MIRRLEESIAN OPTIMAL TAXATION: THEORY AND NUMERICAL SOLUTIONS

Dushko Josheski[1]

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Abstract

In this paper Mirrleesian optimal taxation will be reviewed.Models in optimal tax theory typically posit that the tax system should maximize a social welfare function subject to a government budget constraint, considering how individuals respond to taxes and transfers.James Mirrlees (1971) launched the second wave of optimal tax models by suggesting a way to formalize the planner's problem that deals explicitly with unobserved heterogeneity among taxpayers.There are static and dynamc versions of this model and we will review them or introduce them in this paper.Social welfare is larger when resources are more equally distributed, but redistributive taxes and transfers can negatively affect incentives to work and earn income in the first place. This creates the classical trade-off between equity and efficiency which is at the core of the optimal labor income tax problem.We will describe main theoretical findings in this literature as well as numerical examples with their policy implications.

Keywords: Optimal taxation, Mirrlees tax model, asymmetric information, non-linear tax rates, second-best analysis of taxes
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[1]GoceDelchev University - Stip, North Macedonia, dusko.josevski@ugd.edu.mk

1.Introduction

This paper will review topic from optimal Mirrleesian taxation. In the classical framework initiated by Mirrlees (1971),the theory studies the maximization of a utilitarian social welfare function by a benevolent planner who only observes the pretax labor income of agents whose wages differ, but whose preferences are identical. The other studies have relaxed the assumptions in order to take heterogeneity among agents into account. These studies include:Mirrlees (1976), Saez (2001), Choné and Laroque (2010), see Fleurbaey , Maniquet (2018). Mainly approach is based on asymmetric information. Public policies apply to the individuals on the basis of what the government knows about them. *Second* welfare theorem[2] states, that where a number of convexity and continuity assumptions are satisfied, an optimum is a competitive equilibrium once initial endowments have been suitably distributed.In general, complete information about the consumers for the transfers is required to make the distribution requires, so the question of feasible lump-sum transfers arises here. Usually the optimal tax systems combine flat marginal tax rate plus lump sum grants to all the individuals (so that the average tax rate rises with income even if the marginal does not), Mankiw NG, Weinzierl M, YaganD.(2009).Rigorous derivations of the optimal tax rates include:Atkinson,Stiglitz,(1980);Kaplow,(2008);Mirrlees(1976),Mirrlees(1986);Stiglitz,(1987);Tuomala,(1990). The choice of the optimal redistributive tax involves tradeoffs between three kinds of effects: equity effect (it changes the distribution of income) , the efficiency effect form reducing the incentives, the insurance effect from reducing the variance of individual income streams, Varian,H.R.(1980).Saez (2001) argued that “unbounded distributions are of much more interest than bounded distributions to address high income optimal tax rate problem”. Saez (2001) investigated (four cases)[3] and the optimal tax rates are clearly U-shaped, see Diamond (1998) too.Saez,S.Stantcheva (2016),define social marginal welfare weight as a function of agents consumption, earnings, and a set of characteristics that affect social marginal welfare weight and a set of characteristics that affect utility.Piketty, Saez,Stantcheva(2014),derived optimal top tax rate formulas in a model where top earners respond to taxes through three channels: labor supply, tax avoidance, and compensation bargaining. Dynamic taxation most famous examples in the literature are: Diamond-Mirrlees (1978);Albanesi-Sleet(2006),Shimer-Werning(2008),Ales-Maziero(2009),Golosov-TroshkinTsyvinsky(2011).Sizeable literature in NDPF studies optimal taxation in dynamic settings,(Golosov,Kocherlakota,Tsyvinski(2003),Golosov,Tsyvinski, and Werning (2006), Kocherlakota (2010).Here we will derive optimal linear, non-linear tax rates for top earners and we will derive results in heterogenous preferences

[2] Second fundamental theorem is giving conditions under which a Pareto optimal allocation can be supported as a price equilibrium with lump-sum transfers, i.e. Pareto optimal allocation as a market equilibrium can be achieved by using appropriate scheme of wealth distribution (wealth transfers) scheme (Mas-Colell, Whinston et al. 1995)

[3] Utilitarian criterion, utility type I and II and Rawlsian criterion, utility type I and II.

environment for dynamic taxation. Optimal taxation is not to be confused with Pareto efficient taxes (seeWerning (2007)).

2.Mirrlees framework optimal top tax rate : derivation

The effect of small tax reform in Mirrless (1971) model is examined in Brewer, M., E. Saez, and A. Shephard (2010) ,where indirect utility function is given as :$U(1-\tau,R) = max_w((1-\tau)w + R, z)$, where w represents the taxable income R is a virtual income intercept, and τ is an imposed income tax. Marshalian labor supply is w= $w(1-\tau,R)$, uncompensated elasticity of the supply is given as:$\varepsilon^u = \frac{(1-\tau)}{w}\frac{\partial w}{\partial(1-\tau)}$, income effect is $\eta = (1-\tau)\frac{\partial w}{\partial R} \leq 0$.Hicksian supply of labor is given as:$w^c((1-\tau,u))$, this minimizes the cost in need to achieve slope $1-\tau$, compensated elasticity now is :$\varepsilon^c = \frac{(1-\tau)}{w}\frac{\partial w^c}{\partial(1-\tau)} > 0$, Slutsky equation now becomes:$\frac{\partial w}{\partial(1-\tau)} = \frac{\partial w^c}{\partial(1-\tau)} + z\frac{\partial z}{\partial R} \Rightarrow \varepsilon^u = \varepsilon^c + \eta$, where η represents income effect :$\eta = (1-\tau)\frac{\partial w}{\partial R} \leq 0$.With small tax reform taxes and revenue change i.e.:$dU = u_c \cdot [-wdt + dR] + dw[(1-\tau)u_c + u_z] = u_c \cdot [-zdt + dR]$.Change of taxes and its impact on the society is given as:$dU_i = -u_c dT(w_i)$. Envelope theorem here says :$U(\theta) = \max_x F(x,\theta), s.t. c > G(x,\theta)$, and the preliminary result is :$U'(\theta) = \frac{\partial F}{\partial \theta}\left(x^*(\theta), \theta - \lambda^*(\theta)\frac{\partial G}{\partial \theta}x^*(\theta), \theta\right)$. Government is maximizing :

$$0 = \int G'(u^i)u_c^i \cdot \left[(W - w^i) - \frac{\tau}{d(1-\tau)}eW\right] \quad (1)$$

1. mechanical effect is given as:$dM = [w - w^*]d\tau$,
2. welfare effect is :$dW = -\bar{g}dM = -\bar{g}[w - w^*]$, and at last
3. the behavioral response is :$dB = -\frac{\tau}{1-\tau} \cdot e \cdot wd\tau$.

And let's denote that:

$$dM + dW + dB = d\tau\left[1 - \bar{g}[w - w^*] - e\frac{\tau}{1-\tau} \cdot w\right] \quad (2)$$

When the tax is optimal these three effects should equal zero i.e. $dM + dW + dB = 0$ given that:$\frac{\tau}{1-\tau} = \frac{(1-\bar{g})[w-w^*]}{e \cdot z}$, and we got $\tau = \frac{1-\bar{g}}{1-\bar{g}+a \cdot e}, a = \frac{w}{w-w^*}$,and $dM = d\tau[w - w^*] \ll dB = d\tau \cdot e\frac{\tau}{1-\tau} \cdot w, when\ w^* > w^T$, where w^T is a top earner income. Pareto distribution is given as:

$$1 - F(w) = \left(\frac{k}{w}\right)^a; f(w) = a \cdot \frac{k^a}{w^{1+a}} \quad (3)$$

$a$is a thickness parameterand top income distribution is measured as:

$$w(w^*) = \frac{\int_{z^*}^{\infty} sf(s)ds}{\int_{z^*}^{\infty} f(s)ds} = \frac{\int_{z^*}^{\infty} s^{-a}ds}{\int_{z^*}^{\infty} s^{-a-1}ds} = \frac{a}{(a-1)} \cdot w^* \quad (4)$$

