The GUIDELINE of mathSTEM METHOD

Teaching mathematics in STEM context for STEM students

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mathSTEM
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(2019 – 2021)

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TABLE OF CONTENT

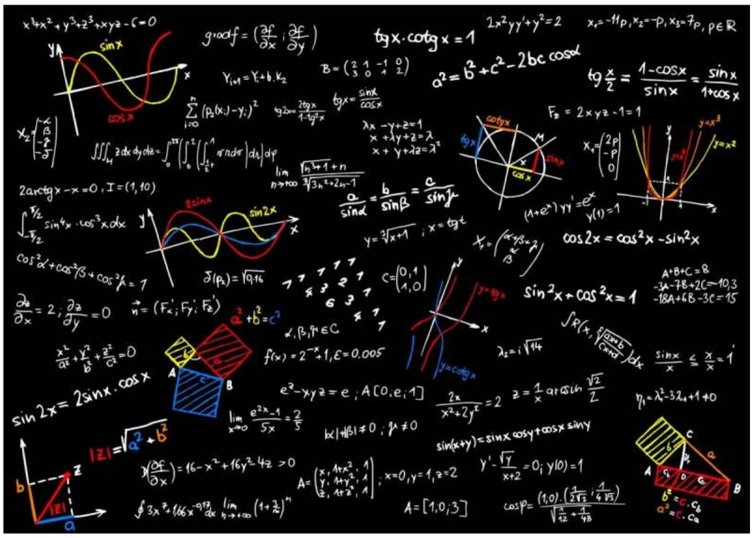
INTRODUCTION AND BACKGROUND

The Importance of Mathematics in the education

We know that mathematics is very exact discipline and studying it needs thoroughly studying all its concepts and their relations. Moreover, math knowledge is crucial for acquiring knowledge in many other disciplines. Mathematics is important in solving various problems in science, techniques, engineering, economy, ecology, as well as solving various real life and everyday problems. The ability for understanding mathematics and mathematical judgment is crucial for the future of the students. In today complex society, learning and understanding mathematics and natural sciences has become necessary for full development of everyone. Despite that, mathematics also helps students to have better problem-solving skills. Mathematics is challenge for many students.

But, most of the students usually find mathematics as difficult, abstract and boring discipline, probably because of the close relations among its concepts and their properties. Most of them consider it as useless subject based on useless complicated calculations. Thus, they become victims in the process of their education. They are actually victims of the outdated syllabuses and boring teaching methods, not victims of the boring subject like mathematics is considered to be, because mathematics can be made very interesting, which depends of the approach of teaching and learning. Very often students face up with the neediness of mathematics when they finish studies and start a carrier, actually when they face up with problems that cannot be solved without math skills and knowledge.

To be changed this situation, to be changed the students' attitude that they study useless and difficult calculations on the math lessons, no doubly it should be changed the teachers' approach in teaching math, firstly. Different environments have significant impact in creating students' attitude towards mathematics, but the main contribution is determined in the class environment. Math teachers are those who have to build awareness among students about the importance of mathematics and possibilities to apply it in their education and future carrier development.



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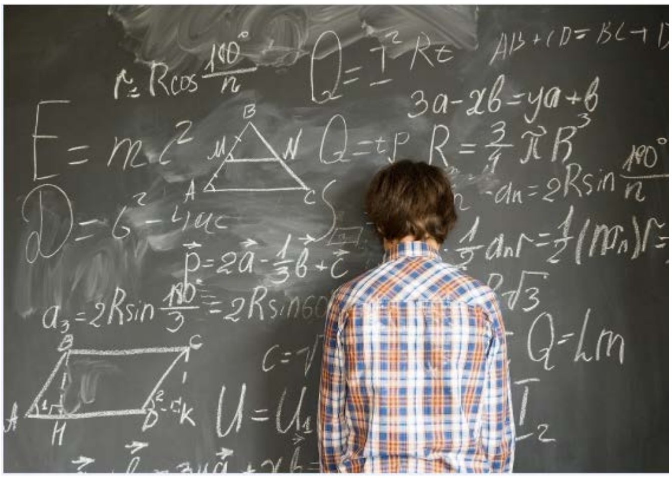
Mathematics is important for the students on different study programs, but especially for students that study science, technology and engineering. The importance of mathematics in the recent period is increased because of the huge application of the computers, information technologies, modeling and simulation, but at the same time computers and information technologies can help in better learning and acquiring mathematics.

