λ_t is Lagrange multiplier on the borrowing constraint, and the complementary-slackness condition,$\lambda k_{t+1} = 0$ with state transition functions. For the one sector growth model studied by Huggett (1997),steady state equilibrium object is aggregate capital stock and the transition path aggregate equilibrium object is the time sequence of the aggregate capital stock. Now about Euler equation here.

3.1 Euler equation

Here following lemma applies (see Achdou et al.,2022)

Lemma 2: The consumption and savings policy functions $c_j(a)$ and $s_j(a)$ for $j = 1, 2..$ corresponding to HJB equation : $\rho v_j(a) = \max_c u(c) + v_j'(a)(y_j + ra - c) + \lambda_j\left(v_{-j}(a) - v_j(a)\right)$ which is maximized at : $0 = -\frac{d}{da}[s_j(a)g_j(a)] - \lambda_j g_j(a) + \lambda_{-j}g_{-j}(a)$ is given as:

$$(\rho - r)u'\left(c_j(a)\right) = u''\left(c_j(a)\right)c_j'(a)s_j(a) + \lambda_j(u'\left(c_{-j}(a)\right) - u'\left(c_j(a)\right)$$
$$s_j(a) = y_j + ra - c_j(a)$$

(14)

Proof:

differentiate$\rho v_j(a) = \max_c u(c) + v_j'(a)(y_j + ra - c) + \lambda_j\left(v_{-j}(a) - v_j(a)\right)$ with respect to a and use that $v_j'(a) = u'\left(c_j(a)\right)$ and hence $v_j''(a) = u''\left(c_j(a)\right)c_j'(a)$ ■

The differential equation:

$$(\rho - r)u'\left(c_j(a)\right) = u''\left(c_j(a)\right)c_j'(a)s_j(a) + \lambda_j(u'\left(c_{-j}(a)\right) - u'\left(c_j(a)\right)$$
$$s_j(a) = y_j + ra - c_j(a)$$

is and Euler equation, the right hand side$(\rho - r)u'\left(c_j(a)\right)$ is expected change of marginal utility of consumption $\frac{\mathbb{E}_t[du'(c_j(a_t))]}{dt}$.This uses Ito's formula to Poisson process:

$$\mathbb{E}_t[du'(c_j(a_t)] = \left[u''(c_j(a_t)c_j'(a_t)s_j(a_t) + \lambda_j\left(u'(c_{-j}(a_t)\right) - u'\left(c_j(a_t)\right)\right]dt \quad (15)$$

So, this equation

$$(\rho - r)u'\left(c_j(a)\right) = u''\left(c_j(a)\right)c_j'(a)s_j(a) + \lambda_j(u'\left(c_{-j}(a)\right) - u'\left(c_j(a)\right)$$

$$s_j(a) = y_j + ra - c_j(a)$$

can be written in more standard form:

$$\frac{\mathbb{E}_t[du'(c_j(a_t)]}{dt} = (\rho - r)dt \quad (16)$$

Generalized Euler equations when W is defined recursively $W_{t+1} = R(W_t - c_t)$ previously we should define that $\sum_{t=1}^{\infty} R^{-t+1}c_t \leq W_1$ and gross interest rate $R = r + 1$; are given in the following form:

$$u'(c_t) = R\left[\beta\delta\left(\frac{\partial c_{t+1}(W_{t+1})}{\partial W_{t+1}}\right) + \delta\left(1 - \frac{\partial c_{t+1}(W_{t+1})}{\partial W_{t+1}}\right)\right]u'(c_{t+1}) \quad (17)$$

Where $\left[\beta\delta\left(\frac{\partial c_{t+1}(W_{t+1})}{\partial W_{t+1}}\right) + \delta\left(1 - \frac{\partial c_{t+1}(W_{t+1})}{\partial W_{t+1}}\right)\right]$ is the effective discount factor, also $c_{t+1}(W_{t+1})$ represents the optimal consumption choice. With uncertainty Euler equation will become:

$$u'(c_t) = \beta R\hat{E}\,[u'(c_{t+1})|I_t]$$

Where $\hat{E}\,[u'(c_{t+1})|I_t]$ represents the agents, expectation given the information set I_t.Now, taking 2nd order approx.to marginal utility in $t + 1$ around c_t gives:

$$\hat{E}\left[\frac{c_{t+1} - c_t}{c_t}|I_t\right] = \sigma_t\left(1 - (\beta R)^{-1}\right) + \frac{1}{2}\phi_t\hat{E}[(c_{t+1} - c_t)^2|I_t] \quad (18)$$

Where $\phi_t = -\frac{c_t u'''(c_t)}{u''(c_t)}$ is a coefficient of relative prudence (see Dynan (1991), expected consumption growth that rises with the real interest rate and falls with impatience. In continuous time previous would be:

$$\frac{\dot{c}_t}{c_t} = \sigma_t(r - \rho) \quad (19)$$

Where $\sigma_t = -\frac{u'(c_t)}{c_t u''(c_t)}$; and $c_{t+\Delta t} = c_t + \Delta c_t, \beta = 1 - \rho\Delta t$; $\Delta t \to 0$ .Now, let's consider that $\varepsilon_{t+1} = u'(c_{t+1}) - (\beta R)^{-1}u'(c_t)$ as in Hall (1978) . It was pointed by Hall (1978) that this equation $u'(c_t) = \beta R\hat{E}\,[u'(c_{t+1})|I_t]$ implies that $\hat{E}[\varepsilon_{t+1}z_t|I_t] = z_t\hat{E}[\varepsilon_{t+1}|I_t]$ for any $z_t \in I_t$.

