HETEROGENOUS AGENTS AND INCOMPLETE MARKETS: AN EXPLORATION

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Abstract

This paper will review the issue of heterogeneity of agents and incomplete markets in macroeconomics. Central idea of this paper is the notion that representative agent models were wrong turn for modern macroeconomics especially for general equilibrium model (some individuals are some are not liquidity constrained) and that central problems of macroeconomics cannot arise in representative agent models (debt, bankruptcy, asymmetric information) as has also being criticized by Stiglitz (2017).And finally main motivation for this paper were Achdou et al.(2022) who developed algorithm for "solving equilibria in Aiyagari–Bewley–Huggett economy" and Krusell-Smith (1998) comparison of economy behavior when incomplete markets (heterogeneous agents) and complete markets (representative agent) economy.

Key words: *Heterogenous agents, incomplete markets, Huggett economy, Krusell-Smith economy, Aiyagari model*

JEL Classification: D14, D31, E21

INTRODUCTION

Heterogeneity of agents is relevant, and it could provide answers for the welfare questions that are crucial in macroeconomics. In a way it is a critique on representative agents' models. Lucas (1987) showed that for standard preferences, aggregate fluctuations have a very small impact on the welfare of a representative consumer. There are many heterogenous agents (HA) models, however in this paper we will stick to Huggett model (Huggett (1993)), and Krusell-Smith model (Krusell,Smith (1998)).Huggett model was based on the enormous literature that up until then was done on "….heterogenous-agent-incomplete-insurance models of asset pricing….", some of the references here include : Bewley (1980), Lucas (1980), Taub (1988). Models with heterogeneous agents have become a workhorse in macroeconomics since the seminal work of Bewley (1986), Hopenhayn (1992), Huggett (1993) and Aiyagari (1994). More complete review of this literature could be read in Heathcote et al.(2009).And why heterogeneity of agents is of interest in macroeconomics? This same question is asked and answered partially by Boppart et al. (2018).Marginal decisions made by households, regarding: consumption, hours worked, and investments in various types of assets "vary quite substantially" in population. As an example, study of previous Boppart et al. (2018) mentions: Johnson, Parker, Souleles (2004), who provide evidence on departure from permanent income hypothesis when agents are heterogenous, and Misra and Surico (2014) "for estimating the heterogeneity in responses across households". Real world problems such as inequality see Piketty (2014), and theoretical problems such as optimal taxation with heterogenous agents see Chien,Wen (2020),Ragot,Grand(2017), Bassetto et al.(1999),Brito etal.(1995),Stiglitz(1982),Arnott,Stiglitz(1988),Akerlof(1978),Diamond,Mirrlees(1978),Weiss (1976). Storesletten et al. (2001) showed that liquidity-constrained households are hit particularly hard by aggregate productivity shocks. Arrow (1951) and Arrow,Debreu (1954),proved that competitive equilibrium in Arrow-Debreu economy is Pareto optimal and discovered class of convex Arrow-Debreu economies for which competitive equilibria always exist. In the case of incomplete (see Geanakoplos (1990)) markets this equilibrium may (will) not be efficient see Geanakoplos (1986) or the will be suboptimal constrained. This paper will review previously mentioned issues will be doing so by using derivations and some examples from modern macroeconomic literature such as Achdou et al.(2022), which is the main paper that motivated as to review this area of macroeconomics.

MATERIAL AND METHODS

Hamilton-Jacobi Equation (HJB), Fokker-Planck equation (F-P) and Huggett economy

A crucial question here is how to model income. First, income can be modeled as a Poisson process, that allows income to take two values. Second, income can be modeled as a diffusion process, allowing that the income to take many values. In particular, the case in which the income process follows a two-stage Poisson process: $y_t \in \{ y_1, y_2 \}$ with $y_1 < y_2$. Here the income jumps from state 1 to state 2 with intensity λ_1 and vice versa with intensity λ_2 .Now, how does consumers in this model chooses optimal consumption? They maximize the lifetime present value utility function subject to the dynamic of individual wealth, the borrowing constraint, and the income process. When the agent solves his optimization problem, he takes as given the evolution of the equilibrium of the interest rate. The underlying assumption is that the agent is a price-taker. The next step is to set up and solve the equilibrium of this economy. The equilibrium is represented by a system of partial differential equations (PDEs). To solve this PDEs system, we need first to solve the Hamilton-Jacobi-Bellman equation (HJB) given an interest rate, and then to solve the Fokker-Planck equation (KFPE), and hence the equilibrium in the bond market. Now, we can update the value of the interest rate and start the loop again until we find the equilibrium interest rate. From these equations, we can find the consumption and savings policy functions and the stationary distribution of wealth. HJB equation was a result of the theory of dynamic programing pioneered by Richard Bellman (namely Bellman(1954),Bellman(1957),Bellman, Dreyfus,(1959)).HJB equation is modeled as in Achdou et al.(2022). The deterministic optimal control problem is given as:

equation 1

$$V(x_0) = \max_{u(t)_{t=0}^{\infty}} \int_0^{\infty} e^{-\rho t} h(x(t), u(t) dt \text{ s.t. } \dot{x}(t) = g\big(x(t)\big), u(t), u(t) \in U\,; t \geq 0, x(0) = x_0$$

In previous expression:$\rho \geq 0$ is the discount rate, $x \in X \subseteq \mathbb{R}^m$ is a state vector; $u \in U \subseteq \mathbb{R}^n$ is a control vector, and $h: X \times U \to R$. The value function of the generic optimal control problem satisfies the Hamilton-Jacobi-Bellman equation, i.e.:

equation 2

$$\rho V(x) = \max_{u \in U} h(x, u) + V'(x) \cdot g(x, u)$$

In the case with more than one state variable $m > 1$, $V'(x) \in \mathbb{R}^m$ is the gradient of the value function. Now for the derivation of the discrete-time Bellman eq. we have: time periods of length Δ ,discount factor $\beta(\Delta) = e^{-\rho\Delta}$, here we can note that $\lim_{\Delta \to \infty} \beta(\Delta) = 0$ and $\lim_{\Delta \to 0} \beta(\Delta) = 1$.Now that discrete Bellman equation is given as:

equation 3

$$V(k_t) = \max_{c_t} \Delta U(c_t) + e^{-\rho\Delta} V(k_{t+\Delta}) \text{ s.t. } k_{t+\Delta} = \Delta[F(k_t) - \delta k_t - c_t] + k_t$$

For a small $\Delta = 0$ we can make: $e^{-\rho\Delta} = 1 - \rho\Delta,$ so that $V(k_t) = \max_{c_t} \Delta U(c_t) + (1 - \rho\Delta,)V(k_{t+\Delta})$,if we subtract $(1 - \rho\Delta,)V(k_t)$ from both sides and divide by Δ and manipulate the last term we get : $\rho V(k_t) = \max_{c_t} \Delta U(c_t) + (1 - \rho\Delta,)[V(k_{t+\Delta}) - V(k_t)\,]$ we get :

equation 4

$$\rho V(k_t) = \max_{c_t} \Delta U(c_t) + (1 - \rho\Delta,)\frac{[V(k_{t+\Delta}) - V(k_t)\,]}{k_{t+\Delta} + k_t}\,\frac{k_{t+\Delta} - k_t}{\Delta}$$

If $\Delta \to 0$ then $\rho V(k_t) = \max_{c_t} \Delta U(c_t) + V'(k_t)\ \dot{k}_t$. Hamilton-Jacobi-Bellman equation in stochastic settings is given as:

equation 5

$$V(x_0)) = \max_{u(t)_{t=0}^{\infty}} \mathbb{E}_0 \int_0^\infty e^{-\rho t} h\big(x(t), u(t)\big)dt \text{ s.t.} dx(t) = g\big(x(t), u(t)\big)dt + \sigma\big(x(t)\big)dW(t)\,, u(t) \in U\,; t \geq 0, x(0) = x_0$$