Empirically $\alpha \in [1.5,3]$, $\tau = \frac{1-\bar{g}}{1-\bar{g}+\alpha \cdot e}$.General non-linear tax without incomeeffects is given as:

$$\frac{T'(w_n)}{1-T'(w_n)} = \frac{1}{e}\left(\frac{\int_n^\infty (1-g_m)dF(m)}{w_n h(w)}\right) = \frac{1}{e}\left(\frac{1-H(w_n)}{w_n h(w_n)}\right) \cdot (1-G((w_n)) \quad (5)$$

Where elasticity or efficiency $e = \left[\frac{1-\tau}{w}\right] \times \frac{dw}{d(1-\tau)}$. Where $G((w_n) = \frac{\int_n^\infty g_m dF(m)}{1-F(n)}$,and $g_m = G'(u_m)/\lambda$ this is welfare weight of type m.But non-linear tax witn income effecttakes into account small tax reform where tax rates change from $d\tau$to$[w^*, w^* + dw^*]$.Every tax payer with income w>w^*pays additionaly$d\tau dw^*$valued by$(1-g(w))d\tau dw^*$.Mechanical effect is :

$$M = d\tau dw^* \int_{z^*}^\infty (1-g(w))d\tau dw^* \quad (6)$$

Total income response is :$I = d\tau dw^* \int_{z^*}^\infty \left(-\eta_Z \frac{T'(w)}{1-T'(w)}(w)\right) h(w)dw$. Change at the taxpayers form the additional tax is :$dz = -\varepsilon_{(z)}^c \frac{T''dz}{1-T'} - \eta \frac{d\tau dw^*}{1-T'(w)} \Rightarrow -\eta \frac{d\tau dw^*}{1-T'(w)+z\varepsilon_{(w)}^c T''(w)}$, if one sums up all effects can be obtained:

$$\frac{T'(w)}{1-T'(w)} = \frac{1}{\varepsilon_{(z)}^c}\left(\frac{1-H(w^*)}{z^* h(w^*)}\right) \times \left[\int_{z^*}^\infty (1-g(w))\frac{h(w)}{1-H(w^*)}dz + \int_{z^*}^\infty -\eta \frac{T'(w)}{1-T'(w)} \frac{h^*(w)}{1-H(w^*)}dw\right] \quad (7)$$

With linear tax:$\frac{\dot{z}_n}{z_n} = \frac{1+\varepsilon_{(n)}^u}{n}$ and with non-linear tax:

$$\frac{\dot{w}_n}{w_n} = \frac{1+\varepsilon_{(n)}^u}{n} - \dot{w}_n \frac{T''(w_n)}{1-T''(w_n)} \varepsilon_{w(n)}^c \quad (8)$$

Optimal tax formula here if $dM + dW + dB = 0$ is given as : $\tau = \frac{1-\bar{g}}{1-\bar{g}+\alpha \cdot e}$; $\alpha = \frac{w}{w-w^*}$ where $\bar{g} = \frac{\int g_i \cdot w_i}{w \cdot \int g_i}$ and$g_i = G'(u')u_c^i$.

2.1 Formal derivation of optimal non-linear tax rates with no income effects

This point actually follows Mirrleees (1971) and Diamond (1998) , in deriving non-linear optimal tax rate with no-income effects.Utility function is quasi linear:

$$u(c,l) = c - v(l) \quad (9)$$

c is disposable income and the utility of suuply of labor $v(l)$ is increasing and convex in l. Earnings equal $w = nl$ where n represents innate ability. CDF of skills distribution is $F(n)$, it's PDF is $f(n)$and support range is $[0,\infty)$. Government cannot observe abilities instead it can set taxes asa function of labor income $c = w - \tau(w)$.Individual n chooses l_n to maximize :

$$\max(nl - \tau n(l) - v(l)) \quad (10)$$

When marginal tax rate τ is constant, the labor supply f-ction is given as: $l \to l\left(n(1-\tau)\right)$ and it is implicitly defined by the $n(1-\tau) = v'(l)$. And $\frac{dl}{d\left(n(1-\tau)\right)} = \frac{1}{v''(l)}$, so the elasticity of the net-of-tax rate $1-\tau$ is:

$$e = \frac{\left(\frac{n(1-\tau)}{l}\right) dl}{d\left(n(1-\tau)\right)} = \frac{v'(l)}{lv''(l)} \quad (11)$$

As there are no income effects this elasticity is both the compensated and the uncompensated elasticity.The government maximizes SWF :

$$W = \int G(u_n) f(n) dn \;\; s.t. \int cnf(n)dn \le \int nlnf(n)dn - E(\lambda) \quad (12)$$

u_n denotes utility, $w_n = nl_n$ denotes earnings, c_n denotes consumption or disposable income, and $c_n = u_n + v(l_n)$.By using the envelope theorem and the FOC for the individual, u_n satisfies following:

$$\frac{du_n}{dn} = \frac{lnv'(ln)}{n} \quad (13)$$

Now the Hamiltonian is given as:

$$\mathcal{H} = \left[G(u_n) + \lambda \cdot \left(nl_n - u_n - v(l_n)\right)\right] f(n) + \phi(n) \cdot \frac{lnv'(ln)}{n} \quad (14)$$

In previous $\phi(n)$ is the multiplier of the state variable. The FOC with respect to l is given as:

$$\lambda \cdot \left(n - v'(l_n)\right) + \frac{\phi(n)}{n} \cdot \left[v'(l_n) + l_n v''(l_n)\right] = 0 \quad (14)$$

FOC with respect to u is given as:

$$-\frac{d\phi(n)}{n} = \left[G'(u_n) - \lambda\right] \quad (15)$$

If integrated previous expression gives: $-\phi(n) = \int_n^\infty [\lambda - G'(u_m)] f(m) dm$ where the transversality condition $\phi(\infty) = 0$,and $\phi(0) = 0$, and $\lambda = \int_0^\infty G'(u_m) f(m) dm$and social marginal welfare weights $\frac{G'(u_m)}{\lambda} = 1$. Using this equation for $\phi(n)$ and all previous $n - v'(ln) = n\tau'(w_n)$, and that

$$\frac{\left[v'(l_n) + l_n v''(l_n)\right]}{n} = \left[\frac{v'(l_n)}{n}\right]\left[1 + \frac{1}{e}\right] \quad (16)$$

We can rewrite FOC with respect to l_n as:

$$\frac{\tau'(w_n)}{1-\tau'(w_n)} = \left(1 + \frac{1}{e}\right) \cdot \left(\frac{\int_n^\infty (1-g_m) dF(m)}{nf(n)}\right) \quad (17)$$

In previous expression $g_m = \frac{G'(u_m)}{\lambda}$ which is the social welfare on individual m . The formula was derived in Diamond (1998) . If we denote $h(w_n)$ as density of earnings at w_n if the nonlinear tax system were replaced by linearized tax with marginal tax rate $\tau = \tau'(w_n)$ we would have that following equals $h(w_n) dw_n = f(n) dn$ and $f(n) = h(w_n) l_n (1+e)$,henceforth $nf(n) = w_n h(w_n)(1+e)$ and we can write previous equation as:

$$\frac{\tau'(w_n)}{1-\tau'(w_n)} = \frac{1}{e}\cdot\left(\frac{\int_n^\infty (1-g_m)dF(m)}{w_n h(w_n)}\right) = \frac{1}{e}\cdot\left(\frac{1-H(w_n)}{w_n h(w_n)}\right)\cdot\left(1-G(w_n)\right) \quad (18)$$

In the previous expression $G(w_n) = \int_n^\infty \frac{dF(m)}{1-F(n)}$ is the average social welfare above w_n .If we change variables from $n \to w_n$, we have $G(w_n) = \int_{w_n}^\infty \frac{g_m dH(w_m)}{1-H(w_n)}$.The transversality condition implies $G(w_0 = 0) = 1$.