3.2 Back to Huggett (1997): sequential equilibrium

This sequence is sequential equilibrium: $c_t(k,e), \lambda_t(k,e), k'_t(k,e)$

$$
\begin{gathered}
c_t(k,e)^{-\sigma} = \beta(1+r_{t+1})E[c_{t+1}(k'_t(k,e),e')^{-\sigma}|e] + \lambda(k,e) \\
k'_t(k,e)\lambda_t(k,e) = 0, \lambda_t(k,e) \geq 0; k'_t(k,e) \geq 0 \\
c_t(k,e) + k'_t(k,e) = k(1+r_t) + w_t e
\end{gathered} \quad (20)
$$

Market clearing conditions are:

$$
\begin{gathered}
r_t = \alpha k_t^{\alpha-1} - \delta \\
w_t = (1-\alpha)K_t^{\alpha} \\
K_t = \int k\phi_t(dk, de)
\end{gathered} \quad (21)
$$

ϕ_t are consistent with the transitions implied by policy functions and exogenous shocks[12].The first feature of this model is that there is a continuum of agents in the economy experiencing idiosyncratic labor endowment shocks. The endowment uncertainty is such that there is uncertainty for individual agents but no uncertainty over the aggregate labor endowment. Next on three plots we will presents the results for this model.

Figure 3 Huggett model transition path (GDSGE)

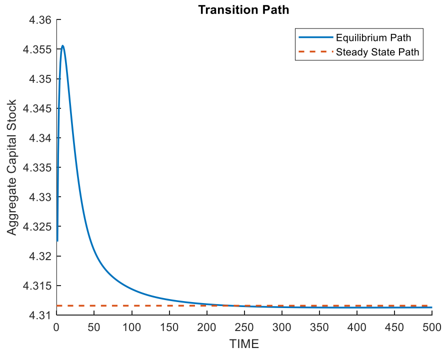
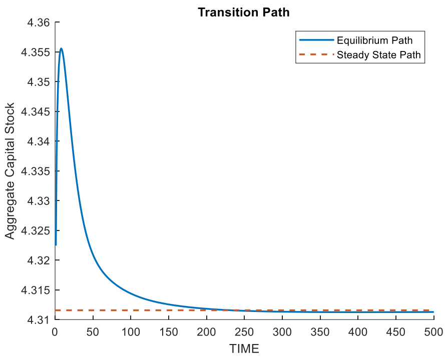


Figure 4Transition path Huggett model



[12] A steady-state equilibrium is a sequential equilibrium with time-invariant equilibrium objects.

Figure 5 Huggett model savings decision

4.Huggett (1993) economy and credit crunch in Huggett economy per (Gustavo Mellior)

As in Achdou et al.(2022), two functions v_1, v_2 at I discrete points in the space dimension a_i , $i = 1, \dots, I$. Equispaced grids are denoted by Δa_ias the distance by the grid points, and shot hand notation used is $v_{i,j} \equiv v_j(a_i)$ and so on. Backward difference approximation is given as:

$$\begin{cases} v_j'(a_i) \approx \frac{v_{i+1,j}-v_{i,j}}{\Delta a} \equiv v_{i,j,F}' \\ v_j'(a_i) \approx \frac{v_{i+1,j}-v_{i-1,j}}{\Delta a} \equiv v_{i,j,B}' \end{cases} \quad (22)$$

Two basic equations to explain Huggett economy are :

$$\begin{cases} \rho v_1(a) = \max_c u(c) + v_1'(a)(z_1 + ra - c) + \lambda_1\big(v_2(a) - v_1(a)\big) \\ \rho v_2(a) = \max_c u(c) + v_2'(a)(z_2 + ra - c) + \lambda_2\big(v_1(a) - v_2(a)\big) \end{cases} \quad (23)$$

Where $\rho \geq 0$ represents the discount factor for the future consumption c_t (Individuals have standard preferences over utility flows), a represents wealth in form of bonds that evolve according to : $\dot{a} = y_t + r_t a_t - c_t$ where

y_t is the income of individual, which is endowment of economy's final good, and r_t represents the interest rate. Equilibrium in this Huggett (1993) economy is given as:

$$\int_{\underline{a}}^{\infty} ag_1(a,t)da + \int_{\underline{a}}^{\infty} ag_2(a,t)da = B \qquad (24)$$

Where in previous expression $0 \leq B \leq \infty$ and when $B = 0$ that means that bonds are zero net supply. So the finite difference method approx. to $\begin{cases} \rho v_1(a) = \max_c u(c) + v_1'(a)(z_1 + ra - c) + \lambda_1(v_2(a) - v_1(a)) \\ \rho v_2(a) = \max_c u(c) + v_2'(a)(z_2 + ra - c) + \lambda_2(v_1(a) - v_2(a)) \end{cases}$ is given as:

$$\begin{gathered} \rho v_{i,j} = u(c_{i,j}) + v_{i,j}'(z_j + ra_i + c_{i,j}) + \lambda_j(v_{i,-j} - v_{i,j}), j = 1,2 \\ c_{i,j} = (u')^{-1}(v_{i,j}') \end{gathered} \qquad (25)$$

Here will be presented two main approaches for solving Huggett (1993) model and problem numerically. This part is based on : Rouwenhorst (1995) and also in Kopecky ,Suen (2010).Now, e_t is a two-state Markov process $e_t \in \{e_l, e_h\}$ and that transition probabilities are given as following:

$$\Gamma = \begin{bmatrix} \gamma & 1-\gamma \\ 1-\gamma & \gamma \end{bmatrix} \qquad (26)$$

