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About the solutions of the Modified Lorenz system

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Introduction

Lorenz system

$$\dot{x} = \sigma(y - x)$$
$$\dot{y} = x(r - z) - y$$
$$\dot{z} = xy - bz$$
$$\sigma, r, b > 0$$

with the initial values $x_0 = x(0), y_0 = y(0), z_0 = z(0)$

Modified Lorenz system

$$\dot{x} = \sigma(y - x)$$
$$\dot{y} = x(r - z) - y$$
$$\overset{(5)}{z} = -(A+b)\overset{(4)}{z} + (B - Ab)\dddot{z} - (C - Bb)\ddot{z} + (D - Cb)\dot{z} + Dbz$$
$$\sigma, r, b > 0,\ A = 1 + \sigma + b, B = \sigma(r - z_0) - x_0{}^2, C = \sigma\ x_0 y_0, D = -\sigma^2 y_0{}^2$$

with the initial values $x_0 = x(0), y_0 = y(0), z_0 = z(0)$ $, z_p = \overset{(p)}{z}(0)$ $, p \in \{1,2,3,4\}.$

[1] Zlatanovska, Biljana and Dimovski, Donco (2020) A modified Lorenz system: Definition and solution. Asian-European Journal of Mathematics, 13 (08). ISSN 1793-5571 | (online) 1793-7183

$$\mathbf{x(t)} = e^{-\frac{\sigma+1}{2}t}\Bigg[C_1\sum_{n=0}^{\infty}(-1)^n \underbrace{\int_0^t\int_0^t L(t)dt^2\cdots\int_0^t\int_0^t L(t)dt^2}_{n\text{-double integrals}} + C_2\left(t+\sum_{n=1}^{\infty}(-1)^n \underbrace{\int_0^t\int_0^t L(t)dt^2\int_0^t\int_0^t L(t)dt^2\cdots\int_0^t\int_0^t tL(t)dt^2}_{n\text{-double integrals}}\right)\Bigg],$$

$$\mathbf{y(t)} = e^{-\frac{\sigma+1}{2}t}\Bigg\{C_1\Bigg(\frac{\sigma-1}{2\sigma}\sum_{n=0}^{\infty}(-1)^n \underbrace{\int_0^t\int_0^t L(t)dt^2\cdots\int_0^t\int_0^t L(t)dt^2}_{n\text{-double integrals}} + \frac{1}{\sigma}\sum_{n=0}^{\infty}(-1)^{n+1}\int_0^t L(t)dt \underbrace{\int_0^t\int_0^t L(s)ds^2\cdots\int_0^t\int_0^t L(t)dt^2}_{n\text{-double integrals}}\Bigg) + C_2\Bigg(\frac{\sigma-1}{2\sigma}\left(t+\sum_{n=1}^{\infty}(-1)^n \underbrace{\int_0^t\int_0^t L(t)dt^2\cdots\int_0^t\int_0^t tL(t)dt^2}_{n\text{-double integrals}}\right) + \frac{1}{\sigma}\left(1-\int_0^t tL(t)dt+\sum_{n=1}^{\infty}(-1)^{n+1}\int_0^t L(t)dt \underbrace{\int_0^t\int_0^t L(t)dt^2\cdots\int_0^t\int_0^t tL(t)dt^2}_{n\text{-double integrals}}\right)\Bigg)\Bigg\},$$

$$\mathbf{z(t)} = K_1e^{-bt}+K_2e^{\lambda_2 t}+K_3e^{\lambda_3 t}+K_4e^{\lambda_4 t}+K_5e^{\lambda_5 t}.$$

[2] Zlatanovska, Biljana and Dimovski, Donco (2017) *Systems of differential equations approximating the Lorenz system.* In: The Fourth Conference of Mathematical Society of the Republic of Moldova, June 28 – July 2 2017, Chisinau, Moldova.

The third equation of the Modified Lorenz System:

$$\overset{(5)}{z} = -(A+b)\overset{(4)}{z} + (B-Ab)\dddot{z} - (C-Bb)\ddot{z} + (D-Cb)\dot{z} + Dbz$$

$$\sigma, r, b > 0,\ A = 1+\sigma+b > 0,\ B = \sigma(r-z_0) - x_0^{\,2},\ C = \sigma\, x_0 y_0,$$

$$D = -\sigma^2 y_0^{\,2} \le 0$$

is a fifth-order linear homogeneous differential equation with the constant coefficients. Its characteristic equation is

$$\lambda^5 + (A+b)\lambda^4 - (B-Ab)\lambda^3 + (C-Bb)\lambda^2 - (D-Cb)\lambda - Db = 0$$