2.2 Optimal linear tax formula

First modern treatment of optimal linear tax was provided by Sheshinski (1972).Optimal linear tax formulae is given as:

$$\int_0^\infty \tau(w)f(n)dn = \int_0^\infty (w-\alpha-\beta w)f(n)dn = 0 \quad (19)$$

$f(n)$ is PDF of ability n, α is a tax parameter and is a lump-sum tax if $\alpha < 0$ and tax-subsidy if $\alpha > 0$ given to an individual with no income.$1-\beta$ is a marginal tax rate i.e. $0 \le \beta \le 1$ so that marginal tax rate is non negative in the linear tax function which is $\tau(w) = -\alpha + (1-\beta)w$, after tax consumption is $c(w) = w - \tau(w) = \alpha + \beta w$.Optimal labor supply is given as: $\ell = \hat{\ell}(\beta n, \alpha)$.If λ is the lowest elasticity of labor supply function and it is equal to $\lambda = \lim_n \inf\left[\frac{\beta}{\hat{\ell}}\frac{\partial \hat{\ell}}{\partial \beta}\right]$ so that $\frac{\beta}{\hat{\ell}}\frac{\partial \hat{\ell}}{\partial \beta} \ge \lambda$. Revenue maximizing linear tax rate is given as: $\frac{\tau^*}{1-\tau^*} = \frac{1}{e}$ or $\tau^* = \frac{1}{1+e}$.Government FOC given $SWF = \int \omega_i G\left(u^i(1-\tau)w^i + \tau w(1-\tau) - E, w^i\right)df(i)$ is :

$$0 = \frac{dSWF}{d\tau} = \int \omega_i G'(u_i)u_c^i \cdot \left((w-w^*) - \tau\frac{dw}{d(1-\tau)}\right)df(i) \quad (20)$$

Social marginal welfare weight g_i is given as: $g_i = \frac{\omega_i G'(u_i)u_c^i}{\int \omega_j G'(u_j)u_c^j df(j)}$.So that optimal linear tax formula is:

$$\tau = \frac{1-\bar{g}}{1-\bar{g}+e} \quad (21)$$

where $\bar{g} = \frac{\int g_i \cdot w_i df(i)}{w}$.

2.3 Diamond ABC formula

Here in this paragraph a Diamond (1988) formula has been derived. Welfare weights are distributed with a CDF: $\Psi(n)$ and PDF :$\psi(n)$. The government maximization function is (objective function) is given as:

$$\int_{\underline{n}}^{\overline{n}} u(n)\psi(n)dn \quad (22)$$

Now by assumption $\int_{\underline{n}}^{\overline{n}} u(n)\psi(n)dn = 1$, which implies that $\lambda = 1$,λ aggregates thesocial welfare weights across the entire economy.

$$\lambda = \int_{\underline{n}}^{\overline{n}} \Psi u(n)\psi(n)dn \tag{23}$$

FOC can be found as previously, form the Hamiltonian $\mathcal{H} = [\Psi(u_n) + \lambda \cdot (nl_n - u_n - v(l_n))]\psi(n) + \phi(n) \cdot \frac{lnv'(ln)}{n}$. In previous $\phi(n)$ is the multiplier of the state variable. The FOC with respect to l is given as:$\lambda \cdot (n - v'(l_n)) + \frac{\phi(n)}{n} \cdot [v'(l_n) + l_n v''(l_n)] = 0$.FOC with respect to u is given as:

$$-\frac{d\phi(n)}{n} = [\Psi(u_n) - \lambda] = -\phi'(n) - \lambda f(n) \tag{24}$$

Or alternatively:$-\phi(n) = \int_n^{\overline{n}}(f(n) - \Psi(n))dn = \Psi(n) - F(n)$

$$\frac{\tau'(w_n)}{1 - \tau'(w_n)} = \left(\frac{1+e}{e}\right) \cdot \left(\frac{\psi(n) - F(n)}{nf(n)}\right) \tag{25}$$

To write ABC formula we divide and multiply by $1 - F(n)$:

$$\frac{\tau'(w_n)}{1 - \tau'(w_n)} = \underbrace{\left(\frac{1+e}{e}\right)}_{A(n)} \cdot \underbrace{\left(\frac{\psi(n) - F(n)}{1 - F(n)}\right)}_{B(n)} \cdot \underbrace{\left(\frac{1 - F(n)}{nf(n)}\right)}_{C(n)} \tag{26}$$

Where $A(n) = \frac{1+e}{e}$ is the elasticity and efficiency argument, $B(n) = \frac{\psi(n) - F(n)}{1 - F(n)}$ measures the desire for redistribution, $C(n) = \frac{1 - F(n)}{nf(n)}$ measures the thickness on the right tail of distribution. In the Rawlsian case $\Psi(n) = 1$ previous formula will converge to:

$$\frac{\tau'(w_n)}{1 - \tau'(w_n)} = \left(\frac{1+e}{e}\right) \cdot \left(\frac{1 - F(n)}{nf(n)}\right) \tag{27}$$

2.4Formal derivation of optimal non-linear tax rates with income effects

Utility function takes form $\tilde{u}(c,l) = u(c) - v(l)$ where $u'(c) > 0; u''(c) \leq 0$.Elasticity of labor supply is :

$$\frac{v'(l)}{u'(c)} = (1 - \tau'(w))n \tag{28}$$

The uncompensated response of labor supply is given as:

$$\frac{\partial l^u}{\partial(1 - \tau'(w))n} = \frac{u'(c) + l(1 - \tau'(w))nu''(c)}{v''(l) - (1 - \tau'(w))^2 n^2 u''(c)} \tag{29}$$

And uncompensated elasticity is implied:

$$\varepsilon^u = \frac{\frac{u'(c)}{l} + \frac{v'(l)^2}{u'(c)^2}u''(c)}{v''(l) - \frac{v'(l)^2}{u'(c)^2}u''(c)} \tag{30}$$

The response of labor to income changes is given as:

$$\frac{\partial l}{\partial y} = \frac{(1-\tau'(w)nu''(c)}{v''(l)-(1-\tau'(w))^2 n^2 u''(c)} \quad (31)$$

By using the Slutsky equation we have:

$$\frac{\partial l^c}{\partial(1-\tau'(w))n} = \frac{u'(c)+l(1-\tau'(w))nu''(c)}{v''(l)-(1-\tau'(w))^2 n^2 u''(c)} - \frac{l(1-\tau'(w))nu''(c)}{v''(l)-(1-\tau'(w))^2 n^2 u''(c)} = \frac{u'(c)}{v''(l)-(1-\tau'(w))^2 n^2 u''(c)} \quad (32)$$

Henceforth :

$$\varepsilon^c = \frac{\frac{v'(l)}{l}}{v''(l)-(1-\tau'(w))^2 n^2 u''(c)} \quad (33)$$

Here everything is as previous except now we cannot replace $c(n)$ in the resource constraint by using def. of indirect utility here we will define consumption as expenditure function $\tilde{c}(\tilde{u}(n), w(n), n)$.Previous resource constraint for this economy with no income effects was:

$$\int_{\underline{n}}^{\overline{n}} c(n)f(n)dn \geq \int_{\underline{n}}^{\overline{n}} w(n)f(n)dn - E \quad (34)$$

So this new function we will differentiate w.r.t.$\tilde{u}(n), w(n)$.Indirect utility is defined as :

$$\tilde{u}(n) = u(\tilde{c}(n)) - v\left(\frac{w^*(n)}{n}\right) \quad (35)$$

At optimum conditions that hold are:

$$d\tilde{u}(n) = u'(\tilde{c}(n))d\tilde{c}(n) \quad (36)$$

$$0 = u'(\tilde{c}(n))d\tilde{c}(n) - \frac{1}{n}v'\left(\frac{w^*(n)}{n}\right)dw^*(n)$$

If we rearrange we will get :

$$\frac{d\tilde{c}(n)}{d\tilde{u}(n)} = \frac{1}{u'(\tilde{c}(n))}$$

$$\frac{d\tilde{c}(n)}{dw^*(n)} = \frac{v'\left(\frac{w^*(n)}{n}\right)}{nu'(\tilde{c}(n))} \quad (37)$$

Hamiltonian for this problem is given as:

$$\mathcal{H} = [G(u(n) + \lambda(w(n) - \tilde{c}(\tilde{u}(n), w(n), n)]f(n) + \phi(n)\frac{w(n)}{n^2}v'\left(\frac{w(n)}{n}\right) \quad (38)$$

FOC's are given a

$$\begin{cases} \frac{\partial \mathcal{H}}{\partial w(n)} = \lambda \left[1 - \frac{v'\left(\frac{w(n)}{n}\right)}{nu'(c(n))}\right] f(n) + \frac{\phi(n)}{n^2}\left[v'\left(\frac{w(n)}{n}\right) + \frac{w(n)}{n} v''\left(\frac{w(n)}{n}\right)\right] = 0 \\ \frac{\partial \mathcal{H}}{\partial u(n)} = \left[G'(u(m) - \frac{\lambda}{u'c(m)}\right] f(n)d = -\phi'(n) \end{cases} \quad (39)$$