Where in previous autocorrelation is given as: $2\gamma - 1$. Now about the two-state Euler equation process:

1. In the low earnings state:

$$(e_l + ra - h(a,e_l))^{-\sigma} = \beta r\{\gamma[e_l + rh(a,e_l) - h(h(a,e_l),e_l]^{-\sigma} + (1-\gamma)[e_h + rh(a,e_l) - h(h(a,e_l),e_h]^{-\sigma}\} \qquad (27)$$

2. In the high earning state:

$$e_h + ra - h(a,e_h)^{-\sigma} = \beta r\{\gamma[e_h + rh(a,e_h) - h(h(a,e_h),e_h]^{-\sigma} + (1-\gamma)[e_l + rh(a,e_h) - h(h(a,e_h),e_l]^{-\sigma}\} \qquad (28)$$

With exogenous grid savings function is approximated as:

$[e_l + ra_k - y]^{-\sigma} = \beta r\left\{\gamma[e_l + ry - \hat{h}(y, \theta^{l,0}]^{-\sigma} + (1-\gamma)[e_h + ry - \hat{h}(y, \theta^{h,0}]\right\}$

(29)

Where in previous: $\theta^0 = [\theta^{l,0}, \theta^{h,0}]$ these are vectors, and$\hat{h}(a^k, \theta) = \theta_k, \forall k$ and $\theta^l, \theta^h \in \mathbb{R}^N$. This MATLAB code and its algorithm explanation are due to Gustavo Mellior (Kent Uni.2016) and those files can be found at Benjamin Moll web site: https://benjaminmoll.com/codes/ [13]. To solve for y we have:

$$y = \frac{a^m f_k^l(a^{m+1};\theta^0) - a^{m+1} f_k^l(a^m;\theta^0)}{f_k^l(a^{m+1};\theta^0) - f_k^l(a^m;\theta^0)} \quad (30)$$

Now:

$f_k^l(y, \theta^0) = [e^l + ra - y]^{-\sigma} = \beta r\left\{\gamma[e^l + ry - \hat{h}(y; \theta^{l,0}]^{-\sigma} + (1-\gamma)[e_h + ry - \hat{h}(y, \theta^{h,0}]\wedge - \sigma\right\}$

(31)

Where $f_k^l(y, \theta^0)$ is the FOC function at $a = a^k; e = e^l$.With the method of endogenous grid we have:

$$a = \frac{[A^l(y,\theta^0)]^{-\frac{1}{\sigma}} - e^l + y}{r} \quad (32)$$

Savings grid is $y = \{y^1, y^2, \dots, y^n\}$ and $y^1 = \underline{a}; y^n = \bar{a}$. We can define here:

$A^l(y; \theta^0) = \beta r\left\{\gamma[e^l + ry - \hat{h}(y; \theta^{l,0}]^{-\sigma} + (1-\gamma)[e_h + ry - \hat{h}(y, \theta^{h,0}]^{-\sigma}\right\} = 0$

(33)

[13] In Bernanke et al.(1991) credit crunch is defined as:"..We define a bank credit crunch as a significant leftward shift in the supply curve for bank loans, holding constant both the safe real interest rate and the quality of potential borrower.."A credit crunch (credit squeeze, credit tightening; credit crisis) is a sudden reduction in the general availability of loans or a sudden tightening of the conditions required to obtain a loan from banks. A credit crunch generally involves a reduction in the availability of credit independent of a rise in official interest rates.

And for $A^l(y;\theta^0)$ we have:

$$A^h(y;\theta^0) = \beta r\left\{\gamma\left[e^h + ry - \hat{h}(y;\theta^{h,0}\right]^{-\sigma} + (1-\gamma)\left[e_l + ry - \hat{h}(y,\theta^{l,0}\right]^{-\sigma}\right\} = 0 \quad (34)$$

Now, the density function can be discretized:

$$f_{i,j}^1 = f_{i,j}^1 + \pi(\lambda^j|\lambda^k)\frac{a_{i+1} - h(a_l,\lambda^k)}{a_{i+1} - a^i} f_{l,k}^0$$
$$f_{i+1,j}^1 = f_{i+1,j}^1 + \pi(\lambda^j|\lambda^k)\frac{h(a_l,\lambda^k) - a^i}{a_{i+1} - a^i} f_{l,k}^0 \quad (35)$$
$$\sum_{i=1}^n \sum_{j=1}^m f_{i,j}^1 = 1$$

Economy is described in the text as before, and when credit crunch occurs a household with assets $\underline{a}_{t_0}$ will find itself below the new borrowing limit, and it will reduce consumption by Δa and it moves closer to a_T. And in this example $\underline{a}_{t_0} + 3\Delta a = \underline{a}_T$

$$\Delta a = s_1(\underline{a}_{t_0}) = z_1 + r(\underline{a}_T - 3\Delta a) - c_1(\underline{a}_{t_0});$$
$$\Delta a = s_1(\underline{a}_{t_0} + \Delta a) = z_1 + r(\underline{a}_T - 2\Delta a) - c_1(\underline{a}_{t_0} + \Delta a);$$
$$\Delta a = s_1(\underline{a}_{t_0} + 2\Delta a) = z_1 + r(\underline{a}_T - \Delta a) - c_1(\underline{a}_{t_0} + 2\Delta a);$$
$$0 = s_1(\underline{a}_{t_0} + 3\Delta a) = z_1 + r(\underline{a}_T) - c_1(\underline{a}_{t_0} + 3\Delta a) \quad (36)$$