In previous expression $x \in \mathbb{R}^m; u \in \mathbb{R}^n$. HJB equation without derivation is :

equation 6

$$\rho V(x) = \max_{u \in U} h(x, u) + V'(x)g(x, u) + \frac{1}{2}V''(x)\sigma^2(x)$$

In the multivariate case: for fixed x we define $m \times m$ covariance matrix, $\sigma^2(x) = \sigma(x)\sigma(x)'$ which is a function of $\sigma^2 \colon \mathbb{R}^m \to \mathbb{R}^m \times \mathbb{R}^m$. HJB equation now is given as:

equation 7

$$\rho V(x) = \max_{u \in U} h(x, u) + \sum_{i=1}^{m} \frac{\partial V(x)}{\partial x_i} g_i(x, u) + \frac{1}{2} \sum_{i=1}^{m} \sum_{j=1}^{m} \frac{\partial^2 V(x)}{\partial x_i \partial x_j} \sigma_{ij}^2(x)$$

In vector notation previous is given as:

equation 8

$$\rho V(x) = \max_{u \in U} h(x, u) + \nabla_x V(x) \cdot g(x, u) + \frac{1}{2} tr\big(\Delta_x V(x)\sigma^2(x)\big)$$

Where $\nabla_x V(x)$: gradient of V (dimension $m \times 1$) ; $\Delta_x V(x)$: Hessian matrix of V (dimension $m \times m$).By Ito's lemma[1]:

equation 9

$$df(x) = \left(\sum_{i=1}^{n} \mu_i(x) \frac{\partial f(x)}{\partial x_i} + \frac{1}{2} \sum_{i=1}^{m} \sum_{i=1}^{m} \sigma_{ij}^2(x) \frac{\partial^2 f(x)}{\partial x_i \partial x_j}\right) dt + \sum_{i=1}^{m} \sigma_I(x) \frac{\partial f(x)}{\partial x_i} dW_i$$

In vector notation:

equation 10

$$df(x) = \left(\nabla_x f(x) \cdot \mu(x) + \frac{1}{2} tr\big(\Delta_x f(x)\sigma^2(x)\big)\right) dt + \nabla_x f(x) \cdot \sigma(x) dW$$

[1] Itô's lemma is an identity used in Itô calculus to find the differential of a time-dependent function of a stochastic process. It serves as the stochastic calculus counterpart of the chain rule, see Kiyosi Itô (1951).

Now for the Kolmogorov Forward (Fokker-Planck[2]) equation we have folowing: let x be a scalar diffusion

equation 11

$$dx = \mu(x)dt + \sigma(x)dW, x(0) = x_0$$

Let's suppose that we are interested in the evolution of the distribution of $x, f(x,t)$ and $\lim_{t\to\infty} f(x,t)$ .So, given an initial distribution $f(x,0) = f_0(x), f(x,t)$ satisfies PDE :

equation 12

$$\frac{\partial f(x,t)}{\partial t} = -\frac{\partial}{\partial x}[\mu(x)f(x,t)] + \frac{1}{2}\frac{\partial^2}{\partial x^2}[\sigma^2(x)f(x,t)]$$

Previous PDE is called "Kolmogorov Forward Equation" or "Fokker-Planck Equation".

Corollary 1: if a stationary equilibrium exists $\lim_{t\to\infty} f(x,t) = f(x)$, it satisfies ODE

equation 13

$$0 - \frac{d}{dx}[\mu(x)f(x)] + \frac{1}{2}\frac{d^2}{dx^2}[\sigma^2(x)f(x)]$$

In the multivariate case Kolmogorov Forward Equation is given as:

equation 14

$$\frac{\partial f(x,t)}{\partial t} = -\sum_{i=1}^{m}\frac{\partial}{\partial x_i}[\mu(x)f(x,t)] + \frac{1}{2}\sum_{i=1}^{m}\sum_{j=1}^{m}\frac{\partial^2}{\partial x^2}[\sigma_{ij}^2(x)f(x,t)]$$

Finite difference method and HJB equation

As in Achdou et al.(2022), two functions v_1, v_2 at I discrete points in the space dimension a_i , $i = 1, \dots, I$. Equispaced grids are denoted by Δa_ias the distance by the grid points, and shot hand notation used is $v_{i,j} \equiv v_j(a_i)$ and so on. Backward difference approximation is given as:

equation 15

$$\begin{cases} v_j'(a_i) \approx \dfrac{v_{i+1,j} - v_{i,j}}{\Delta a} \equiv v_{i,j,F}' \\ v_j'(a_i) \approx \dfrac{v_{i+1,j} - v_{i-1,j}}{\Delta a} \equiv v_{i,j,B}' \end{cases}$$

Two basic equations to explain Huggett economy are :

equation 16

$$\begin{cases} \rho v_1(a) = \max_c u(c) + v_1'(a)(z_1 + ra - c) + \lambda_1\big(v_2(a) - v_1(a)\big) \\ \rho v_2(a) = \max_c u(c) + v_2'(a)(z_2 + ra - c) + \lambda_2\big(v_1(a) - v_2(a)\big) \end{cases}$$

Where $\rho \geq 0$ represents the discount factor for the future consumption c_t (Individuals have standard preferences over utility flows), a represents wealth in form of bonds that evolve according to :

equation 17

$$\dot{a} = y_t + r_t a_t - c_t$$

y_t is the income of individual, which is endowment of economy's final good, and r_t represents the interest rate. Equilibrium in this Huggett (1993) economy is given as:

[2] See Fokker (1914), Planck (1917), Kolmogorov (1931).

equation 18

$$\int_{\underline{a}}^{\infty} a g_1(a,t)da + \int_{\underline{a}}^{\infty} a g_2(a,t)da = B$$

Where in previous expression $0 \leq B \leq \infty$ and when $B = 0$ that means that bonds are zero net supply. So the finite difference method approx. to $\begin{cases} \rho v_1(a) = \max_c u(c) + v_1'(a)(z_1 + ra - c) + \lambda_1\big(v_2(a) - v_1(a)\big) \\ \rho v_2(a) = \max_c u(c) + v_2'(a)(z_2 + ra - c) + \lambda_2\big(v_1(a) - v_2(a)\big) \end{cases}$ is given as:

equation 19

$$\rho v_{i,j} = u(c_{i,j}) + v_{i,j}'(z_j + ra_i + c_{i,j}) + \lambda_j(v_{i,-j} - v_{i,j}), j = 1,2$$
$$c_{i,j} = (u')^{-1}(v_{i,j}')$$

Euler equation

Here following lemma applies see [Achdou et al.(2022)]

Lemma 1: The consumption and savings policy functions $c_j(a)$ and $s_j(a)$ for $j = 1,2$.. corresponding to HJB equation : $\rho v_j(a) = \max_c u(c) + v_j'(a)(y_j + ra - c) + \lambda_j\left(v_{-j}(a) - v_j(a)\right)$ which is maximized at : $0 = -\frac{d}{da}[s_j(a)g_J(a)] - \lambda_j g_j(a) + \lambda_{-j g_{-j}}(a)$ is given as:

equation 20

$$(\rho - r)u'\left(c_j(a)\right) = u''\left(c_J(a)\right)c_J'(a)s_j(a) + \lambda_j(u'\left(c_{-j}(a)\right) - u'\left(c_j(a)\right)$$
$$s_J(a) = y_J + ra - c_j(a)$$