For the multiplier $\phi'(n)$ the equilibrium value is given as:

$$\phi(n) = \int_{\underline{n}}^{\overline{n}} \left[G'(u(m) - \frac{\lambda}{u'c(m)}\right] f(m)dm \quad (40)$$

With the definition of the two elasticities we can write :

$$\left[v'\left(\frac{w(n)}{n}\right) + \frac{w(n)}{n} v''\left(\frac{w(n)}{n}\right)\right] = v'\left(\frac{w(n)}{n}\right)\left[1 + \frac{w(n)}{n} \frac{v''\left(\frac{w(n)}{n}\right)}{v'\left(\frac{w(n)}{n}\right)}\right] = v'^{\left(\frac{w(n)}{n}\right)}\left(\frac{1+\varepsilon^u}{\varepsilon^c}\right) \quad (41)$$

The optimal tax formula then will become :

$$\frac{\tau'(w_n)}{1-\tau'(w_n)} = \left(\frac{1+e^u}{e^c}\right) \cdot \left(\frac{\eta(n)}{nf(n)}\right); \eta(n) = \frac{u'(c(n)\phi(n))}{\lambda}. \quad (42)$$

3.Golosov et al.(2016) framework: heterogenous preferences

This economy is described by $t+1$ periods denoted by $t = 0,1,..,t+1$.Agents preferences are described by a time separable utility function over consumption c_t and labor l_t , and discount factor $\beta \in (0,1)$, and expectation operator in period $t = 1$,E_0 and utility function $u: \mathbb{R}_+^2 \to \mathbb{R}$. Where ; $E_0 \sum_{t=0}^{t+1} \beta^t(c_t, l_t)$.In period$t = 0$ agent skills are θ_0 and the distribution of those skills is $F(\theta_0)$.In period $t+1$;$t \geq 1$ skills follow Markov process $F_t(\theta_t|\theta_{t-1})$,where θ_{t-1} represents skill realization, and PDF is $f_t(\theta_t|\theta_{t-1})$ .People retire at period $\hat{t}$ in which case $F_t(0|\theta) = 1 \forall t, \wedge \forall t \geq \hat{t}$.

Assumption 1.$\forall t \geq \hat{t}$,pdf is differentiable with $f_t' \equiv \frac{\partial f_t}{\partial \theta}$ and $f_{2,t}' \equiv \frac{\partial f_t}{\partial \theta_{t-1}}$,where $\forall \theta_{t-1}$, where $\psi(\theta|\theta_{t-1}) = \frac{\theta_{t-1}\int_\theta^\infty \frac{\partial f_t}{\partial \theta_{t-1}}(x|\theta_{t-1})dx}{\theta f_t(\theta|\theta_{t-1})}$, is bounded one sided $|_\theta^\infty \forall \theta$ and this limit is finite :$\lim_{\theta\to\infty} \frac{1-F_t(\theta|\theta_{(t-1)})}{\theta f_t(\theta|\theta_{t-1})}$.

If previous process is AR(1) then ψ is equal to autocorrelation of the shock process $\forall \theta$. Skills are non-negative $\theta_t \in \Theta = \mathbb{R}^+$, $\forall t$. Agent types are also persistent like in Hellwig (2021) :

$$\Theta(\theta|\theta_{t-1}) = \frac{\frac{\partial f_t(\theta_t|\theta_{t-1})}{\partial \theta_{t-1}}}{f_t(\theta|\theta_{t-1})} \quad (43)$$

Where $\frac{\partial f_t(\theta_t|\theta_{t-1})}{\partial \theta_{t-1}} = -\rho \frac{\partial f_t(\theta_t|\theta_{t-1})}{\partial \theta_t}$,when $\rho = 0$, θ_t is i.i.d. and when $\rho = 1$ θ_t is random walk with persistence.

Assumption 2. Single crossing condition strictly decreasing:$\frac{u_{c\theta}}{u_c} - \frac{u_{y\theta}}{u_y} > 0$

Where $\boldsymbol{y}$ are the earnings of the agent. Social planer evaluates welfare by Pareto weights $\boldsymbol{\alpha}: \boldsymbol{\Theta} \rightarrow \mathbb{R}_{+}$ .Then $\boldsymbol{\alpha}$ is normalized to 1 $\int_0^\infty \boldsymbol{\alpha}(\boldsymbol{\theta})\boldsymbol{dF_0}(\boldsymbol{\theta}) = \mathbf{1}$ Social welfare is given by:

$$SWF = \int_0^\infty \alpha(\theta)\left(E_0 \sum_{t=0}^{t+1} \beta^t(c_t, l_t)\right) dF_0(\theta) \tag{44}$$

Assumption 3. $\boldsymbol{u}$ is continuous and twice differentiable in both arg. and satisfies $\boldsymbol{u_c} > 0; \boldsymbol{u_l} < 0; \boldsymbol{u_{cc}} \leq \mathbf{0}; \boldsymbol{u_{ll}} \leq \mathbf{0}$, and $\frac{\partial}{\partial\theta}\frac{u_y\left(c,\frac{y}{\theta}\right)}{u_c\left(c,\frac{y}{\theta}\right)}$ .There the optimal allocation solve mechanism design problem as in Golosov, Kocherlakota, Tsyvinski (2003):

$$\max_{c_{t(\theta_t)}, y_{t(\theta_t)}; \theta_t \in \Theta_t; t \in (0,\hat{t})} \int_0^\infty \alpha(\theta)\left(E_0\left(\sum_{t=0}^{t+1} \beta^t \left(c_t(\theta_t), \frac{y_t(\theta_t)}{\theta_t}\right) | \theta_t\right)\right) dF_0(\theta) \tag{45}$$

s.t. IC (incentive compatibility) constraint :

$$E_0\left(\sum_{t=0}^{t+1} \beta^t u\left(c_t(\theta_t), \frac{y_t(\theta_t)}{\theta_t}\right) | \theta_t\right) \geq E_0\left(\sum_{t=0}^{t+1} \beta^t u\left(c_t(\sigma^t(\theta_t)), \frac{y_t(\sigma^t((\theta_t)))}{\theta_t}\right) | \theta_t\right), \forall \sigma\hat{t} \in \sum^{t}, \sigma^t \in \sigma^{\hat{t}}, \theta \in \Theta \tag{46}$$

and feasibility constraint:

$$\int_0^\infty E_0\left\{\sum_{t=0}^{\hat{t}} R^{-t} c_t(\theta_t) | \theta_t\right\} dF_0(\theta) \leq \int_0^\infty E_0\left\{\sum_{t=0}^{\hat{t}} R^{-t} y_t(\theta_t) | \theta_t\right\} dF_0(\theta) \tag{47}$$

Now, $\boldsymbol{\omega}(\widehat{\boldsymbol{\theta}}, \boldsymbol{\theta})$ is state variable following Fernandes,Phelan (2000).Dynamic generalization of Envelope condition of Mirrlees (1971) and Milgrom and Segal (2002) , Kapicka (2013), Williams (2011), Pavan, Segal and Toikka (2014).So now we have:

$$\begin{cases} u\left(c_t(\theta_t), \frac{y_t(\theta_t)}{\theta_t}\right) + \beta\omega_{t+1}(\theta_{t-1}|\theta_t) \geq u\left(c_t(\theta_{t-1}, \hat{\theta}), \frac{y_t(\theta_{t-1}, \hat{\theta})}{\theta_t}\right) + \beta\omega_{t+1}(\theta_{t-1}, \hat{\theta}|\theta_t), \forall\hat{\theta}, \theta \in \Theta, \forall t, \\ \omega_{t+1}(\theta_{t-1}, \hat{\theta}|\theta_t) = E_t\left\{\sum_{s=t+1}^{\hat{t}} \beta^{s-t-1} u\left(c_s, (\hat{\theta}_s), \frac{y_s(\hat{\theta}_s)}{\theta_s} | \theta_t\right)\right\} \end{cases} \tag{48}$$