When credit crunch occurs previous will be modified to reduce borrowing limit by $3\Delta a$

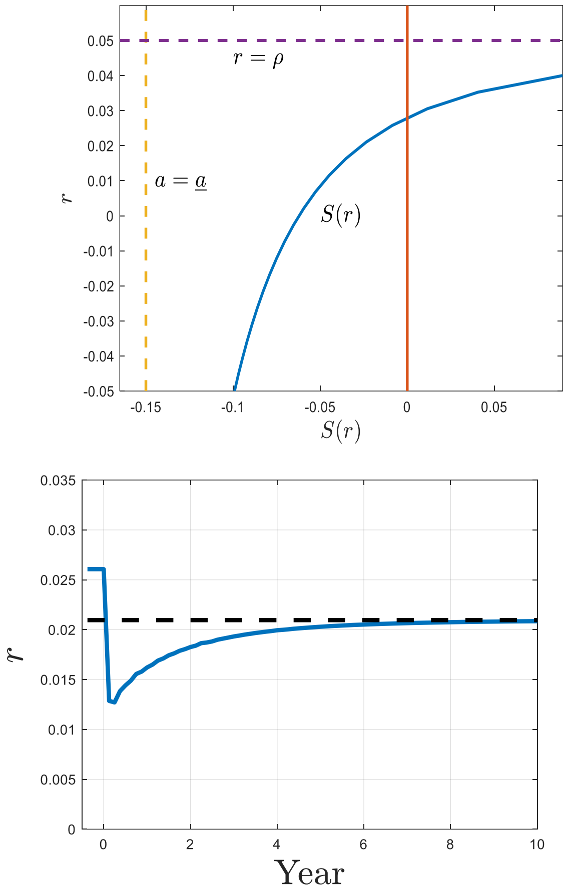
$$\bar{c}_{1,1} - z_1 + r\underline{a}_{t_0} - \Delta a; \qquad \bar{c}_{2,1} = z_1 + r(\underline{a}_{t_0} + \Delta a) - \Delta a;$$
$$\bar{c}_{3,1} = z_1 + r(\underline{a}_{t_0} + 2\Delta a) - \Delta a; \qquad \bar{c}_{\underline{a}_T}, 1 = z_1 + r\underline{a}_T; \qquad \bar{v}'_{i,j} = u'(\bar{c}'_{i,j});$$

$$v_{i,j} = v'_{i,j}\mathbb{1}_{s_F>0} + \mathrm{v}'_{i,j}\mathbb{1}_{(s_B<0)} + \bar{\mathrm{v}}'_{i,j}\mathbb{1}_{s_B>0>s_F} \quad (37)$$

In this example parameters of the model are : $s = 2; \rho = 0.05;$
$z_1 = 0.12;\ z_2 = 0.25;\ z = [z_1, z_2];\ la_1 = 1.15;\ la_2 = 1,$
$la = [la_1, la_2]; r_0 = 0.03; r_{min} = 0.001;\ r_{\max} = 0.045;$

$I = 800$; Equilibrium Found, Interest rate =0.0261. In the next photo equilibrium interest rate and supply of borrowings (loans) priced by that rate are depicted (Huggett model Credit crunch interest rate Response of r(t) after a credit crunch from $\underline{a}_{t_0} = -0.1692$ to $\underline{a}_T = -0.15$):

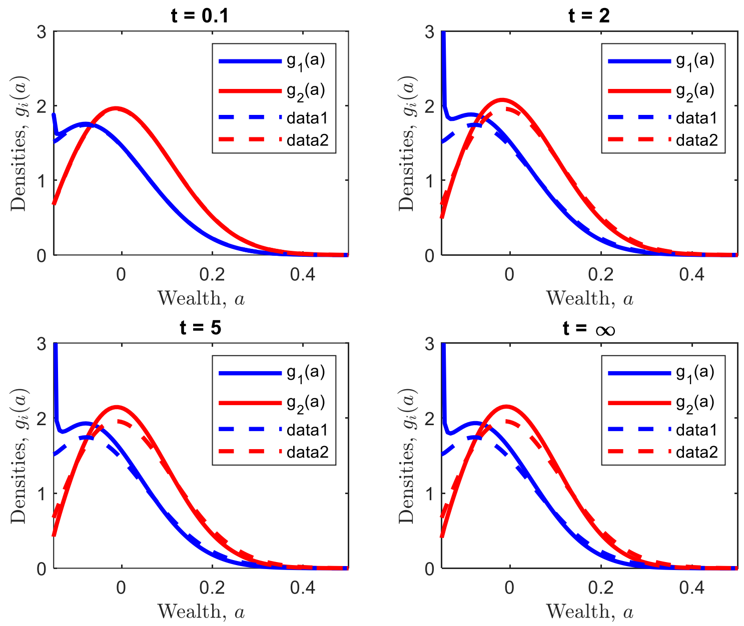
Figure 6 equilibrium interest rate



Source: Author's calculations based on code available at: https://benjaminmoll.com/codes/

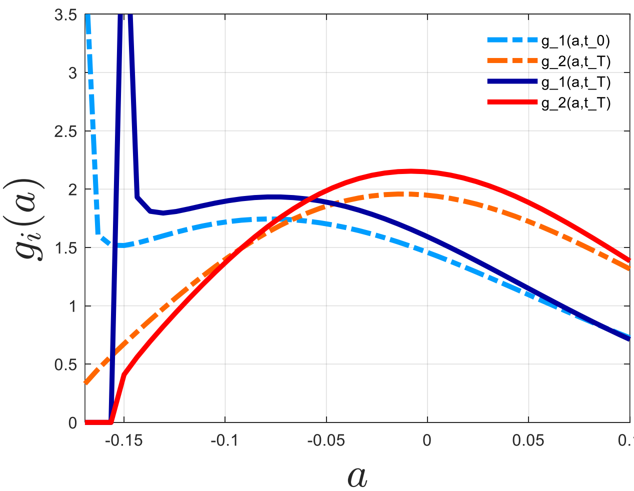
Next we will plot Huggett transitional model of credit crunch and distribution of wealth in this economy.