Proof : differentiate $\rho v_j(a) = \max_c u(c) + v_j'(a)(y_j + ra - c) + \lambda_j\left(v_{-j}(a) - v_j(a)\right)$ with respect to a and use that $v_j'(a) = u'\left(c_j(a)\right)$ and hence $v_j''(a) = u''\left(c_j(a)\right)c_j'(a)$ ∎

The differential equation $\begin{matrix}(\rho - r)u'\left(c_j(a)\right) = u''\left(c_J(a)\right)c_J'(a)s_j(a) + \lambda_j(u'\left(c_{-j}(a)\right) - u'\left(c_j(a)\right) \\ s_J(a) = y_J + ra - c_j(a)\end{matrix}$ is and Euler equation , the right hand side$(\rho - r)u'\left(c_j(a)\right)$ is expected change of marginal utility of consumption $\frac{\mathbb{E}_t[du'(c_j(a_t))]}{dt}$.This uses Ito's formula to Poisson process:

equation 21

$$\mathbb{E}_t[du'(c_j(a_t)] = \left[u''(c_j(a_t)c_j'(a_t)s_j(a_t) + \lambda_j\left(u'(c_{-j}(a_t)\right) - u'\left(c_j(a_t)\right)\right]dt$$

So, this equation $\begin{matrix}(\rho - r)u'\left(c_j(a)\right) = u''\left(c_J(a)\right)c_J'(a)s_j(a) + \lambda_j(u'\left(c_{-j}(a)\right) - u'\left(c_j(a)\right) \\ s_J(a) = y_J + ra - c_j(a)\end{matrix}$ can be written in more standard form:

equation 22

$$\frac{\mathbb{E}_t[du'(c_j(a_t)]}{dt} = (\rho - r)dt$$

CARA utility with borrowing constraint

Assumption 1: The CARA coefficient $\mathcal{R}(c) := -\frac{u''(c)}{u'(c)}$ when wealth $a \to \underline{a}$ approaches lower borrowing limit is given as see Achdou et al.(2022):

equation 23

$$\underline{\mathcal{R}} :== \lim_{a \to \underline{a}} \frac{u''(y_1 + ra)}{u'(y_1 + ra)} < \infty$$

This is also known as the Arrow–Pratt measure of absolute risk aversion (ARA), after the economists Arrow (1965), and Pratt (1964).

Marginal propensity to consume (MPC) and Marginal propensity to save (MPS)

Definition 1: Marginal propensity to consume (MPC) is defined as:

equation 24

$$MPC_{j.\tau}(a) = c'_{j,\tau}(a) = \mathbb{E}\left[\int_0^\tau c_j(a_t)dt \,\middle|a_0 = a, y_0 = y_j)\right]$$

Similarly, MPS is given as $MPS_{j.\tau}(a) = s'_{j,\tau}(a) = \mathbb{E}\left[a_\tau \middle| a_0 = a, y_0 = y_j)\right]$

Assumption 2: if we define Euler equation and budget constraint as:

equation 25

$$\frac{\dot{c}}{c} = \frac{1}{\gamma}(r - \rho); \dot{a} = ra - c$$

We must remember that $u(c) = \frac{c^{1-\gamma}}{1-\gamma}; \gamma > 0$; so now savings and consumption are:

equation 26

$$\dot{a}(t) = -\eta a(t);$$
$$\eta := \frac{\rho - r}{\gamma}$$
$$c(t) = (r + \eta)a(t)$$

So the wealth is given as:

equation 27

$$a(\tau) = a_0 e^{-\eta t}, \tau \geq 0$$

CARA utility with upper borrowing constraint

Assumption 3. Here we are assuming that: $\rho < r\,; y_1 < y_2$ and that CRRA is given as:

equation 28

$$\mathcal{R}(c) = -\frac{cu''(c)}{u'(c)}$$

$\exists \overline{a} < \infty$ such that $s_j(a) < 0$; $\forall a \geq \overline{a}; j = 1,2..$and $s_2(a) \sim \psi_2(\overline{a} - a)$ as $a \to \overline{a}$ for some constant ψ. Asymptotic movement of wealth of some individual is given as :

equation 29

$$\dot{a}(\tau) = a(\tau) - \overline{a} \sim e^{-\psi_2 \tau}(a_0 - \overline{a})$$

In case $u(c) = \frac{c^{1-\gamma}}{1-\gamma}$ individual policy functions are linear in a. In the asymptotic case where $a \to \infty$ satisfy: $s_j(a) \sim \frac{r-\rho}{\gamma} a$, so $c_j(a) \sim \frac{\rho-(1-\gamma)r}{\gamma} a$.

First part of this proposition where $-\frac{cu''(c)}{u'(c)}$ is bounded above $\forall c$ rules out exponential utility function, $u(c) = -\frac{1}{\theta} e^{-\theta c}; \theta > 0$. This is like Aiyagari (1994). While the second part $s_j(a) \sim \frac{r-\rho}{\gamma} a$, $c_j(a) \sim \frac{\rho-(1-\gamma)r}{\gamma} a$ is same as in Benhabib,Bisin, Zhu (2015),see Achdou et al.(2022).

Now, since $\underline{a} = -\frac{y}{r}$,consumption and saving policy functions are given as:

equation 30

$$s(a) = \frac{r-\rho}{\gamma}\left(a + \frac{y}{r}\right) ;$$
$$c(a) = \frac{\rho-(1-\gamma)r}{\gamma}\left(a + \frac{y}{r}\right)$$

Krussel, Smith (1998) explain that linearity of consumption and saving policy functions with CRRA utility functions, explains their finding that the business cycle properties of baseline heterogeneous agent model are virtually indistinguishable from its representative agent counterpart. Now MPC and MPS will be given as: $MPS_\tau(a) = e^{-\eta\tau} \approx 1 - \eta\tau$ and $MPC_\tau(a) = 1 - e^{-\eta\tau} + \tau r \approx \tau(\eta + r), \eta \coloneqq \frac{\rho-r}{\gamma}$.

Lemma 2. The conditional expectation of consumption $c_{j,\tau}(a)$ defined previously as $c'_{j,\tau}(a) = \mathbb{E}\left[\int_0^\tau c_j(a_t)dt \,\middle|\, a_0 = a, y_0 = y_j\right]$ can be computed as $c_{j.\tau}(a) = \mathcal{P}_j(a, 0)$.In previous expression $\mathcal{P}_j$ satisfies system of two PDE's.

equation 31

$$0 = c_j(a) + \partial_a\mathcal{P}_j(a,\tau)s_j(a) + \lambda_j\left(\mathcal{P}_{-j}(a,\tau) - \mathcal{P}_j(a,\tau)\right) + \partial_\tau\mathcal{P}_j(a,\tau), j = 1{,}2..\mathcal{P}_j(a,\tau) = \forall a$$

Proof per Achdou et al.(2022) follows directly from application of Feynman-Kac formula for computing conditional expectations as solutions to PDE's. So, since $c'_{j,\tau}(a) = \mathbb{E}\left[\int_0^\tau c_j(a_t)dt \,\middle|\, a_0 = a, y_0 = y_j\right]$ and if A is infinitesimal generator (Feller process or Levy process, or Ornstein–Uhlenbeck process):

1. **Feller process**-Let E be a LCCB (locally compact with countable base) and $E \subset \mathbb{R}^n, \exists n \in N$ and $C_0(E) = C_0(E, \mathbb{R})$ be the space of continuous function that vanishes in inf. A Feller semigroup $C_0(E)$ is a family of positive linear operators $T_\tau, \tau \geq 0$ on $C_0(E)$
   - ✓ $T_0 = Id; \|T_\tau\|; \forall\tau \in T$ i.e. $\{T_\tau\}_{\tau\in T}$ is a family of contracting maps
   - ✓ $T_{\tau+s} = T_\tau \circ T_s$ (the semigroup property)
   - ✓ $\lim_{t\downarrow 0}\|T_\tau f - f\| \forall f \in C_0(E)$

See Revuz et al.(2005).