which solutions are $\lambda_{1/2/3/4/5} = k(A, B, C, D, b)$

For the solutions $\lambda_{1/2/3/4/5}$ the system (2) of seventh order can be transformed in the following systems of differential equations (SDE):

$$\begin{aligned} &\dot{x} = \sigma(y-x) \\ &\dot{y} = x(r-z) - y \\ &\dot{z} - \lambda_1 z = 0 \end{aligned} \qquad (3)$$

$$\begin{aligned} &\dot{x} = \sigma(y-x) \\ &\dot{y} = x(r-z) - y \\ &\overset{(2)}{z} - (\lambda_1+\lambda_2)\dot{z} + \lambda_1\lambda_2 z = 0 \end{aligned} \qquad (4)$$

$$\begin{aligned} &\dot{x} = \sigma(y-x) \\ &\dot{y} = x(r-z) - y \\ &\overset{(3)}{z} - (\lambda_1+\lambda_2+\lambda_3)\overset{(2)}{z} + (\lambda_1\lambda_2+\lambda_1\lambda_3+\lambda_2\lambda_3)\dot{z} - \lambda_1\lambda_2\lambda_3 z = 0 \end{aligned} \qquad (5)$$

$$\begin{aligned} &\dot{x} = \sigma(y-x) \\ &\dot{y} = x(r-z) - y \\ &\overset{(4)}{z} - (\lambda_1+\lambda_2+\lambda_3+\lambda_4)\overset{(3)}{z} + \\ &(\lambda_1\lambda_2+\lambda_1\lambda_3+\lambda_1\lambda_4+\lambda_2\lambda_3+\lambda_2\lambda_4+\lambda_3\lambda_4)\overset{(2)}{z} - \\ &(\lambda_1\lambda_2\lambda_3+\lambda_1\lambda_2\lambda_4+\lambda_1\lambda_3\lambda_4+\lambda_2\lambda_3\lambda_4)\dot{z} + \lambda_1\lambda_2\lambda_3\lambda_4 z = 0 \end{aligned} \qquad (6)$$

with the initial values $a_0=x(0), b_0=y(0), c_0=z(0)$, $\dot{z}(0) = c_1$, $\overset{(2)}{z}(0) = c_2$, $\overset{(3)}{z}(0) = c_3$.

[3] Zlatanovska, Biljana and Piperevski, Boro (2022) *A particular solution of the third-order shortened Lorenz system via integrability of a class of differential equations.* Asian-European Journal of Mathematics, 15 (10). ISSN 1793-5571, (online) 1793-7183

$$\dot{x} = \sigma(y - x)$$
$$\dot{y} = x(r - z) - y$$
$$\dot{z} - \lambda_1 z = 0$$

[4] Zlatanovska, Biljana and Piperevski, Boro and Kocaleva, Mirjana and Miteva, Marija (2023) *A particular solution to the special case of a fourth order shortened Lorenz system.* In: Proceedings of the CODEMA 2022, 25-28 Sept 2022, Ohrid, R.N.Macedonia.

$$\begin{aligned} &\dot{x} = \sigma(y - x) \\ &\dot{y} = x(r^* - z) - y \\ &\dot{z} = u \\ &\dot{u} = (m_1 + m_2)u - m_1 m_2 z \\ &\sigma, r^* > 0,\ m_1, m_2 \in \mathbb{R} \end{aligned} \qquad (6)$$

with the initial values $x_0 = x(0), y_0 = y(0), z_0 = z(0), u_0 = u(0) = \dot{z}(0) = z_1$.

$$m_1, m_2 = 2m_1 \in \mathbb{R}$$

Solving the algebraic equation

Theorem 1. The algebraic equation of the 5th degree

$$\lambda^5+(A+b)\lambda^4-(B-Ab)\lambda^3+(C-Bb)\lambda^2-(D-Cb)\lambda-Db=0,$$
$$\sigma,r,b>0,\ A=1+\sigma+b>0, B=\sigma(r-z_0)-x_0{}^2, C=\sigma\ x_0y_0, D=-\sigma^2y_0{}^2\le 0$$

has one solution $\lambda_1=-b$.

Indeed by dividing this equation of the 5th degree with $\lambda+b$ the algebraic equation of the 4th degree

$$\lambda^4+A\lambda^3-B\lambda^2+C\lambda-D=0$$
$$\sigma,r,b>0,\ A=1+\sigma+b>0, B=\sigma(r-z_0)-x_0{}^2, C=\sigma\ x_0y_0, D=-\sigma^2y_0{}^2\le 0$$

is obtained.