First and second derivative of utility are: $\boldsymbol{w}(\boldsymbol{\theta}) = \boldsymbol{\omega}(\boldsymbol{\theta}|\boldsymbol{\theta})$ and $\boldsymbol{w_2}(\boldsymbol{\theta}) = \boldsymbol{\omega_2}(\boldsymbol{\theta}|\boldsymbol{\theta})$.The value function takes form of:

$$\begin{cases} V_t(\hat{w},\hat{w}_2,\underline{\theta}) = \min_{c,y,w,w_2} \int_0^\infty \left(c(\theta)-y(\theta)+\frac{1}{R}V_{t+1}(w(\theta),w_2(\theta),\theta)\right) f_t(\theta|\underline{\theta})\, d\theta, s.t. \\ \dot{u}(\theta) = u_\theta\left(c(\theta),\frac{y(\theta)}{\theta}\right)+\beta w_2(\theta), \hat{w} = \int_0^\infty u(\theta) f_t(\theta|\underline{\theta})d\theta, \hat{w}_2 = \int_0^\infty u(\theta) f_{2,t}(\theta|\underline{\theta})d\theta \\ u(\theta) = u\left(c(\theta),\frac{y(\theta)}{\theta}\right)+\beta w(\theta) \end{cases} \quad (49)$$

Labor $\left(1-\tau_t^y(\theta_t)\right)$ and savings distortions $\left(1-\tau_t^s(\theta_t)\right)$ are defined as:

$$1-\tau_t^y(\theta_t) \equiv \frac{-u_l\left(c_t(\theta_t),\frac{y_t(\theta_t)}{\theta_t}\right)}{\theta_t u_c\left(c_t(\theta_t),\frac{y_t(\theta_t)}{\theta_t}\right)};\ 1-\tau_t^s(\theta_t) \equiv \frac{1}{\beta R}\frac{u_c\left(c_t(\theta_t),\frac{y_t(\theta_t)}{\theta_t}\right)}{E_t\left\{u_c\left(c_{t+1}(\theta_{t+1}),\frac{y_{t+1}(\theta_{t+1})}{\theta_{t+1}}\right)\right\}} \quad (50)$$

In the case of separable preferences, let $\varepsilon_t(\theta) \equiv \frac{u_{ll,t}(\theta)l_t(\theta)}{u_{l,t}(\theta)}$ is the inverse of Frisch elasticity of labor[4] , and $\sigma_t(\theta) \equiv -\frac{u_{cc,t}(\theta)c_t(\theta)}{u_{c,t}(\theta)}$ represents the intertemporal elasticity of substitution. Preferences are isoelastic: $u(c,l) = \frac{c^{1-\sigma}-1}{1-\sigma} - \frac{l^{1+\varepsilon}}{1+\varepsilon}$.Optimal tax rate here is:

$$\frac{\tau_t^y(\theta)}{1-\tau_t^y(\theta_t)} = (1+\varepsilon)\frac{1-F_0(\theta)}{\theta f_0(\theta)}\int_0^\infty \exp\left(\int_0^x \sigma_t(\tilde{x})\frac{\dot{c}(\tilde{x})}{c_t(\tilde{x})}d\tilde{x}\right)\left(1-\lambda_{1,t}\bar{\alpha}_t(x)u_{c,t}(x)\right) + \beta R\frac{\tau_t^y(\theta)}{1-\tau_t^y(\theta_t)}\frac{A_t(\theta)}{A_{t-1}}\frac{u_{c,t}(\theta)}{u_{c,t-1}}\psi_t(\theta)\ , t>0 \quad (51)$$

In previous expression: $A_t(\theta) = (1+\varepsilon)$; $B_t(\theta) = \frac{1-F_0(\theta)}{\theta f_0(\theta)}$; $C_t(\theta) = \left(\int_0^x \sigma_t(\tilde{x})\frac{\dot{c}(\tilde{x})}{c_t(\tilde{x})}d\tilde{x}\right)\left(1-\lambda_{1,t}\bar{\alpha}_t(x)u_{c,t}(x)\right)$; $D_t(\theta) = \frac{A_t(\theta)}{A_{t-1}}\frac{u_{c,t}(\theta)}{u_{c,t-1}}\psi_t(\theta)$where also : $\lambda_{1,t} = \int_0^\infty \frac{f_t(x)}{u_{c,t}(x)}dx$; $\bar{\alpha}_t(\theta) = \alpha(\theta)\ if\ t=0; \bar{\alpha}_t(\theta) = 1\ if\ t>0$ .In a case when $\sigma = 0$ and $t = 0$ previous optimal labor tax becomes:

$$\frac{\tau_t^y(\theta)}{1-\tau_t^y(\theta_t)} = (1+\varepsilon)\frac{1-F_0(\theta)}{\theta f_0(\theta)}\int_0^\infty (1-\alpha(x))\frac{f_0(x)dx}{1-F_0(\theta)} \quad (52)$$

And if $t>0$ then previous intratemporal components will be equal to zero ($A_t(\theta) = B_t(\theta) = C_t(\theta) = 0$ and optimal marginal tax rate will be qual to intertemporal component

$$\frac{\tau_t^y(\theta)}{1-\tau_t^y(\theta_t)} = \beta R\rho\frac{\tau_t^y(\theta)}{1-\tau_t^y(\theta_t)} \quad (53)$$

[4]The Frisch elasticity measures the relative change of working hours to 1% increase in real wage given the marginal utility of wealth λ.In the steady state benchmark model is given as: $\frac{\frac{dh}{h}}{dw/w} = \frac{1-h}{h}\left(\frac{1-\eta}{\eta}\theta-1\right)^{-1}$

In the case of nonseparable preferences between labor and consumption almost all principles as in the case with separable preferences hold, $\gamma_t(\theta) \equiv \frac{u_{c,l,t}(\theta) l_t(\theta)}{u_{c,t}(\theta)}$ represents the degree of complementarity between consumption and labor, and the MPC from after-tax income on the right upper tail of the distribution $\bar{x} = \lim_{\theta\to\infty} \frac{c_t(\theta)}{(1-\tau_t^y(\theta) y_t(\theta)}$.Labor distortions are :

$$\begin{cases} A_t(\theta) = (1 + \varepsilon(\theta) - \gamma_t(\theta)) \\ C_t(\theta) = \int_\theta^\infty \exp\left(\int_\theta^x \left[\sigma_t(\tilde{x}) \frac{\dot{c}(\tilde{x})}{c_t(\tilde{x})} - \gamma_t(\tilde{x}) \frac{\dot{y}_t(x)}{y_t(\tilde{x})}\right] d\tilde{x}\right)\left(1 - \lambda_{1,t}\bar{\alpha}_t(x) u_{c,t}(x)\right)\frac{f_t(x)dx}{1 - F_t(\theta)} \\ D_t(\theta) = \frac{A_t(\theta)}{A_{t-1}} \frac{u_{c,t}(\theta)}{u_{c,t-1}} \frac{\theta_{t-1}\int_\theta^\infty \exp(-\int_\theta^x \gamma_t(\tilde{x})\frac{d\tilde{x}}{\tilde{x}})\, f_{2,t} * x) dx}{\theta f_t(\theta)} \end{cases} \quad (54)$$

Now about the income and substitution effects, let $\varepsilon_t^u(\theta), \varepsilon_t^c(\theta)$ be the compensated and uncompensated elasticities and the income effect is $\eta_t(\theta) = \varepsilon_t^u(\theta) - \varepsilon_t^c(\theta)$,now we can rewrite labor distortions $A_t(\theta), C_t(\theta)$:

$$\begin{cases} A_t(\theta) = \frac{1 + \varepsilon_t^u(\theta)}{\varepsilon_t^c(\theta)} \\ C_t(\theta) = \int_\theta^\infty \exp\left(g_t; (x; \theta)\right)\left(1 - \lambda_{1,t}\bar{\alpha}_t(x) u_{c,t}(x)\right)\frac{f_t(x)dx}{1 - F_t(\theta)} \end{cases} \quad (55)$$