Figure 7 Huggett model credit crunch transition model



Source: Author's calculations based on code available at: https://benjaminmoll.com/codes/

Figure 8. Initial and terminal distributions of wealth



Source: Author's calculations based on code available at: https://benjaminmoll.com/codes/

5. G (Global) DSGE: A toolbox for Solving DSGE Models with Global Methods: Kiyotaki and Moore (1997), Credit Cycles

This is a model due to Kiyotaki and Moore (1997) paper simply entitled credit cycles. In their seminal, Kiyotaki and Moore (1997) put forth a model of credit cycles in which movements in asset prices interacts with the real side of the economy and produce amplified and persistent effects of shocks to the economy. The original model is relatively simple with risk-neutral agents and one-time unanticipated MIT shocks. Next, we will explain MIT shock.

5.1 MIT shock

Boppart et al. (2018) writes that “MIT shock” is defined as: *“An “MIT shock” is an unexpected shock that hits an economy at its steady state, leading to a transition path back towards the economy’s steady state……”*.Mukoyama (2021) also follows Boppart et al. (2018) definition:”…. the probability of the shock is considered zero, and no prior (contingent) arrangement is possible for the occurrence of the MIT shock”…..The dynamic analysis that was using exogenous shocks or policy changes has been used in the literature with the earlier examples including: Abel, Blanchard (1983), Auerbach, Kotlikoff (1983), and Judd (1985).And more recent examples being: Boppart et al. (2018), Kaplan et al. (2018), Boar ,Midrigan (2020), Guerrieri et al. (2020).

5.2 Back to Kiyotaki, Moore (1997)

Here the economy consists of two production sectors, farming and gathering, with the population of each sector normalized to one. The farmers are more productive but are less patient than the gatherers and thus they tend to borrow from the gatherers in equilibrium. Farmers now maximize:

$$\max_{\{x_t, k_t, b_t\}} \mathbb{E}_0 \left[\sum_t \beta^t \frac{(x_t)^{1-\sigma}}{1-\sigma} \right] \tag{38}$$

Subject to budget constraint: $x_t + q_t k_t + \frac{b_t}{R_t} = y_t + q_t k_{t-1} + b_{t-1}$and the production function is :$y_t = A_t(a + c)k_{t-1}$.The value of land holding in previous is: $q_t k_{t-1}$ and the bond holding is: b_{t-1}. The aggregate TFP shock A_t follows a Markov process. Resources are allocated among consumption x_t, as well as land and bond holdings in the next period. Portion c of the output is non-tradable and must be consumed:$x_t \geq A_t c k_{t-1}$, only remaining portion a is tradable. Collateral constraint for the agent is given as:

$$b_t + \theta q_{1-t} k_t \geq 0 \tag{39}$$

Where q_{1-t} is the lowest possible land price in the next period. And $\theta \leq 1$. Gatherer solves:

$$\max_{\{x'_t, k'_t, b'_t\}} \mathbb{E}_0 \left[\sum_t \beta'^t \frac{(x'_t)^{1-\sigma}}{1-\sigma} \right] \tag{40}$$

Subject to budget constraint: $x'_t + q'_t k'_t + \frac{b'_t}{R_t} = y'_t + q'_t k'_{t-1} + b'_{t-1}$ and concave production function: $y'_t = \underline{A}_t (k'_{t-1})^{\alpha}$. We assume here that: $\underline{A}_t = \delta A_t; \delta \leq 1$.The multiplier on farmer's budget constraint is $\beta^t \lambda_t$ and on the tradability constraint $\beta^t \eta_t$ and on the collateral constraint $\beta^t \mu_t$ the following first order conditions and complementary-slackness conditions are necessary and sufficient for optimality:

$$\begin{gathered}
(x_t)^{-\sigma} - \lambda_t + \eta_t = 0 \\
\eta_t (x_t - c k_{t-1}) = 0 \\
-q_t \lambda_t + \theta q_{1-t} \mu_t + \beta \mathbb{E}[(q_{t+1} + a + c)\lambda_{t+1} - c\eta_{t+1}] = 0 \\
-\frac{1}{R_t} \lambda_t + \mu_t + \beta \mathbb{E}[\lambda_{t+1}] = 0 \\
\mu_t (q \theta q_{1-t} k_t + b_t) = 0 \\
\beta' \mathbb{E}\left[\left(\frac{q_{t+1} + \alpha (k'_t)^{\alpha-1}}{q_t} \right) \left(\frac{x'_{t+1}}{x'_t} \right)^{-\sigma} \right] = 1 \\
\beta' R_t \mathbb{E}_t \left[\left(\frac{x'_{t+1}}{x'_t} \right)^{-\sigma} \right] = 1
\end{gathered} \tag{41}$$