Teachers can exploit different available software to modernize the teaching process, as well as to show students on the math lessons how they can solve practical problems, related to their carrier. Thus, they will modernize students' thinking and students' attitude toward mathematics. Students' negative attitude toward mathematics is one of the main reasons for low success and bed results of the students on each educational level, and unfortunately, it is reality in many countries nowadays.

The classical approach in the teaching creates passive students and to be improved the situation with mathematics and to be increased the interest for studying it, the classical approach must be changed. The most important goal of the mathematics is to develop skills to the students for understanding the abstract mathematical concepts and solving the real-life problems. Teaching mathematics should be done in a way that students are not concerned with the exact mathematical procedures, but more concerned with learning mathematics as a way of thinking and knowing. Teaching mathematics should be done in such a way as to promote the likelihood that students will be able to transfer what they have learned, to other areas. Instruction that aims for transferable skills must provide opportunities for students to make connections between different areas and reflect on what they have learned. Such approach of teaching math and appropriate types of activities should be regular part of math classes. STEM students especially need excellent understanding and excellent results in mathematics, thus they need the implementation of some new methodology in the process of teaching math.



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STEM education – meaning and importance (Why STEM education?)

The acronym STEM was initiative first coined by the National Science Foundation (NSF) nearly two decades ago, in 1990s, [1]. Sometimes the term can refer to any-or-all the fields of Science, Mathematics, Engineering and Technology both, individually and in integrated form. STEM was introduced to describe the focus on developing curriculum to enhance learning in these four areas. STEM (Science, Technology, Engineering, and Math) is an educational framework that brings reality into the classroom by connecting different subjects together in a way that they will relate to both the real and business world and to each other, [2].

But STEM education is far more than just sticking those subject titles together. It's a philosophy of education that embraces teaching skills and subjects in a way that resembles real life. This educational initiative was to provide all students with critical thinking skills that would make them creative problem solvers and ultimately more marketable in the workforce. It is perceived that any student who participates in STEM Education, particularly in the K-12 (kindergarten through 12th grade) setting would have an advantage if they choose not to pursue a post-secondary education or would have an even greater advantage if they did attend college, particularly in a STEM field, [3].

The integration of Science, Technology, Engineering and Mathematics, known as STEM education, is a growing area in developed and developing countries (United Nations Educational, Scientific and Cultural Organization [UNESCO], 2010). In the United States, for example, the Next Generation Science Standards [NGSS] acknowledges the importance and value of integrating the main disciplines identified in the acronym STEM and therefore engineering and technology are now integral parts of science literacy (National Research Center [NRC], 2012), [4]. Bybee (2013), as is cited in [5], clearly articulates that the overall purpose of STEM education is to further develop a STEM literate society. His definition of "STEM literacy" refers to an individual's:

- Knowledge, attitudes, and skills to identify questions and problems in life situations, explain the natural and designed world, and draw evidence-based conclusions about STEM-related issues.
- Understanding of the characteristic features of STEM disciplines as forms of human knowledge, inquiry and design;
- Awareness of how STEM disciplines shape our material, intellectual, and cultural environments; and
- Willingness to engage in STEM-related issues and with the ideas of science, technology, engineering and mathematics as a constructive, concerned, and reflective citizen.