2. **Levy process**- L let be is an infinite divisible random variable $\forall t \in [0, \infty]$
   - ✓ L can be written as the sum of a diffusion, a continuous Martingale and a pure jump process; i.e:

equation 32

$$L_t = at + \sigma B_t + \int_{|x|<1} x d\tilde{N}_\tau + \int_{|x|\geq 1} x dN_\tau(\cdot, dx), \forall t \geq 0$$

In previous expression $a \in \mathfrak{R}$, B_t is the standard Brownian motion, N is defined to be the Poisson random measure of the Lèvy process

   - ✓ Lèvy -Khintchine formula: from the previous property it can be shown that for $\forall\tau \geq 0$ one has that :

equation 33

$$E\left|e^{inL_t}\right| = e\text{^}(-\tau\psi(u)$$
$$\psi(u) = -iau + \frac{\sigma^2}{2}u^2 + \int_{|x|\geq 1}(1 - e^{iux})dv(x) + \int_{|x|<1}(1 + e^{iux} + iux)dv(x)$$

$a \in \Re; \sigma \in [0, \infty); v > 0$ borel measure and σ is Lèvy measure. More so $v(\cdot) = E[N_1(\cdot, A)]$ See Applebaum (2004).

3. **Ornstein–Uhlenbeck process-** The Ornstein-Uhlenbeck process is a stochastic process that satisfies the following stochastic differential equation:

equation 34

$$dx_\tau = k(\theta - x_\tau)d\tau + \sigma dW_\tau$$

$k > 0$ is the mean rate of reversion; θ is the long term mean of the process, $\sigma > 0$ is the volatility or average magnitude, per square-root time, of the random fluctuations that are modelled as Brownian motions.

- ✓ Mean reverting property-where $dx_\tau = k(\theta - x)$:

equation 35

$$\frac{\theta - x_\tau}{\theta - x_0} = e^{-k(\tau-\tau_0)}, x_\tau = \theta + (x_0 - \theta)e^{-k(\tau-\tau_0)}$$

- ✓ Solution for $\forall \tau > s \geq 0$ is given as:

equation 36

$$x_\tau = \theta + (x_s - \theta)e^{-k(\tau-s)} + \sigma\int_s^\tau e^{-k(\tau-u)}dW_u$$

See Jacobsen.M(1996) .So now partial differential equation $\frac{\partial c_{j,\tau}}{\partial \tau} = Ac_{j,\tau}(a) - c_{j,\tau}(\underline{a})(a)$ is the solution to $c'_{j,\tau}(a) = \mathbb{E}\left[\int_0^\tau c_j(a_t)dt \,\middle|a_0 = a, y_0 = y_j)\right]$ ■.

Short note on Feynman -Kac formula

Feynman-Kac formula- Suppose $\exists\mathcal{P}(t,x)$ that satisfies :$\frac{\partial\mathcal{P}}{\partial t} + f(t,x)\frac{\partial\mathcal{P}}{\partial x} + \frac{1}{2}\rho^2(t,x)\frac{\partial^2\mathcal{P}}{\partial x^2} - R(x)\mathcal{P} + h(t,x) = 0$ s.t $\mathcal{P}(t,x) = \psi(x)$. Then $\exists\widetilde{W}(t)$ and a measure Q where solution is given as $\mathcal{P}(t,x) = E_Q[\int_t^T \mathcal{V}(t,u)h(u,x(u))du + \mathcal{V}(t,T)\psi(x(t))|\mathcal{F}_t]; t < T$ $dx(t) = f(t,x(t))dt + \rho(t,x(t))d\widetilde{W}(t)$; $\mathcal{V}(t,u) = \exp\left(-\int_t^u R(x(s)ds\right)$ given that $\int_t^T E_Q\left[\left(\rho(s,x(s))\frac{\partial\mathcal{P}}{\partial x}(s,x(s))\right]^2\middle|\mathcal{F}_t\right]$ .In previous expression $\mathcal{F}_t$ is a σalgebra[3]

Note on "MIT"shocks

Following Boppart et al. (2018)," simple linearization method for analyzing frameworks with consumer heterogeneity and aggregate shocks" was applied to standard RBC model with neutral technology shocks as in Kydland,Prescott (1982),and investment specific as in investment-specific, as in Greenwood et al. (2000). In definition given by Boppart et al. (2018) "MIT shock" is defined as:

"*An "MIT shock" is an unexpected shock that hits an economy at its steady state, leading to a transition path back towards the economy's steady state……*".

Mukoyama (2021) also follows Boppart et al. (2018) definition:"…. the probability of the shock is considered zero, and no prior (contingent) arrangement is possible for the occurrence of the MIT shock"…..The dynamic analysis that was using exogenous shocks or policy changes has been used in the literature with the earlier examples including: Abel,Blanchard (1983),

[3] Let $\mathcal{P}(x)$is a $\mathcal{P}(s)$, then a subset $\sum \subseteq \mathcal{P}(x)$ is σ-algebra if it satisfies: $x \in \sum$, and is considered to be $\cup$, and if $x \in \sum \Rightarrow \overline{x} \in \sum$; and if $x_1, x_2, \ldots \in \sum$ then $x = x_1 \cup x_2$ ….see Rudin (1987).

Auerbach, Kotlikoff (1983), and Judd (1985). And more recent examples being: Boppart et al. (2018), Kaplan et al. (2018), Boar ,Midrigan (2020), Guerrieri et al. (2020).

Transitory dynamics and MIT shocks (an implicit-uncertainty economy): short note

In the Aiyagari version of the model[4]:

equation 37

$$r(t) = F_k(K(t),1)) - \delta$$
$$w(t) = F_l(K(t),1))$$
$$K(t) = \int a g_1(a,t)da + \int a g_2(a,t)da$$

HJB equation is given as:
equation 38

$$\rho v_j(a,t) = \max_c u(c) + \partial_a v_j(a,t)\big(w(t)z_j + r(t)a - c\big) + \lambda_j\left(v_{-j}(a,t) - v_j(a,t)\right) + \partial_t v_j(a,t)$$

Kolmogorov Forward equation is:

equation 39

$$\partial_t q_j(a,t) = -\partial_a\big[s_j(a,t)g_j(a,t)\big] - \lambda_j g_j(a,t) + \lambda_{-j}(a,t) + \lambda_{-j} g_{-j}(a,t)$$
$$s_j(a,t) = w(t)z_j + r(t)a - c_j(a,t), c_j(a,t) = (u')^{-1}\left(\partial_a v_j(a,t)\right)$$

In previous expression a represents the borrowing limit, $g_{j,0}(a)$ represents the initial condition. Now, recall discretized equations for stationary equilibrium:
equation 40

$$\rho(v) = u(v) + A(v)v$$
$$0 = A(v)^{\mathrm{T}} g$$

Transition dynamics is given as:
- First denote $v_{i,j}^n = v_j(a_i t^n)$ and stack into v^n
- Denote $g_{i,j}^n = g_j(a_i, t^n)$ and stack into g^n

Then following applies:
equation 41

$$\rho v^n = u(v^{n+1}) + A\,(v^{n+1})v^n + \frac{1}{\Delta t}(v^{n+1} - v^n)$$
$$\frac{g^{n+1} - g^n}{\Delta t} = A(v)^{\mathrm{T}} g^{n+1}$$

Terminal condition for v is given as: $v^N = v_\infty$ which represents steady state, while initial condition is given as: $g\colon g^1 = g_0$.