For the solution $\lambda_1=-b$, this fifth-order linear homogeneous differential equation with the constant coefficients

$$\overset{(5)}{z}=-(A+b)\overset{(4)}{z}+(B-Ab)\dddot{z}-(C-Bb)\ddot{z}+(D-Cb)\dot{z}+Dbz$$
$$\sigma,r,b>0,\ A=1+\sigma+b>0, B=\sigma(r-z_0)-x_0{}^2, C=\sigma\ x_0y_0, D=-\sigma^2y_0{}^2\le 0$$

has a solution

$$z(t)=K_1e^{-bt}+K_2e^{\lambda_2 t}+K_3e^{\lambda_3 t}+K_4e^{\lambda_4 t}+K_5e^{\lambda_5 t}$$

where $K_i, i=1,2,3,4,5$ are constants.

Our algebraic equation of 4th-degree

$$\lambda^4 + A\lambda^3 - B\lambda^2 + C\lambda - D = 0$$

$$\sigma, r, b > 0,\ A = 1 + \sigma + b > 0, B = \sigma(r - z_0) - x_0{}^2, C = \sigma\ x_0 y_0, D = -\sigma^2 y_0{}^2 \le 0$$

is the quartic equation.

In mathematical literature, the formulas for obtaining solutions for the quartic equation exist as the Ferrari-Cardano method. This quartic equation is the highest-order equation that can be solved in a general case where the coefficients can take any value. In most cases, the quartic equations are solved using the numerical methods or the computer algorithms.

Because, the coefficients *A*, *B*, *C* and *D* depend from the parameters $\sigma, r, b > 0$ and the initial values $x_0 = x(0), y_0 = y(0), z_0 = z(0)$, these formulas are complex for use in our case of the quartic equation.

This is a reason, why special cases of the quartic equation will have been considered.

Theorem 2. If the initial value $y_0 = 0$ then the solution of this differential equation

$$\overset{(5)}{z} = -(A+b)\overset{(4)}{z} + (B - Ab)\dddot{z} - (C - Bb)\ddot{z} + (D - Cb)\dot{z} + Dbz$$

$$\sigma, r, b > 0,\ A = 1 + \sigma + b > 0, B = \sigma(r - z_0) - x_0^{\ 2}, C = \sigma\ x_0 y_0,$$

$$D = -\sigma^2 y_0^{\ 2} \le 0$$

has a form $z(t) = K_1 e^{-bt} + K_2 + K_3 t + K_4 e^{\lambda_4 t} + K_5 e^{\lambda_5 t}, K_i = const, i = 1,2,3,4,5$

where

$$\lambda_{4/5} = \frac{-(1+\sigma+b) \pm \sqrt{(1+\sigma+b)^2 + 4\sigma(r - z_0) - 4x_0^{\ 2}}}{2}.$$

Indeed, for the $y_0 = 0$ the parameters *C* and *D* are $C = \sigma\ x_0 y_0 = 0, D = -\sigma^2 y_0^{\ 2} = 0$. The quartic equation has the form

$$\lambda^2(\lambda^2 + A\lambda + B) = 0$$

by the solutions $\lambda_{2/3} = 0$ and $\lambda_{4/5} = \frac{-A \pm \sqrt{A^2 + 4B}}{2}, A = 1 + \sigma + b, B = \sigma(r - z_0) - x_0^{\ 2}$.

Theorem 3. If the initial values $x_0 = \pm A, y_0 = \pm\frac{1}{\sigma}$ then this differential equation

$$\overset{(5)}{z} = -(A+b)\overset{(4)}{z} + (B-Ab)\dddot{z} - (C-Bb)\ddot{z} + (D-Cb)\dot{z} + Dbz$$
$$\sigma, r, b > 0,\ A = 1+\sigma+b > 0, B = \sigma(r-z_0) - x_0^{\ 2}, C = \sigma\ x_0 y_0, D = -\sigma^2 y_0^{\ 2} \le 0$$

has the solution

$$z(t) = K_1 e^{-bt} + K_2 e^{\lambda_2 t} + K_3 e^{\lambda_3 t} + K_4 e^{\lambda_4 t} + K_5 e^{\lambda_5 t}$$

where

$$\lambda_{2/3/4/5} = \frac{1}{4}\left[-A \pm \sqrt{A^2 + 4(B+2)} \pm \sqrt{2A^2 + 4B - 8 \mp 2A\sqrt{A^2+4(B+2)}}\right], A = 1+\sigma+b, B = \sigma(r-z_0) - A^2, z_0 = z(0).$$