In an ever-changing, increasingly complex world, it's more important than ever that our nation's youth are prepared to bring knowledge and skills to solve problems, make sense of information, and know how to gather and evaluate evidence to make decisions. These are the kinds of skills that students develop in science, technology, engineering and math-disciplines collectively known as STEM. If we want a nation where our future leaders, neighbors, and workers have the ability to understand and solve some of the complex challenges of today and tomorrow, and to meet the demands of the dynamic and evolving workforce, building students' skills, content knowledge, and fluency in STEM fields is essential. We must also make sure that, no matter where children live, they have access to quality learning environments.

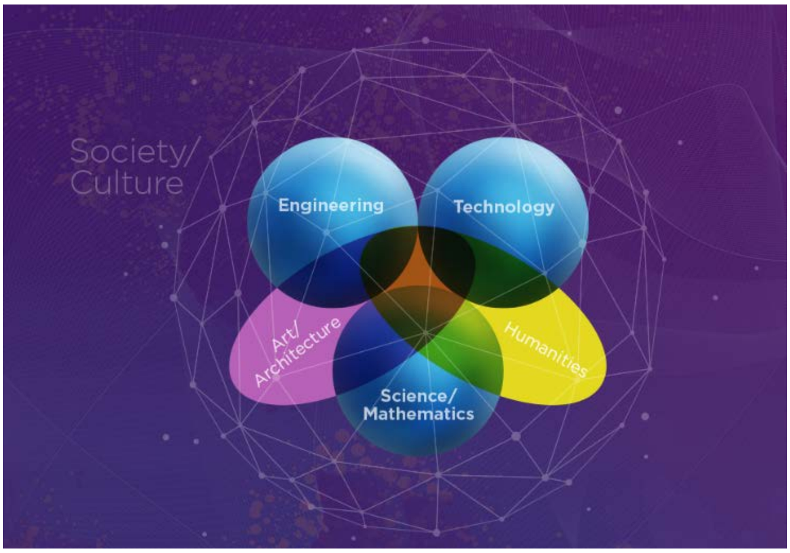
According to [6], the rationale for targeting above mentioned four areas is that while Science and Math are important to achieve a basic understanding of the universe, Engineering and Technology are needed for people to interact with the universe. When we refer to STEM, we are referring to a real-world, inquiry-based exploration of the world around us, with the goal of breaking down artificial barriers between disciplines.

Today, it is indisputable that Science, Technology, Engineering and Mathematics (STEM) are strong drivers of competitive national economies. Science, technology, engineering and mathematics workers play a key role in the sustained growth and stability of the global economy. They are a critical component in development of the people in every country. These necessary components are helping the humanity to be able to follow all enormous changes in all spheres in the society and to win the future. Thus, throughout the world, all nations are investing so much in STEM with the hope of grooming innovative minds to spearhead the development and sustainable growth of their economies.

STEM is connected to a wide range of human activities, including art, architecture, the humanities, and design thinking. This concept and connections are encompassed in the most recent term, **'STEAM'** for science, technology, engineering, *art* and math. The addition of arts to STEM to create STEAM is about incorporating creative thinking and applied arts in real situations. Art isn't just about working in a studio. Art is about discovering and creating ingenious ways of problem solving, integrating principles or presenting information. Picture an architect; they use engineering, math, technology, science and arts to create stunning buildings and structures. Many people feel that adding the A is unnecessary and that the application of creativity and arts is a natural part of STEM, but others like to highlight it. However, the key component of both STEM and STEAM is integration. ***Integrated STEAM education is one way to make learning more connected and relevant for students.***

The main objective of the innovative STEAM learning system is to strengthen students' personal and social abilities in order to familiarize students with the new demands of the labor field and create better opportunities with their futures in mind. STEM education research is a relatively young field with limited but growing empirical research available to inform policy and practice, [7].