Incomplete markets: Arrow securities and Bond markets (per Mukoyama (2021))

In this economy there are two types of consumers type I and type II. Arrow security[5] does not exist for the irregular state although the consumers recognize the possibility of the irregular state in the future. A Type-I consumer's problem is given as:

[4] See lecture notes by Benjamin Moll: https://benjaminmoll.com/lectures/

[5] An Arrow security is an instrument with a fixed payout of one unit in a specified state and no payout in other states, see Arrow (1953)

equation 42

$$\max_{c_1,c_2.\tilde{c}_2,a} u(c_1) + (1-\pi)u(c_2) + \pi u(\tilde{c}_2)$$

s.t. $c_1 + pa = 0; c_2 = 2 + a \ ; \tilde{c}_2 = 2 - \tau$;

where a denotes holding Arrow securities, regular state occurs with probability $1-\pi$, irregular state occurs with probability π where $\pi \in (0,1)$. Type I receives $1-\tau$, Type II consumer receives $(1+\tau)$ where $\tau \in (0,1)$ in irregular state transfer occurs from Type I to type II consumer. Utility $u(\cdot)$ is strictly increasing, strictly concave, and continuously differentiable. Robbin, Joel W. (2010), here states that f is said to be continuous on $\mathbb{R}^l$ if :

equation 43

$$\forall x_0 \in \mathbb{R}^l \forall \epsilon > 0 \ \exists \delta > 0 \ \forall x \in \mathbb{R}^l [|x - x_0| < \delta \ |\Rightarrow f(x) - f(x_0) < \epsilon|]$$

In previous condition ϵ is trimmed price space [6], x_0 is vector parameter, hence why the PDF is of a form $f_{x_0}(x) = (x - x_0)$. Next, for type II consumer we have:

$$\max_{c'_1,c'_2.\tilde{c}'_2,a'} u(c'_1) + (1-\pi)u(c'_2) + \pi u(\tilde{c}'_2)$$

This is the maximization problem for consumer Type II $c'_1 + pa' = 2; c'_2 = a \ ; \tilde{c}'_2 = \tau$.The competitive equilibrium here is :$(c_1, c'_1, c_2, c'_2, \tilde{c}_2, \tilde{c}'_2) = (1,1,1,1,2-\tau,\tau)$. Thus the limit is given as:

equation 44

$$li\,m_{\pi\to 0}(c_1, c'_1, c_2, c'_2, \tilde{c}_2, \tilde{c}'_2) = (1,1,2-\tau,\tau)$$

Where p is the price of Arrow security. In the Bond markets this version of the model is given as with quadratic utility function:

equation 45

$$u(c) = \alpha c - \frac{\gamma}{2}c^2$$

Where $\alpha > 0; \gamma > 0$, the value of $\alpha \gg 0$ so that utility is increasing in c for relevant range.Type I consumer problem in this economy is given as:

$$\max_{c_1,c_2.\tilde{c}_2,b} u(c_1) + (1-\pi)u(c_2) + \pi u(\tilde{c}_2)$$

s.t. $c_1 + qb = 1; c_2 = 1 + b \ ; \tilde{c}_2 = 1 - \tau + b$; where q represents the bond price and b is the bond holding. Now, a type I consumer problem and bond demand after FOC is given as:

equation 46

$$b = \frac{q(\gamma - \alpha) + \alpha - \gamma(1 - \pi\tau)}{\gamma(q^2 + 1)}$$

Type II consumer problem is given as :

$$\max_{c_1,c_2.\tilde{c}_2,b} u(c'_1) + (1-\pi)u(c'_2) + \pi u(\tilde{c}'_2)$$

s.t. $c'_1 + qb' = 1; c'_2 = 1 + b' \ ; \tilde{c}'_2 = 1 - \tau + b'$ .The bond demand for Type II consumer is given as:

equation 47

$$b = \frac{q(\gamma - \alpha) + \alpha - \gamma(1 + \pi\tau)}{\gamma(q^2 + 1)}$$

[6] Trimmed space as a location parameter class of probability functions that is parametrized by scalar or vector valued parameter x_0 which determines distributions or shift of the distribution.

The bond price q demand is zero here is set so that :$b + b' = 0$. Now, $q = 1, (b, b') = \left(\frac{\pi}{2}\tau, -\frac{\pi}{2}\tau\right)$. The resulting consumption functions are :

equation 48

$$(c_1, c_1', c_2, c_2'. \tilde{c}_2, \tilde{c}_2') = \left(1 - \frac{\pi}{2}\tau, 1 + \frac{\pi}{2}\tau, 1 + \frac{\pi}{2}\tau, 1 - \frac{\pi}{2}\tau, 1 + \left(\frac{\pi}{2} - 1\right)\tau, 1 + \left(1 - \frac{\pi}{2}\right)\tau\right)$$

In the limit $\pi \to 0$, the consumption profile when irregular state takes place in period 2 approach:

equation 49

$$\lim_{\pi \to 0}(c_1, c_1', \tilde{c}_2, \tilde{c}_2') = (1,1,1 - \tau, 1 + \tau)$$

Now in an Arrow security economy if there is MIT shock, because the irregular state is not spanned by the Arrow security, the ex-post allocation will be given as: $\tilde{c}_2' = 2 - \tau$; $\tilde{c}_2' = \tau$ where tilde $(\tilde{\ })$ denotes irregular state. The entire ex-post allocation with MIT shock is: $(c_1, c_1', \tilde{c}_2, c_2') = (1,1,2 - \tau, \tau)$. The unique competitive equilibrium before the shock was: $p = 1, a = 1, a' = 1, c_1 = c_1' = c_2 = c_2' = 1$. In the bond economy post MIT shock allocation would be : $\tilde{c}_2 = 1 - \tau; \tilde{c}_2 = 1 + \tau$ .The unique competitive equilibrium before the shock was: $q = 1, b = -1, b' = 1$; $c_1 = c_1' = c_2 = c_2' = 1$.

Krusell-Smith and Ayagari type incomplete markets

In this economy $i \in (0,1)$, $l(s_t) = s_t$ it is i.i.d employment with support $S = \{s_{\min}, s_{\max}\}$,where $s_{\min} > 0$. Now let $\pi'(s'|s) = \Pr(s_{t+1} = s'|s_t = s)$, and $\sum_{s'} \pi(s'|s) = 1, \forall s$ and $\pi(s') = \sum_s \pi(s'|s)\pi(s)$, we normalize $\mathbb{E}(s) = 1$.Preferences are given as:

equation 50

$$\mathbb{E}_0\mathcal{U} = \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t U(c_t)$$

Budget and borrowing constraint are given as:

equation 51

$$c_t + a_{t+1} = w_t s_t + (1 + r_t)a_t - \tau_t$$

Where $a_t = k_t - b_t, c_t \geq 0, k_t \geq 0, b_t \leq \bar{b}_t, a_{t+1} \leq -\bar{b}_t$.The asset grid is $a_{t+1} \in A = \{a^1, a^2, .., a^N\}$. Now in previous $a^1 = -\bar{b}$

equation 52

$$\bar{b} = \inf_{\{s_{t+j}\}_{j=1}^{\infty}} \sum_{j=1}^{\infty} \left(\frac{q_{t+j}}{q_t}\right) [w_{t+j}s_{(t+j)} - \tau_{t+j}] = \sum_{j=0}^{\infty} \left(\frac{q_{t+j}}{q_t}\right) [w_{t+j}s_{min} - r_{t+j}D]$$

$$q_r \equiv \frac{q_{t-1}}{1 + r_t}$$

In equilibirum lets $\Phi_t(a, s) = \Pr(a_t = a, s_t = s)$ which will denote the joint probability of a, s in time period t.The distribution of wealth in period t is given by:

equation 53

$$\psi_t(a) = \sum_{x \in s} \Phi_t(a, s) = \Pr(a_t = a)$$

Market clearing condition is given as: $K_t + D = \sum_{a \in A} a\psi_t(a)$ .Where $D$is exogenous government debt, and K_t is aggregate *per capita* capital. Equilibrium prices are given as:

equation 54

$$r(t) = f'(K_t) - \delta \equiv r(K_t) = k(r_t); w(t) = f(K_t) - f'(K_t)K_t \equiv w(K_t). w_t = \omega(r_t)$$