The total land supply is fixed $\bar{K}$ and the market clearing conditions are given as:

$$\begin{gathered}
b_t + b'_t = 0 \\
k_t + k'_t = \bar{K} \\
x_t + x'_t = Y_t = A_t (a + c) k_{t-1} + \underline{A}_t (k'_{t-1})^{\alpha}
\end{gathered} \tag{42}$$

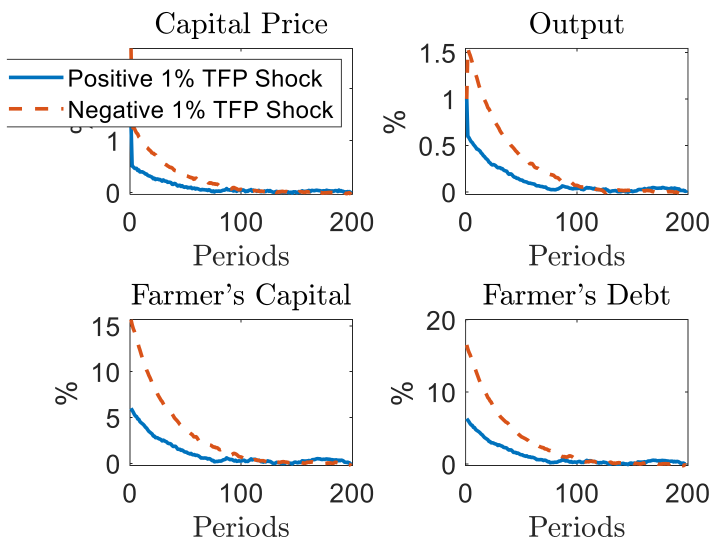
We define recursive equilibrium[14] over two endogenous state variables. The first-one is the farmers' land-holding $\boldsymbol{k_{t-1}}$. The second one is the farmers' financial wealth share defined as:

$$\omega_t = \frac{q_t k_{t-1} + b_{t-1}}{q_t \bar{K}} \qquad (43)$$

In the numerical exercise TFP is i.i.d and $A_t \in \{0.99, 1.0, 1.01\}$ with probability of 1/3 for each state.

In the ergodic distribution, the probabilities for binding collateral constraint conditional on the three values of $A_t \in \{0.92, 0.80, 0.77\}$.Next =, results of simulation of Kiyotaki, Morre (1997) are graphically presented.

Figure 9 Kiyotaki-Moore (1997) The IRFs after positive and negative 1 percent TFP shocks



Source: Author's own calculations based on code available at: https://www.gdsge.com/example/KM1997/KM1997.html#equation-eq-budget-farmer

[14] It has been widely used in exploring a wide variety of economic issues including business-cycle fluctuations, monetary and fiscal policy, trade related phenomena, and regularities in asset price co-movements. This is the equilibrium associated with dynamic programs that represent the decision problem when agents must distinguish between aggregate and individual state variables.

6. Two-asset HANK model: Bayer,Luetticke (2020) method for solving Heterogenous DSGE

Bayer, Luetticke (2020) propose a method for solving Heterogeneous Agent DSGE models that uses fast tools originally employed for image and video compression to speed up a variant of the solution methods proposed by Michael Reiter,see(Perturbation with our reduction for HANK models[15]) . Bayer,Luetticke (2020)method has the following broad characteristics: The model is formulated and solved in discrete time .Solution begins by calculation of the steady-state equilibrium with no aggregate shocks. Both the representation of the consumer's problem and the description of the distribution are subjected to a form of "dimensionality reduction”. This means finding a way to represent these objects efficiently using fewer points. “Dimensionality reduction" of the consumer's decision problem is performed before any further analysis is done. This involves finding a representation of the policy functions using some class of "basis functions”. Dimensional reduction of the joint distribution is accomplished using a "copula”. The method approximates the business-cycle-induced deviations of the individual policy functions from those that characterize the riskless steady-state. This is done using the same basis functions originally optimized to match the steady-state individual (micro) policy function. Now will set up the dynamic recursive problem here. Consider a household problem in presence of aggregate and idiosyncratic risk $\boldsymbol{S_t}$ measures the (exogenous) aggregate state (e.g., levels of productivity and unemployment),$\boldsymbol{s_{it}}$ records agent 's idiosyncratic state (exogenous and endogenous, e.g. employment or assets).$\boldsymbol{\mu_t}$ is the distribution over s at date t (e.g., the wealth distribution).$\boldsymbol{P_t}$ is the pricing kernel. It captures the info about the aggregate state that the consumer needs to know in order to behave optimally.$\boldsymbol{\Gamma}$ defines the budget set. This delimits the set of feasible choices $\boldsymbol{x}$ that the agent can make. The Bellman equation for this problem is:

$$v(s_{it}, S_t, \mu_t) = \max_{x \in \Gamma(s_a, P_t)} u(s_{it}, x) + \beta \mathbb{E} v(s_{it+1}(x, s_{it}), S_{t+1}, \mu_{t+1} \quad (44)$$

And the corresponding Euler equation is:

[15] code available at: https://github.com/econ-ark/BayerLuetticke

$$u'\left(s_{it}, x(s_{it}, S_t, \mu_t)\right) = \beta \mathbb{E}_t R\left(S_t, S_{t+1}, \mu_t, \mu_{t+1}{}'_u\left(s_{i,t+1}, S_{t+1}, \mu_{t+1}\right)\right) \quad (45)$$