In the education, STEM is actually a curriculum based on the idea of educating students in four specific disciplines - science, technology, engineering and mathematics - in an interdisciplinary and applied approach. Rather than teach the four disciplines as separate and discrete subjects, STEM integrates them into a cohesive learning paradigm based on real-world applications. STEM education has been pursued internationally since the mid-2000s, Australia is a relative late adopter, [7].

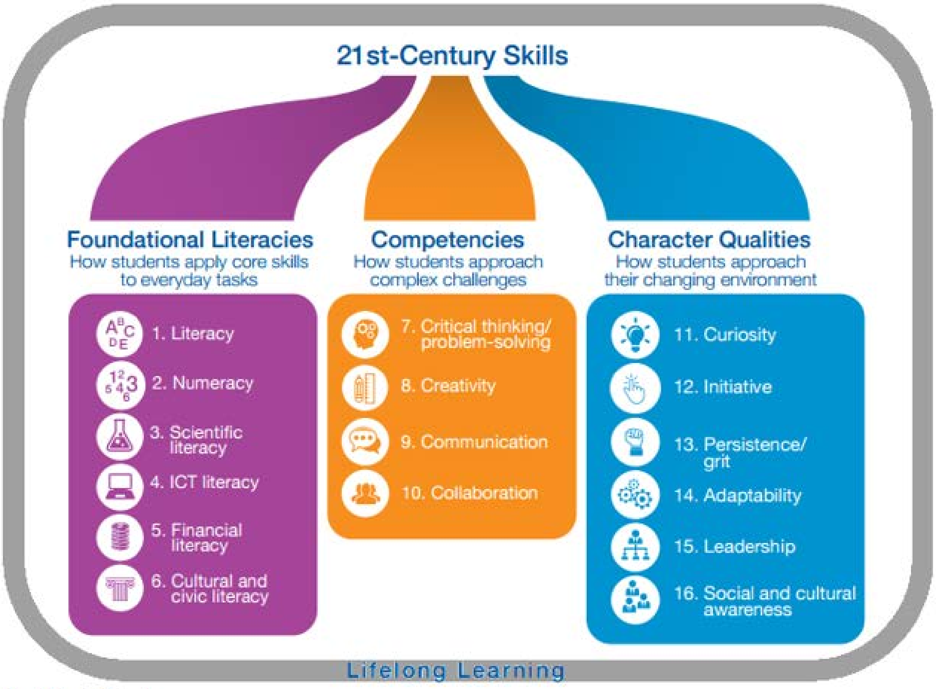


https://talkstem.org/what-is-stem/ (Downloaded: 29.05.2020)

STEM education ensures that students are prepared for a new world, [8]. Incorporating explicit teaching of the capabilities within integrative STEM contexts has the potential to enhance further the outcomes from learning activities, [9]. STEM builds on developing new models of teaching that foster such integrated meaningful learning experiences. Science teachers need to be able to offer learning opportunities that provide their students with authentic learning through provoking their understanding of the various concepts in the various STEM disciplines when working with others and applying their knowledge and skills to solve problems creatively, [10].

The STEM teaching-learning methodology aims to guarantee the development of transversal knowledge, in which the contents of each of these branches is not taught or learned in isolation, but rather is imparted in an interdisciplinary way that ensures contextualized and meaningful learning. STEM learning helps students observe, investigate, innovate, develop curiosity and their imaginations, ask questions, ponder how things work and solve the problems that they face on a daily basis.