In recursive equilibrium suppose that, in equilibrium, the law of motion for the distribution of wealth is some functional Γ s.t.:$\Phi_{t+1} = \Gamma(\Phi_t)$,this means that the evolution of (Φ_t) is deterministic, also $K_t = K(\Phi_t)$; $\bar{b}_t = b(\Phi_t)$ .A recursive equilibrium is given by (V, A, Γ):

equation 55

$$V(a, s, \Phi) = \max U(c) + \beta \sum_{(s' \in S)} V(a', s', \Phi')\pi(s'|s)$$

s.t. $a' = w(\Phi')s' + [1 + r(\Phi')][a - c] - r(\Phi')D; 0 \leq c \leq a, a' \in A(\Phi); \Phi' = \Gamma(\Phi); A(a, s, \Phi) = argmax\{..\}$. Where Γ is generated by A that is , Γ maps $\Phi \to \Phi'$ so that :

equation 56

$$\Phi'(a', s') = \sum_{s \in S} \Phi(a, s) 1_{[A(a,s,\Phi)=a']} \pi(s, s')$$

Now capital plus debt equal to :

equation 57

$$\begin{aligned} K_{t+1} + D = \sum_{a' \in A} a'\psi_{t+1}(a') \\ &= \sum_{a' \in A} \sum_{s' \in S} a'\Phi_{t+1}(a', s') \\ &= \sum_{a' \in A} \sum_{s' \in S} \sum_{s \in S, a \in A} \Phi(a, s) 1_{[A(a,s,\Phi)=a']} \pi(s, s') \\ &= \sum_{s \in S, a' \in A} \sum_{s' \in S} a' 1_{[A(a,s,\Phi_t)=a']} \Phi_t(a, s) \sum_{s' \in S} \pi(s'|s) = \sum_{s \in S, a \in A} A(a, s, \Phi_t)\Phi_t(a, s) \end{aligned}$$

Steady-state capital, interest rate and wage are given as:

equation 58

$$K = \int a d\Phi(a) - D; r = r(K); w = w(K)$$

Aiyagari steady state is given as:

equation 59

$$r_t = r, w_t = w = \omega(r)$$

$$\bar{b}_t = \bar{b} = \min\left\{b, \frac{wl_{\min}}{r} - D\right\} \equiv \bar{b}(w, r, D)$$

We define that $x_t \equiv a_t + \bar{b}; z_t \equiv wl_t + (1 + r)a_t + \bar{b} - \tau$ it follows that $z_t \equiv wl_t(1 + r)x_t - \zeta$,where z_t are the total resources in the economy available at time t and x_{t+1} is investment in t and

equation 60

$$\zeta \equiv r\bar{b} + \tau = r[\bar{b} + D] = \zeta(w, r, D)$$

If $-\Delta\bar{b} = -\Delta D$ then ζ is independent of D. Individual consumption and resources of individual are given as:

equation 61

$$c_t = z_t - x_{t+1}\,; z_{t+1} = ws_{t+1} + (1+r)x_{t+1} - \zeta$$

Value function in terms of z is:

equation 62

$$V(z) = \max_{0\leq 0\leq z} U(z-x) + \beta\sum V(z')\pi(s')$$

s.t. $z' \equiv ws' - \zeta + (1+r)x$. Abou the optimal consumption individual wealth dynamics in this economy is given as:

equation 63

$$c_t = m\cdot\left[(1+r)a_t + w_t s_t + h - (t+1) = m\cdot\left[z_T + (h_{t+1} - \bar{b}\right]\right]$$

Where h_{t+1} is the present value of labor income and m is the marginal propensity to consume out of effective wealth and $m \in (0,1)$ and $h_{t+1} \geq \bar{b}$, so that $c_T = \bar{c} + m\cdot z_t$ where $\bar{c} > 0$ and $m \in (0,1)$.Now in Krusell-Smith dynamics:]= approximate constrained equilibrium is given as:

equation 64

$$V(a,s,\mathrm{m}) = \max U(c) + \beta\sum_{(s'\in S)} V(a',s',\mathrm{m}')\pi(s'|s)$$
$$V(a,s,\Phi) = \max U(c) + \beta\sum_{(s'\in S)} V(a',s',\Phi')\pi(s'|s)$$

s.t. $a' = w(\Phi')s' + [1+r(\Phi')][a-c] - r(\Phi')D; c\geq a, a' \in A(\Phi); \mathrm{m}' = \widehat{\mathrm{G}}(\mathrm{m}); A(a,s,\mathrm{m}) = argmax\{..\}$.Now, given that $\Phi\_)$ ad the rule A , compute $\{m_t, \Phi_t\}_{t=0}^{\infty}$ by :

equation 65

$$\Phi_{t+1}(a,s) = \sum_{s\in S}\Phi_t(a,s)1_{[\hat{A}(a,s,m_t)=a']}\pi(s,s')$$

And $\varepsilon_t \equiv m_{t+1} - \hat{G}(m_t)$ are very small.

RESULTS AND DISCUSSION

Consumption savings problem and endogenous labor supply per Achdou et al.(2022).

This section uses MATLAB codes used for computation in Achdou et al.(2022)m and published in Benjamin Moll web site :Benjamin Moll Heterogeneous Agent Models in Continuous Time London School of economics and political science, See: https://benjaminmoll.com/codes/ . The aim here is to depict graphically what was written previously theoretically about Huggett (1993) model. This problem outlined here is consumption-savings problem and endogenous labor supply. Here individuals solve:

equation 66

$$\max_{\{c_t,l_t\}_{t\geq 0}} \mathbb{E}_0\int_0^{\infty} u(c_t,l_t)dt\; s.t.\,\dot{a}_t = wz_t l_t + ra_t - c_t\,; a_t \geq \underline{a}$$

Where, $z_t \in \{z_1, z_2\}$ follows a two step Poisoon process with intensities λ_1, λ_2. Now individuals endogenously choose labor supply l,here we assume that period utility function is given as:

equation 67

$$u(c,l) = \frac{c^{(1-\gamma)}}{1-\gamma} - \frac{l^{1+\frac{1}{\varphi}}}{1+\frac{1}{\varphi}}$$

HJB equation here is :

equation 68

$$\rho v_j(a) = \max_{c,l} u(c,l) + v_i'(a)(wz_j l + ra - c) + \lambda_j \left(v_{-j}(a) - v_j(a)\right), j = 1,2$$