$g_t = \int_\theta^x \left\{\frac{-\eta_t(\tilde{x})}{\varepsilon_t^c(\tilde{x})} \frac{\dot{y}_t}{y_t}\tilde{x} - \sigma_t(\tilde{x}) \frac{\left(1-\tau_t^y(\tilde{x})\right)\dot{y}_t - \dot{c}_t}{c_t}\tilde{x}\right\} d\tilde{x}$, $A_t(\theta), C_t(\theta)$ are similar in their dependence on $\varepsilon_t^u(\theta), \varepsilon_t^c(\theta)$ as in Saez (2001). Preferences here are given as in Greenwood, Hercowitz,, Huffman (1988): $u(c, l) = \frac{1}{1-v}\left(c - \frac{1}{1+\varepsilon} l^{1+\frac{1}{\varepsilon}}\right)$.Labor distortions here are given as:

$$\frac{\tau_t^y(\theta)}{1 - \tau_t^y(\theta_t)} \sim \left[a \frac{1}{1 + \frac{1}{\varepsilon}} - \varepsilon \frac{-\bar{\sigma}(1 - \bar{x})}{\bar{x}}\right]^{-1} ; \theta \to \infty \quad (56)$$

3.1 Dynamic Mirrlees taxation: two period example

Government computes allocations subject to IC constraints and then implicit taxes are inferred from the resulting wedges between marginal rates of substitution (MRS) and marginal rates of transformation (MRT). Assumption of the model here are:

1. Workers are heterogenous plus random
2. The government does not observe individual skills, but it knows the distribution of skills *apriori*
3. There are no *apriori* restrictions on fiscal policy *e.g. lump-sum taxes are available -possible
4. Government can commit

5. Preferences are separable between consumption and leisure (government should be able to observe marginal utility of consumption)
6. There is no aggregate uncertainty

Without aggregate uncertainty perfect consumption insurance is possible (everybody gets the same consumption). However, if government cannot observe the skills.Assumptions here are:

1. $\exists$ continuum of workers who live in 2 period and the maximization problem is :

$$\max E(u(c_1) + v(n_1) + \beta[u(c_2) + v(n_2)] \qquad (57)$$

2. Skills production is :$y = \theta \cdot n$

y represents observable output,θ are skills, n is effort/labor. Furthermore: θ_i is only observed by the agent i at the beginning of period,$\Pi_1(i)$ represents period 1 distribution of skills, and here $\Pi_2(j|i)$ is the conditional distribution of skills 2. Government maximization problem is given as:

$$\max_{\substack{c_1(i),c_2(i)\\ y_1(i),y_2(i)}} \sum_i \left\{u(c_1, l_{i,j} + v\left(\frac{y_1(i)}{\theta_1(i)}\right) + \beta \sum_j \left[u(c_2, l_{ij}) + v\left(\frac{y_2(i)}{\theta_2(i)}\right)\right]\right\}\Pi_2(j|i)\Pi_1(i) \text{s.t.}$$

1) Resource constraint :

$$\sum_i \left\{[c_i, l_{i,j} + \frac{1}{R}\sum_j c_2, l_{ij}\Pi_2(j|i)]\ \Pi_1(i)\right\} + G_1 + \frac{1}{R}G_2 \leq \sum_i \left[y_1(i) + \frac{1}{R}\sum_j y_2(i,j)\Pi_2(j|i)\right]\Pi_1(i) + Rk_1 \qquad (58)$$

2) Incentive compatibility constraints are given below:

$$u(c_1 l_{ij}) + v\left(\frac{y_i l_{ij}}{\theta l_{ij}}\right) + \beta\sum_i\left[u(c_2, l_{i,j}) + v\left(\frac{y_2(i,j)}{\theta_2(i,j)}\right)\right]\Pi_2(j|i) \geq u\left(c_1 l(i_r) + v\left(\frac{y_i l(i_r)}{\theta(i)}\right) + \beta\sum_j (u(c_2(i_r, j_r)) + v\left(\frac{y_2(i_r, j_r)}{\theta_2(i,j)}\right)\Pi_2(j|i)\right) \qquad (59)$$

3. Revelation principle: Government asks what your skill is and allocates consumption plus labor contingent on your answer. So now here we have i_r-which denotes first-period skills report (which depends on realized i) and j_r-which represents the 2nd period skills report (which depends on realized j). Characterization of optimum

Let's consider the following simple variational argument:

1) Fix a 1st period realization i and a hypothetical optimum $c_1^*(i), c_2^*(i)$.
2) Increase 2nd period utility uniformly across 2nd period realizations :$u(\tilde{c}_2(i,j;\Delta) \equiv u(c_2^*(i,j)) + \Delta$
3) Hold total utility constant by decreasing 1st period utility by $\beta\Delta$: $u(\tilde{c}_1(i,j,\Delta)) = u(c_1^*(i)) - \beta\Delta$
4) Note that this variation does not affect IC constraint and only the resource constraint is potentially affected.
5) Therefore, for $c_1^*(i); c_2^*(i)$ to be optimal, $\Delta = 0$ must minimize resources expended on the allocation.

One can express the resource costs of the perturbed allocation as follows:

$$\tilde{c}_i(i;\Delta) + R^{-1}\sum_j \tilde{c}_2(i,j,\Delta)\Pi(j|i) = u^{-1}(u(c_1(i) - \beta\Delta) + R^{-1}\sum_j u^{-1}(u(c_2(i,j) + \Delta)\Pi(j|i)$$

FOC evaluated at $\Delta = \mathbf{0}$ is as follows:

$$\frac{\mathbf{1}}{u'(c_1(i))} = \frac{\mathbf{1}}{\beta R}\sum_j \frac{\mathbf{1}}{u'(c_2(i,j))}\Pi_2(j|i) \qquad (60)$$

Previous equation is inverse Euler equation, $\frac{d}{dx}f^{-1}(x) = \frac{1}{f'(x)}$. We outline three cases as follows: 1. Skills observable $\Rightarrow u'(c_1) = \beta R u'(c_2)$;2.Skills unobservable $\Rightarrow u'(c_1) = \beta R u'(c_2)$ but not random constant overtimes,3.Skills observable plus random: $\frac{1}{u'(c_1)} = \frac{1}{\beta R}E\left[\frac{1}{u'(c_2)}\right] > \frac{1}{\beta R}\frac{1}{Eu'(c_2)} \Rightarrow u'(c_1(i)) < \beta R\, E[u'(c_2(i,j))] \Rightarrow \tau_k > 0$.Previous is Jensen's inequality. Intuition here is that savings affects incentive to work, so government needs to discourage savings to prevent the flowing deviation by highly-skilled: 1) save more today; 2) work less tomorrow. Some other features of optimal fiscal policy are:

1) On average wealth taxes across individuals are zero ex-ante
2) However, they depend on future labor income-if labor income is below average, your capital tax is positive. If your labor income is above average, then your capital tax is negative.
3) So this tax or this fiscal policy might be regressive for incentive reasons

The fact that the capital tax varies in this regressive way makes investment risky and creates a positive risk premium[5]. This explains how $\tau_k > 0$

4. Arguments against capital taxation

The argument against taxing income from capital most often rests on two results one of them is Atkinson and Stiglitz (1976) demonstrated the following theorem known as Atkinson, Stiglitz theorem:

Theorem: Commodity taxes cannot increase social welfare if utility functions are weakly separable in consumption goods versus leisure and the subutility of consumption goods is the same across individuals, i.e.,$u_i(c_1,\dots,c_k,w) = u_i(v(c_1,\dots,c_k),w)$ with the subutility function $v(c_1,\dots,c_k)$ homogenous across individuals.Laroque (2005) and Kaplow (2006) provided simple proof.Atkinson-Stglitz (1976) employ nonlinear taxation in a static model in which individuals have utility defined over a number of consumer goods and labor. Hence, theyprove that if labor is weakly separable from all the consumer goods, then the consumer goods should be taxed at the same rate. Consequently, only labor income need be taxed. Under weak separability, the government cannot achieve any distributional goals with differential taxation of the consumer goods that it cannot achieve with a tax on labor.The second result is due separately to Kenneth Judd (1985) and Christophe Chamley (1986).

[5]The risk premium is the rate of return on an investment over and above the risk-free or guaranteed rate of return. To calculate risk premium, investors must first calculate the estimated return and the risk-free rate of return.