Indeed for $x_0 = \pm A, y_0 = \pm\frac{1}{\sigma}$, the quartic equation can transform to an algebraic symmetric equation of the 4th-degree

$$\lambda^4 + A\lambda^3 - B\lambda^2 + A\lambda + 1 = 0,$$

because $A = 1+\sigma+b, B = \sigma(r-z_0) - x_0^{\ 2} = \sigma(r-z_0) - A^2, C = \sigma x_0 y_0 = \sigma(\pm A)(\pm\frac{1}{\sigma}) = A, D = -\sigma^2 y_0^{\ 2} = -\sigma^2(\pm\frac{1}{\sigma})^2 = -1$

By mathematical transformations and using the replacement $y = \lambda + \frac{1}{\lambda}$, the equation

$$y^2 + Ay - (B+2) = 0$$

is obtained with solutions $y_{1/2} = \frac{-A \pm \sqrt{A^2 + 4(B+2)}}{2}.$

Theorem 4. If for the initial values $x_0 = x(0), y_0 = y(0), z_0 = z(0)$ and the parameters σ and *r*, the following relations

$$x_0{}^2 + \sigma z_0 = \sigma r \quad \text{and} \quad y_0 = 0$$

are satisfied then the differential equation

$$\overset{(5)}{z} = -(A+b)\overset{(4)}{z} + (B - Ab)\dddot{z} - (C - Bb)\ddot{z} + (D - Cb)\dot{z} + Dbz$$

$$\sigma, r, b > 0,\ A = 1 + \sigma + b > 0, B = \sigma(r - z_0) - x_0{}^2, C = \sigma\, x_0 y_0, D = -\sigma^2 y_0{}^2 \le 0$$

has the solution

$$z(t) = K_1 e^{-bt} + K_2 e^{-(1+\sigma+b)t} + K_3 + K_4 t + K_5 t^2, K_i = const, i = 1,2,3,4,5.$$

Indeed for $x_0{}^2 + \sigma z_0 = \sigma r$ and $y_0 = 0$, the quartic equation has the form

$$\lambda^3(\lambda + A) = 0,$$

because

$$A = 1 + \sigma + b,\ B = \sigma(r - z_0) - x_0{}^2 = 0,\ C = \sigma x_0 y_0 = 0,\ D = -\sigma^2 y_0{}^2 = 0.$$

Its solutions are

$$\lambda_2 = -A < 0, \lambda_{3/4/5} = 0.$$

Theorem 5. If for the initial values $x_0 = x(0), y_0 = y(0), z_0 = z(0)$ and the parameters σ, *r* and *b,* the following relations

$$x_0 = 1+\sigma+b \quad , \quad y_0 = \frac{(1+\sigma+b)^2}{16\sigma} \text{ , and } z_0 = r - \frac{5(1+\sigma+b)^2}{8\sigma}$$

are satisfied then the quartic equation

$$\lambda^4 + A\lambda^3 - B\lambda^2 + C\lambda - D = 0$$

$$\sigma, r, b > 0,\ A = 1+\sigma+b > 0, B = \sigma(r - z_0) - x_0{}^2, C = \sigma\ x_0 y_0, D = -\sigma^2 y_0{}^2 \le 0$$

has one real root with multiplicity-4

$$\lambda = -\frac{1+\sigma+b}{4}$$

In this case, the solution of this differential equation

$$\overset{(5)}{z} = -(A+b)\overset{(4)}{z} + (B - Ab)\dddot{z} - (C - Bb)\ddot{z} + (D - Cb)\dot{z} + Dbz$$

$$\sigma, r, b > 0,\ A = 1+\sigma+b > 0, B = \sigma(r - z_0) - x_0{}^2, C = \sigma\ x_0 y_0, D = -\sigma^2 y_0{}^2 \le 0$$

has a form

$$z(t) = K_1 e^{-bt} + (K_2 + K_3 t + K_4 t^2 + K_5 t^3) e^{-\frac{(1+\sigma+b)}{4}t}, K_i = const, i = 1,2,3,4,5.$$

Example. By using the parameters $\sigma = 5; r = 25; b = 0,8$ and the initial values $x_0 = 10; y_0 = 0; z_0 = 8$

the solution

$$z(t) = K_1 e^{-0,8t} + K_2 + K_3 t + e^{-3,4t}\left(C_1 \cos\frac{\sqrt{13,76}}{2} t + C_2 \sin\frac{\sqrt{13,76}}{2} t\right), K_i, C_1, C_2 = const, i = 1,2,3$$

of this differential equation

$$\overset{(5)}{z} = -(A+b)\overset{(4)}{z} + (B - Ab)\dddot{z} - (C - Bb)\ddot{z} + (D - Cb)\dot{z} + Dbz$$

$$\sigma, r, b > 0,\ A = 1 + \sigma + b > 0, B = \sigma(r - z_0) - x_0^{\,2}, C = \sigma\, x_0 y_0,$$

$$D = -\sigma^2 y_0^{\,2} \le 0$$

is obtained.