FOC's are:

equation 69

$$u_c\left(c_j(a), l_j(a)\right) = v_j'(a)$$

$$-u_l\left(c_J(a), l_j(a)\right) = v_j'(a) w z_j$$

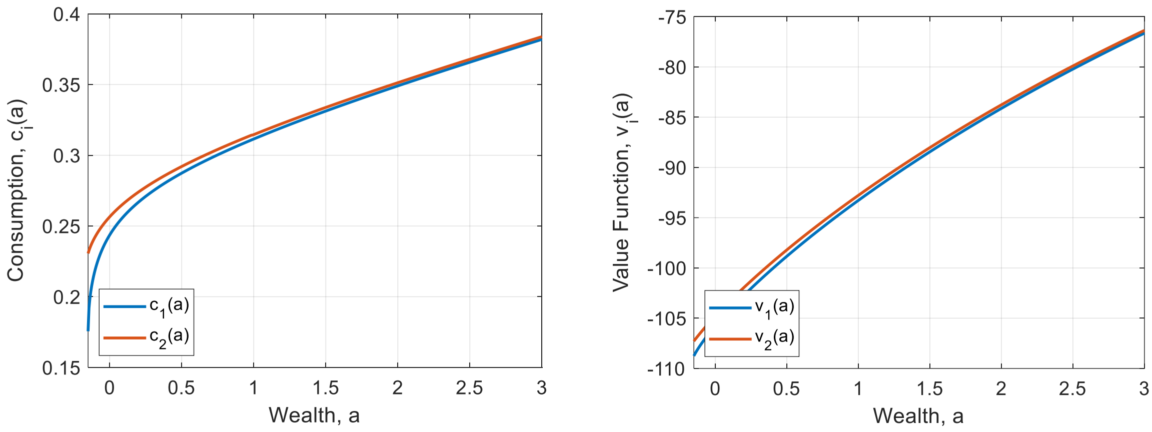
Intra-temporal FOC is given as:

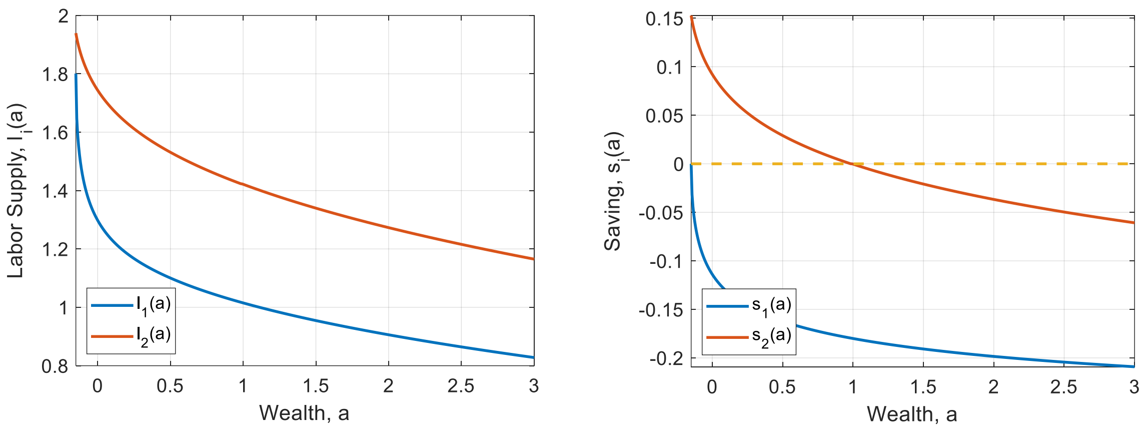
equation 70

$$-\frac{u_l\left(c_j(a), l_j(a)\right)}{u_c\left(c_j(a), l_j(a)\right)} = wz_j$$

Parameters here are : $s = 2$; $\rho = 0.05$; $r = 0.03$; $z_1 = 0.1$; $z_2 = 0.2$; $z = [z_1, z_2]$; $la_1 = 1.5$; $la_2 = 1$; $la = [la_1, la_2]$;etc. Now resulting graph Figure 1 shows wealth-consumption; wealth-value function; wealth-labor supply, and wealth-saving.

Figure 1 Consumption savings problem and endogenous labor supply per Achdou et al.(2022).





Source: solved with Benjamin Moll codes https://benjaminmoll.com/codes/

Credit crunch in Huggett economy (per to Mellior, Gustavo)

This MATLAB code and its algorithm explanation are due to Gustavo Mellior (Kent Uni.2016) and those files can be found at Benjamin Moll web site: https://benjaminmoll.com/codes/ .What is credit crunch? In Bernanke et al.(1991) credit crunch is defined as:"..*We define a bank credit crunch as a significant leftward shift in the supply curve for bank loans, holding constant both the safe real interest rate and the quality of potential borrower*.."A credit crunch (credit squeeze, credit tightening; credit crisis) is a sudden reduction in the general availability of loans or a sudden tightening of the conditions required to obtain a loan from banks. A credit crunch generally involves a reduction in the availability of credit independent of a rise in official interest rates. Economy is described in the text as before, and when credit crunch occurs a household with assets $\underline{a}_{t_0}$ will find itself below the new borrowing limit, and it will reduce consumption by Δa and it moves closer to a_T. And in this example $\underline{a}_{t_0} + 3\Delta a = \underline{a}_T$

equation 71

$\Delta a = s_1\left(\underline{a}_{t_0}\right) = z_1 + r\left(\underline{a}_T - 3\Delta a\right) - c_1\left(\underline{a}_{t_0}\right)$;$\Delta a = s_1\left(\underline{a}_{t_0} + \Delta a\right) = z_1 + r\left(\underline{a}_T - 2\Delta a\right) - c_1\left(\underline{a}_{t_0} + \Delta a\right)$;$\Delta a = s_1\left(\underline{a}_{t_0} + 2\Delta a\right) = z_1 + r\left(\underline{a}_T - \Delta a\right) - c_1\left(\underline{a}_{t_0} + 2\Delta a\right)$;$0 = s_1\left(\underline{a}_{t_0} + 3\Delta a\right) = z_1 + r\left(\underline{a}_T\right) - c_1\left(\underline{a}_{t_0} + 3\Delta a\right)$

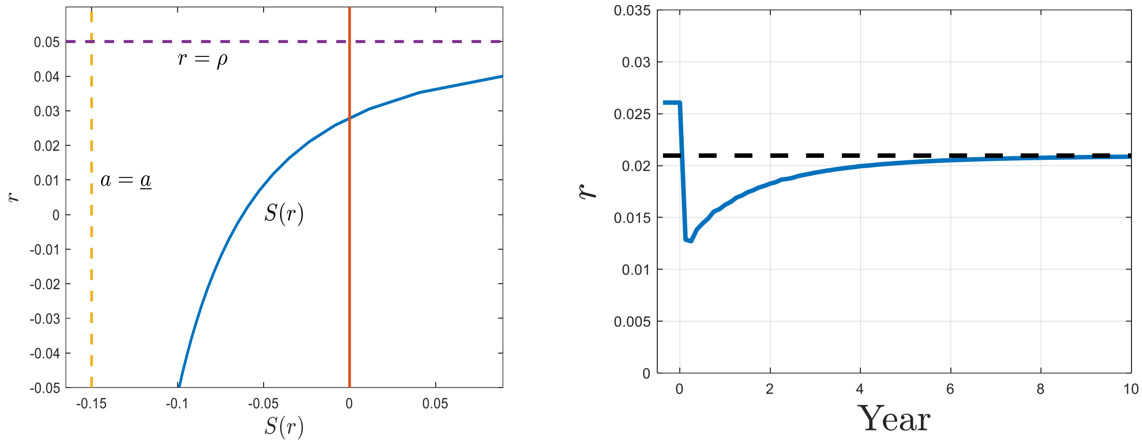
When credit crunch occurs previous will be modified to reduce borrowing limit by $3\Delta a$

equation 72

$\bar{c}_{1,1} - z_1 + r\underline{a}_{t_0} - \Delta a$; $\bar{c}_{2,1} = z_1 + r\left(\underline{a}_{t_0} + \Delta a\right) - \Delta a$; $\bar{c}_{3,1} = z_1 + r\left(\underline{a}_{t_0} + 2\Delta a\right) - \Delta a$; $\bar{c}_{\underline{a}_T}, 1 = z_1 + r\underline{a}_T$; $\bar{v}'_{i,j} = u'\left(\bar{c}'_{i,j}\right)$; $v_{i,j} = v'_{i,j}\mathbb{1}_{S_F>0} + \mathrm{v}'_{i,j}\mathbb{1}_{(S_B<0)} + \bar{\mathrm{v}}'_{i,j}\mathbb{1}_{S_B>0>S_F}$