Theorem: In a representative consumer, infinite horizon Ramsey model with linear taxes, the tax on income from capital should be zero in the long run.
The intuition here is that $(1+r) = MRT$ while $(1+r)(1-\tau) = \sum_{i-1}^{n} MRS$, and time period in that economy is $t = 1, \dots .T$ since $\frac{\sum_{i-1}^{n} MRS}{MRT} = 1$,$T \to \infty \to \frac{\sum_{i-1}^{n} MRS}{MRT} = 1$ i.e. $\left[\frac{1+r}{(1+r)(1-\tau)}\right]^{T}$ becomes large as $T \to \infty$ .We may dismiss the Atkinson/Stiglitz result on the grounds that labor is almost certainly not weakly separable,but the Judd/Chamley result is more difficult to ignore. But it is not clear why the relationship between consumption today and in the distant future should be so heavily distorted (Tresch (2014)).

5. Numerical solution of linear and nonlinear top-earners marginal tax rates

Here we are utilizing this equation :$\tau = \frac{1-\bar{g}}{1-\bar{g}+e}$. The first column of the table follows realistic scenario with elasticity of range $e = 0.25$, as in Saez et al., (2012) and Chetty, (2012) , and Piketty, Saez (2013).The second column is with estimates in range $e = 0.5$ which is high range elasticity scenario and a third scenario is $e = 1$ which is well above estimates in the current literature.

Table 2 Linear optimal tax rates per Piketty, Saez (2013)

	$e = 0.25$		$e = 0.5$		$e = 1$	
	$\bar{g}$	τ	$\bar{g}$	τ	$\bar{g}$	τ
Rawlsian revenue maximizing rate	0	0.8	0	0.67	0	0.50
Utilitarian CRRA=1 $u_c = \frac{1}{c}$	0.61	0.61	0.54	0.48	0.44	0.36
Median voter I $\frac{w_{median}}{w_{average}}$	0.7	0.55	0.7	0.38	0.7	0.23
Median voter II $\frac{w_{median}}{w_{average}}$	0.75	0.50	0.75	0.33	0.75	0.20
very low tax country 10%	0.97	0.1	0.94	0.1	0.88	0.1
low tax country 35%	0.87	0.35	0.807	0.35	0.46	0.35
high tax country 50%	0.75	0.5	0.5	0.5	0	0.5

Source: Author's calculation

The first row of table 1 is Rawlsian criterion with $\bar{g} = 0$. The second row is utilitarian criterion with coefficient of risk aversion (CRRA) equal to one and social marginal welfare weights are proportional to $u_c = \frac{1}{c}$ where $c = (1-\tau)w + R$ where R is disposable income. Chetty (2006) proved and showed that $CRRA = 1$ is consistent with empirical labor supply behavior and that is a reasonable benchmark. First scenario with $e = 0.25$ shows that revenue maximizing tax rate is 80% which is higher even for the countries with highest marginal tax rate which is around 50%. The optimal tax rate under Utilitarian criterion is 61%. The optimal tax rate for median earner is 55% or 38% under

$e = 0.5$ and 36% under $e = 1$. In the examples with very low tax country one can see that a tax rate of 10% is optimal in a situation where $g = 0.97$i.e. in a country with very low redistributive tastes. A tax rate of 50% would be optimal in a country with $\bar{g} = 0.75$. A high elasticity estimate $e = 0.5$ would generate tax rate of 67% above current rates in every country. The median voter tax rate in such a situation would be 38%, Utilitarian criterion generate tax rate of 48% in this situation. In the unrealistically high elasticity scenario $e = 1$ the revenue maximizing tax rate is 50% which is about the current rate in countries with highest $\frac{Tax}{GDP}$ ratios.

Example 1 with non-linear taxes

Here we are using this exact formula for calculation: : $\bar{\tau} = \frac{1-\bar{g}}{1-\bar{g}+\bar{\varepsilon}^u+\bar{\varepsilon}^c(\alpha-1)}$,and we get table that consists of three global columns with supposed elasticities (uncompensated) $\varepsilon_u \in (0, 0.2, 0.5)$ and supposed compensated elasticities $\varepsilon_c \in (0.2, 0.5, 0.8)$.

Table 3 Non-linear income taxes under different uncompensated and compensated elasticities

	$\varepsilon_u = 0$			$\varepsilon_u = 0.2$			$\varepsilon_u = 0.5$		
$\varepsilon_c =$	0.2	0.5	0.8	0.2	0.5	0.8	0.2	0.5	0.8
$\bar{g} = 0$									
a=1.5	0.91	0.80	0.71	0.77	0.69	0.63	0.63	0.57	0.53
a=2	0.83	0.67	0.56	0.71	0.59	0.50	0.59	0.50	0.43
a=2.5	0.77	0.57	0.45	0.67	0.51	0.42	0.56	0.44	0.37
$\bar{g} = 0.25$									
a=1.5	0.88	0.75	0.65	0.71	0.63	0.56	0.56	0.50	0.45
a=2	0.79	0.60	0.48	0.65	0.52	0.96	0.52	0.43	0.37
a=2.5	0.71	0.50	0.38	0.60	0.44	0.35	0.48	0.38	0.31
$\bar{g} = 0.5$									
a=1.5	0.83	0.67	0.56	0.63	0.53	0.45	0.45	0.40	0.36
a=2	0.71	0.50	0.38	0.56	0.42	0.33	0.42	0.33	0.28
a=2.5	0.63	0.40	0.29	0.50	0.34	0.26	0.38	0.29	0.23
$\bar{g} = 0.75$									
a=1.5	0.71	0.50	0.38	0.45	0.36	0.29	0.29	0.25	0.22
a=2	0.56	0.33	0.24	0.38	0.26	0.20	0.26	0.20	0.16
a=2.5	0.45	0.25	0.17	0.33	0.21	0.15	0.24	0.17	0.13

Source: Author's calculation

Another example with non-linear U-shaped taxes as per Diamond (1998).The formulae that we are using here is :$\tau' = \frac{(e^{-1}+1)(1-g)}{[a+(e^{-1}+1)(1-g)]}$

Table 3Non-linear income tax rates as per Diamond (1998) and authors own calculations

	$g = 0$			$g = 0.25$			$g = 0.5$			$g = 0.975$		
a= e	0.5	1.5	5	0.5	1.5	5	0.5	1.5	5	0.5	1.5	5
0.2	0.9	0.8	0.5	0.9	0.7	0.4	0.8	0.6	0.3	0.2	0.09	0.0

	2		5	0	5	7	6	7	8	3		3
0.5	0.86	0.67	0.38	0.82	0.60	0.31	0.75	0.50	0.23	0.13	0.05	0.01
0.75	0.82	0.61	0.32	0.78	0.54	0.26	0.70	0.44	0.19	0.10	0.03	0.01
1	0.80	0.57	0.29	0.75	0.50	0.23	0.67	0.40	0.17	0.09	0.03	0.01
1.5	0.77	0.53	0.25	0.71	0.45	0.20	0.63	0.36	0.32	0.08	0.03	0.01
2	0.75	0.50	0.23	0.69	0.43	0.18	0.60	0.33	0.13	0.07	0.02	0.01

Source: Author's calculation

Form previous table one can see that highest non-linear income taxes are generated with high tastes for redistribution where $g = 0$ and Pareto shape parameter $\alpha = 0.5$ and with labor elasticity $e = 0.2$.Generated tax rates are $\tau \in (0.92, 0.88, 0.55)$ for Pareto shape parameters $\alpha \in (0.5, 1.5, 5)$.For the same elasticities and Pareto shape parameters but with very low almost non-existent redistributive tastes generated low tax rates are: $\tau \in (0.23, 0.09, 0.03)$ respectively. On a very high (unrealistically high) labor elasticities generated are tending to zero $\tau \to 0$ .Next original Mirrlees (1971) paper main result has been simulated.First we start with two graphs presenting the original Mirrleesian taxation idea.