Now when solving for steady state, first we need to discretize the state space by representing the nodes of the discretization in a set of vectors. Such vectors will be represented by an overbar: $\bar{s}_{it}$;$\bar{c}_{it}$. The optimal policy $c(s_{it}; P(\mu))$induces flow utility u_c whose discretization is a vector $\bar{u}_{\bar{c}}$. Π is like an expectation operator. In steady-state discretized Bellman equation is given as:

$$\bar{v} = \bar{u} + \beta \Pi_{\bar{c}} \bar{v}$$
$$\bar{\mu} = \bar{\mu} \Pi_{\bar{c}} \quad (46)$$
$$d\bar{\mu} = d\bar{\mu} \Pi_{\bar{c}}$$

We will define an approximate equilibrium in which: $\bar{c}$ is the vector that defines a linear interpolating policy function c at the state nodes given P and v ,.v is a linear interpolation of $\bar{v}$,$\bar{v}$; $d\bar{\mu}$ solve the approximated Bellman equation subject to the steady-state constraint Markets clear joint requirement on $\bar{c}$, μ , and P ; denoted as $\Phi(\bar{c}, \mu, P) = 0$).

6.1 *Sequential equilibrium (Reiter (2002)*

A 'sequential equilibrium with recursive individual planning' is: A sequence of discretized Bellman equations, such that

$$vt = \bar{u}_{P_t} + \beta_{\Pi c_t} v_{t+1} \quad (47)$$

Previous holds for policy c_t which optimizes with respect to v_{t+1} and P_t and a sequence of "histograms" $d\mu$ (discretized distributions), such that:$d\mu_{t+1} = d\mu_t \Pi_{c_t}$ which holds given the policy h_t , that is optimal given P_t , v_{t+1}.That is, given a histogram describing the distribution in period t , $d\mu_t$, next period's histogram is determined by the transition matrix. Prices, distribution, and policies lead to market clearing. The large system above can be transformed into much smaller system:

$$F\left(\{d\mu_t^1, \ldots, d\mu_t^n\}, S_t, \{d\mu_{t+1}^1, \ldots, d\mu_{t+1}^n\} S_{t+1}, \theta_t, P_t, \theta_{(t+1)}, P_{t+1}\right) =$$

$$\begin{bmatrix} d\bar{C}(\mu_t^1, \ldots, \overline{\mu}_t^n) - d\bar{C}(\mu_t^1, \ldots, \overline{\mu_t}^n)\Pi_{c_t} \\ dct\left[idct(\tilde{\Theta}(\theta_t) - (\bar{u}_{c_t} + \beta\Pi_{c_t} idct\left(\tilde{\Theta}(\theta_{t+1})\right)\right] \\ S_{t+1} - c(S_t, d\mu_t) \\ \Phi(c_t, d\mu_t, P_t, S_t) \end{bmatrix} \quad (48)$$

6.2 *Back to Bayer, Luetticke (2020)*

Let $\bar{\Theta} = dct(\bar{v})$be the coefficients obtained from the DCT[16] of the value function in steady-state. Define an index set I that contains the x percent largest (i.e. most important) elements from $\bar{\Theta}$.Let θ be a sparse vector with non-zero entries only for elements $i \in I$ .Define:

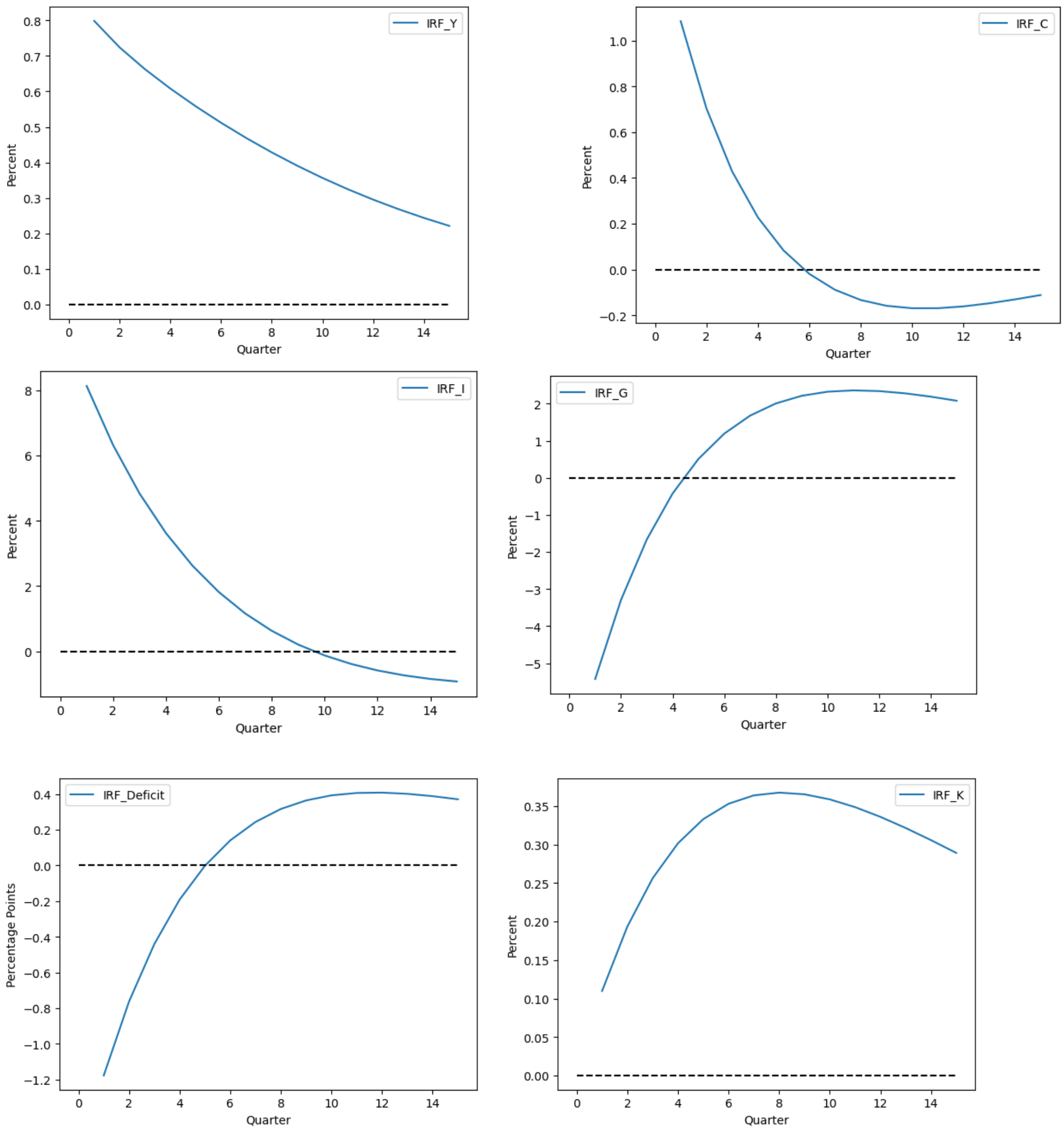
$$\tilde{\theta}\,(\theta_t t) = \{\bar{\Theta}(i) + \theta_t t(i), i \in I \quad \bar{\Theta}(i), else \quad (49)$$