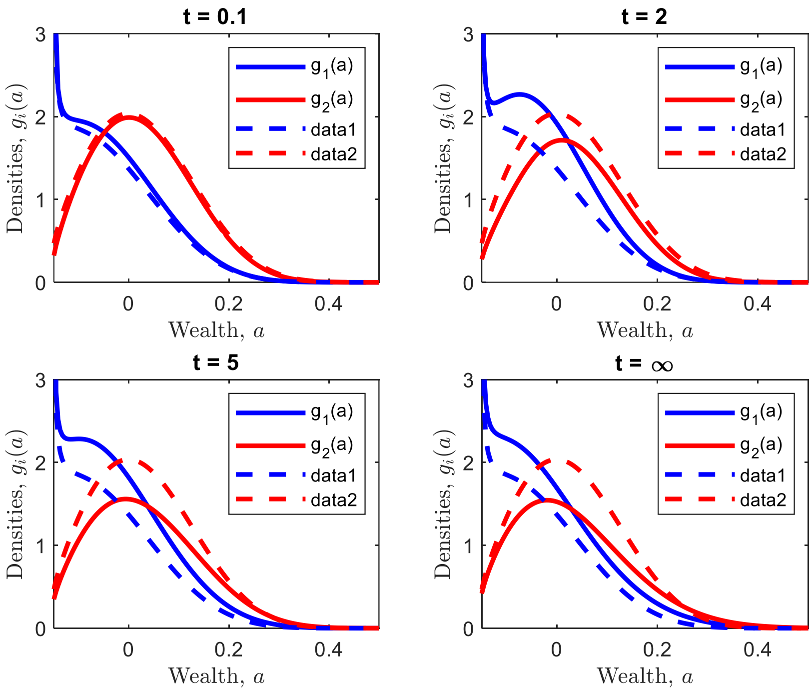
In this example parameters of the model are : $s = 2$;$\rho = 0.05$;$z_1 = 0.12$; $z_2 = 0.25$; $z = [z_1, z_2]$; $la_1 = 1.15$; $la_2 = 1$,$la = [la_1, la_2]$; $r_0 = 0.03$; $r_{min} = 0.001$; $r_{\max} = 0.045$;
$I = 800$; Equilibrium Found, Interest rate =0.0261. In the next photo equilibrium interest rate and supply of borrowings (loans) priced by that rate are depicted:

Figure 2 equilibrium interest rate



Next densities $g_i(a)$ and wealth a are depicted.

Figure 3 wealth distribution and densities



Krusell-Smith program for “Solving the incomplete markets model with aggregate uncertainty using the Krusell-Smith algorithm” by Maliar et al.(2010)

This part is based on a program written by Lilia Maliar, Serguei Maliar and Fernando Valli (2008) which is available online : https://lmaliar.ws.gc.cuny.edu/codes/ .Paper that uses this code is published as [Maliar et al. (2010)](). Parameters set for the model are:

$\beta = 0.99$; - discount factor
$\gamma = 1$; - utility-function parameter
$\alpha = 0.36$; - share of capital in the production function
δ=0.025; - depreciation rate
δ_a=0.01; - $(1 - \delta_a)$ is the productivity level in a bad state, and $(1 + \delta_a)$ is the productivity level in a good state

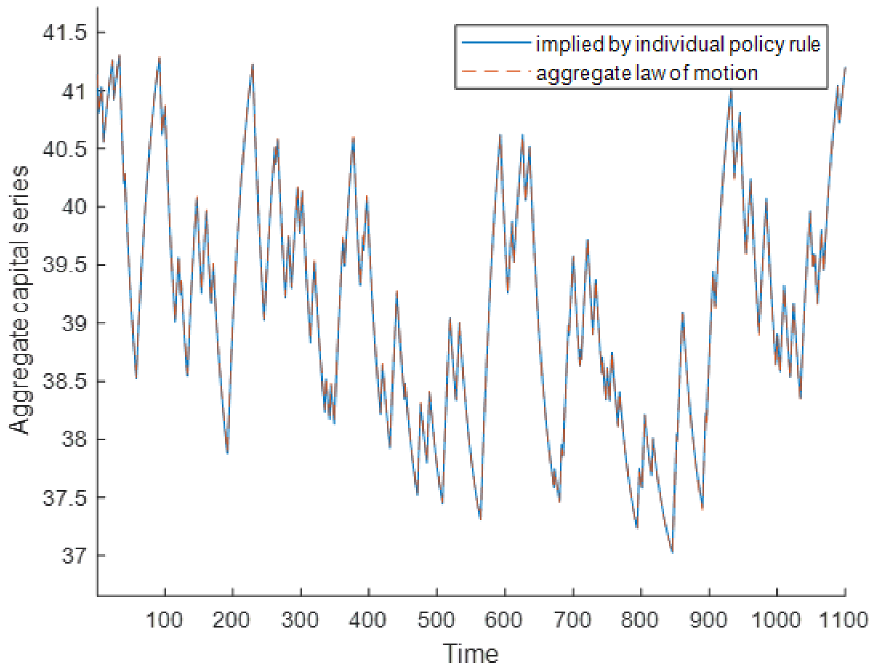
$\mu = 0.15$; - unemployment benefits as a share of wage
$\bar{l}$ =1/0.9; - time endowment; normalizes labor supply to 1 in a bad state
$T = 1100$; - simulation length
$ndiscard = 100$; - number of periods to discard
$nstates_{id} = 2$; - number of states for the idiosyncratic shock
$nstates_{ag} = 2$;- number of states for the aggregate shock
$\epsilon_u = 0$; - idiosyncratic shock if the agent is unemployed
$\epsilon_e = 1$; - idiosyncratic shock if the agent is employed
$ur_b = 0.1$; - unemployment rate in a bad aggregate state
$er_b = (1 - ur_b)$; employment rate in a bad aggregate state
$ur_g = 0.04$; - unemployment rate in a good aggregate state
$er_g = (1 - ur_g)$; -employment rate in a good aggregate state

Transition probability matrix is given as:

$$\pi_{i,j} = \begin{pmatrix} 0.525 & 0.35 & 0.03125 & 0.09735 \\ 0.038889 & 0.836111 & 0.002083 & 0.122917 \\ 0.09375 & 0.03125 & 0.291667 & 0.583333 \\ 0.009115 & 0.115885 & 0.024306 & 0.850694 \end{pmatrix}$$

Now, to compute the aggregate law of motion, we use the stochastic-simulation approach of Krusell and Smith (1998). Results are presented in Figure 4.

Figure 4 Accuracy of the aggregate law of motion with random shocks



CONCLUDING REMARKS

This paper was investigating model with heterogeneity of agents in incomplete markets in Huggett (1993) ,by using examples solved in MATLAB with codes written for paper by Achdou et al.(2022) .Heterogeneity of individuals was also investigated in Krusell-Smith type economy (with aggregate uncertainty) and with MATLAB code written to find solution to aggregate law of motion and its accuracy with stochastic simulation as per Krusell,Smith (1998). So in conclusion of this review of these models' incomplete markets (heterogenous agents) is that there are developed algorithms for numerically solving the equilibria as equilibria do exist in these types of economics although they may be constrained efficient or inefficient. So, this is one temptation for further exploration in this area and shifting away from representative agent models.

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