Figure 1 Mirrleesian taxation: consumption and earnings schedule

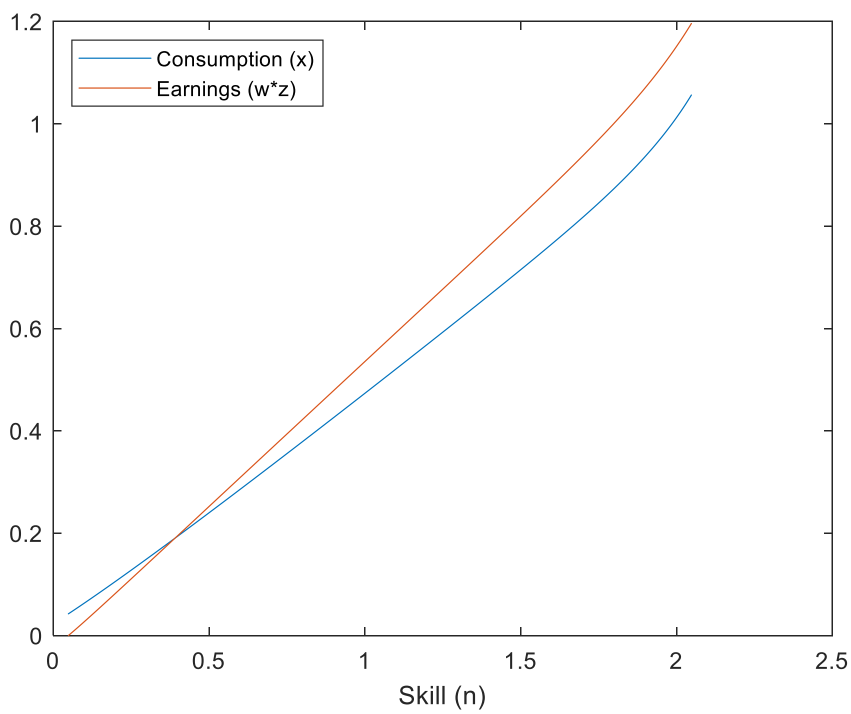
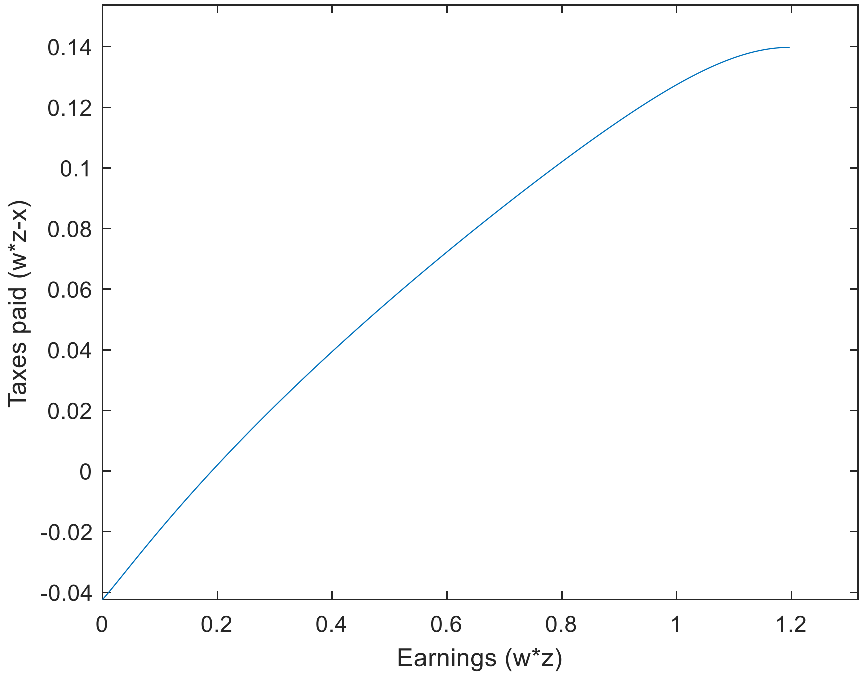


Figure 2Mirrleesian taxation: taxes and earnings schedule



In previous two figures we can see the schedules of taxes and earnings as well as skills and earning in the Mirrlees taxation model. What do studies tell? The compensated elasticity of labor supply with respect to real wage ε_w^* has been estimate approximately to be 0.5 see Gruber, Saez (2002). Gruber, Saez (2002) estimate that for the US taxpayer with incomes above 100K$ have elasticity around 0.57. And those <100K$ have elasticity around 0.2 or even less.Next in table 4 FOC's for the Mirrlees model are presented.

Table 4FOC's for the Mirrlees model

iteration	Func-count	f(x)	Norm of step	First-order optimality
0	3	1.37E-01		
1	6	9.01E-04	0.000224	0.00276
2	9	2.13E-04	2.97E-01	0.000677
3	12	5.02E-08	4.86E-01	9.93E-06
4	15	2.87E-14	6.74E-03	7.50E-09

In the next table 5 skills and consumption of agents that previously were depicted graphically are presented.

Table 5skills, consumption and earnings for the Mirrlees model

F(n)-skills	x-cons.	y-income	x(1-y)	z-earnings
0	0.0424	0	0.0424	0
0.1	0.116	0.3894	0.0708	0.0869
0.5	0.18	0.4382	0.1011	0.1612
0.9	0.2888	0.4686	0.1535	0.2842

0.99	0.4315	0.4841	0.2226	0.4412

Table 6 depicts earning schedule, consumption, average tax rate and marginal tax rate correspondingly.

Table 6average and marginal tax rates for Mirrlees model

z-earnings	x-consumption	average tax rate	marginal tax rate
0	0.0424	-Inf	0.2147
0.05	0.0847	-0.54	0.2336
0.1	0.1271	-0.1558	0.2223
0.2	0.214	0.0273	0.1993
0.3	0.3031	0.0817	0.1824
0.4	0.3937	0.1052	0.1698
0.5	0.4856	0.1171	0.1599

Optimal mirrleesian taxation is flat for a long range of top incomes >1.

6. Conclusion

Optimal tax rates as this paper shows depend on redistributive tastes of the supposedly benevolent social planers. The marginal social welfare weight on a given individual measures the value that society puts on providing an additional dollar of consumption to this individual.As the numerical solutions in the non-linear optimal tax rates showed that high tax rates are obtained when there unrealistically low uncompensated and compensated elasticities, also the shape parameter of Pareto distribution must be lower. For high tax countries e.g. countries with highest tax burden around 50% the area that provides such high tax rates is where compensated elasticity is between 0.2 and 0.5 and uncompensated elasticity and unrealistically high compensated elasticity between 0.5 and 0.8 but medium redistributive tastes $\bar{g} = 0.5$. Or alternatively, if uncompensated elasticity is high $\varepsilon_u = 0.5$ than also the taste for redistribution must be high e.g. $\bar{g} \in (0, 0.25)$.For low tax countries the area where those taxes are provided is in high Pareto distribution parameter and very low taste for redistribution. These are very loose results and are conditioned by themselves and their combinations. In turn there is not straightforward solution to the optimal linear or non-linear labor income tax problem. Pareto efficient tax rates differ from those proposed by Mirrlees (1971). In the dynamic Mirrlees approach, when it comes to the result for capital, capital is taxed to provide more efficient labor supply incentives when there is imperfect information (private distributions of ability unknown to other parties) and as a part of optimal insurance scheme against stochastic earning abilities. Intuition here is that savings affects incentive to work, so government needs to discourage savings to prevent the flowing deviation by highly skilled: 1) save more today; 2) work less tomorrow. That was the second model we reviewed and from there some optimal fiscal policy features are:1) On average wealth taxes across individuals are zero ex-ante ;2) However, they depend on future labor income-if labor income is below average, your capital tax is positive. If your labor

income is above average, then your capital tax is negative. 3) So, this tax or this fiscal policy might be regressive for incentive reasons.So, in general about dynamic Mirrlees approach it can beconcluded that: this approach assumes that agents' abilities to earn income are heterogeneous, stochastic, and private information. Tax instruments ex ante are unrestricted. The model solves for the optimal allocations using dynamic mechanism design (subject only to incentive compatibility constraints) and then considers how to implement these allocations using decentralized tax systems, see alsoStantcheva (2020).This story also has normative element into it. Namely we must not forget principles of horizontal and vertical equity according to neo-classical economics defined by Feldstein (1976) when we define tax systems and marginal tax rates.Feldstein's Horizontal Equity Principle: Two people
with the same utility before tax must have the same utilityafter tax and Feldstein's Vertical Equity Principle (No Reversals): Ifperson i has greater utility than another person j beforetax, then person i must have greater utility than person j after tax.Feldstein's no-reversals principle has important efficiency implications in asecond-best world of imperfect information in which the governmentmight not know how well-off certain people are, and they may have powerful incentive to hide private information about themselves, if the tax laws permitted reversals of utility

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