For the initial values

$$z_1 = \dot{z}(0) = -6,4;\ z_2 = \ddot{z}(0) = 1705,12;\ z_3 = \dddot{z}(0) = -27924,1;\ z_4 = \overset{(4)}{z}(0) = 746387,3$$

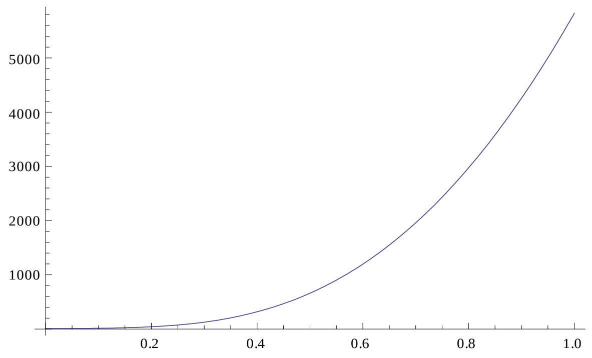
the particular solution has the form

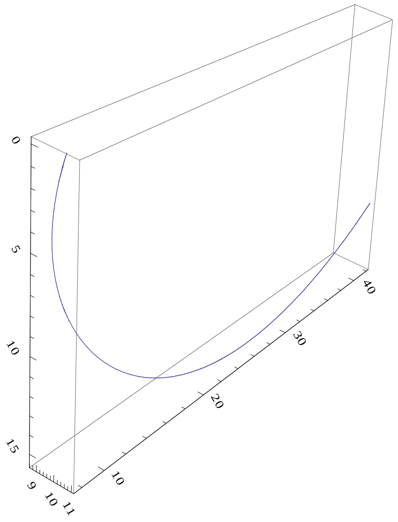
$$z(t) = 89166,7e^{-0,8t} - 82007,9 + 47411,7t + e^{-3,4t}\left(-7150,83\cos\frac{\sqrt{13,76}}{2} t - 214,33\sin\frac{\sqrt{13,76}}{2} t\right).$$

Some examples

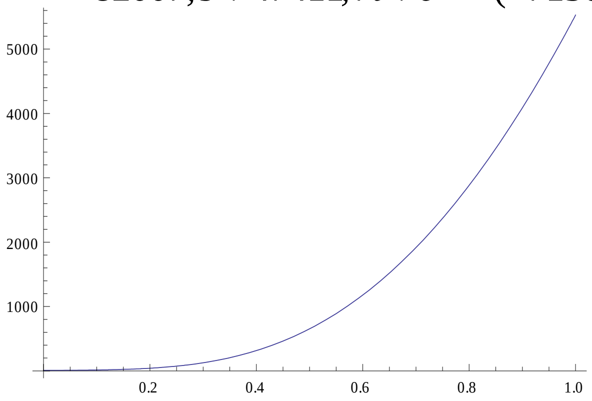
In Mathematica, the solution $z(t)$ and the solution of the MLS are shown by

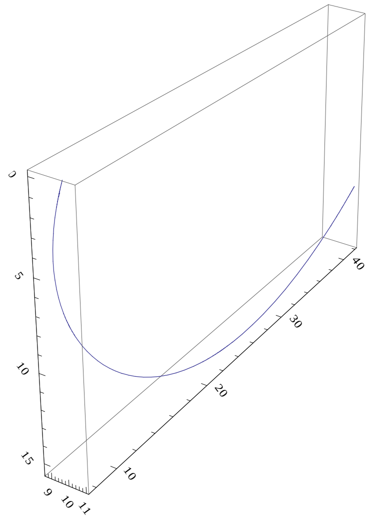
a) Runge-Kutta





b) $z(t) = 89166,7e^{-0,8t} - 82007,9 + 47411,7t + e^{-3,4t}(-7150,83\cos\frac{\sqrt{13,76}}{2}t - 214,33\sin\frac{\sqrt{13,76}}{2}t).$





Conclusion

By using of the theorems for these special cases, the explicit solutions of the Modified Lorenz system is completed.

As further work, it remains to find a way to obtain the solution of the Modified Lorenz system for the other cases, which do not cover in this paper.

Thank You For Your Attention