This assumes that the basic functions with least contribution to representation of the function in levels, make no contribution at all to its changes over time. For the two asset HANK model we have: consumption c, CRRA parameter ξ,CES consumption bundles η,Frisch elasticity γ,and two assets: liquid bonds b ,and lower bound $\underline{b}$.Borrowing constraint: $R_b(b < 0) = R^B(b > 0) + \bar{R}$.

Idiosyncratic productivity shock is h,if $h = 0$ entrepreneur receives profits Π,otherwise $h = 1; h \sim AR(1)$,ρ_h persistence parameter,c^h idiosyncratic risk,wage W cost of capital $r + \delta$, Rotemberg price setting: quadratic adjustment cost scaled by $\frac{\eta}{2\kappa}$; discount factor β, Investment subject to Tobin's q adjustment cost ϕ, Government spending G, ρ_G intensity of repaying government debt: $\rho_G = 1$ implies roll-over,and tax$T$ .results frm simulation are presented in following page.

[16] A discrete cosine transform (DCT) expresses a finite sequence of data points in terms of a sum of cosine functions oscillating at different frequencies.

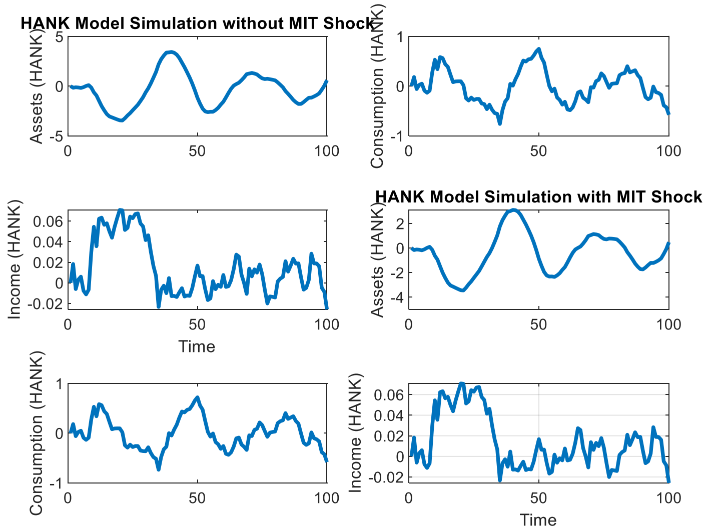
Figure 10 HANK IRF'S.



Source: Authors' calculations based on a code available at: https://github.com/econ-ark/BayerLuetticke

On the next page we will present HANK model with and without MIT shock.

Figure 11. HANK model in discrete time with MIT and without MIT shock.



Source: Authors' calculations

7. Conclusion

When savings and assets are plotted one against in Winberry (2018), there is a positive association between savings and assets for employed and unemployed. Same goes when consumption is plotted against agents assets except now this association is concave. When comes to distribution of assets (wealth) and mass of households in the economy there is not much difference between histogram and parametric family[17]In contrast to most existing work, Winberry (2018) method does not rely on the dynamics of the distribution being well-approximated by a small number of moments, substantially expanding the class of models which can be feasibly computed. In Huggett (1997) steady states and transition paths in heterogeneous Agent converge in

[17] Parametric family or a parameterized family is a family of objects (a set of related objects) whose differences depend only on the chosen values for a set of parameters.

around 200-250 periods. In credit crunch model for Huggett (1993), interest rate converges to equilibrium in 4-6 years after the shock. In Kiyotaki,Moore(1997) capital price ,output farmers' capital and farmers debt are transitioning to steady-state for about 100-200 periods after positive or negative TFP shock. The two-asset HANK (Heterogeneous Agent New Keynesian) model by Bayer,Luetticke (2020),proposed an extension of Reiter's method to solve heterogeneous agent models with aggregate risk by perturbation. This method does not rely on the dynamics of the distribution being well-approximated by a small number of moments, substantially expanding the class of models which can be feasibly computed.

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