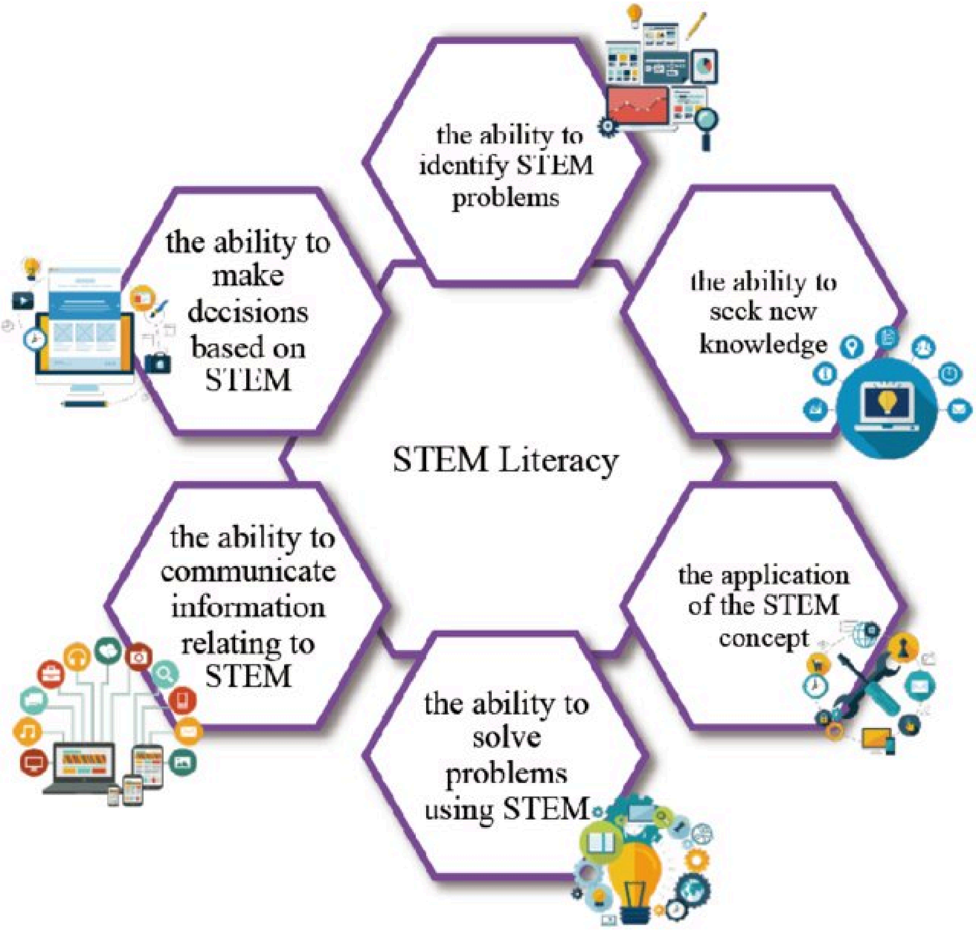
Furthermore, in the education, strong STEM programs are regarded as critical in developing students with twenty-first century competences (knowledge, skills and values), [11]. So-called 21st century skills, like problem-solving ability, communication skills, entrepreneurial, critical and creative thinking are considered to be essential for successful participants in the society. All of these competences are prerequisite students' further studies in STEM areas, their taking up of related careers and ventures into entrepreneurship and inventions, [12].



https://www.enterrasolutions.com/blog/stem-education-helps-teach-skills-necessary-for-21st-century-success/ (downloaded 01.06.2020)

Technology literacy, productivity, social skills, communication, flexibility and initiative, curiosity, decision making, leadership, entrepreneurship, acceptance of failure and more are other skills attained through STEM education. By carrying out project-based work, students are able to gain a clear understanding of the daily reality as well as learn content that is appropriate for the 21st century. This system encourages students to make their own decisions, develop their creative skills and evaluate their own achievements along the way. It helps students to be respectful, reflexive and emotional young people and promotes a critical spirit that makes them the active protagonists in their own teaching-learning process. Through this system, students are able to become democratic, participatory, supportive, happy and committed young people.

STEM it's a crucial component toward building critical-thinking and creativity skills for our nation's learners. For the evolving world of work, students need to develop these skills so that they become curious, analytical thinkers who are our future problem-solvers. STEM-centered learning provides a powerful avenue for building those skills. In the last 20 years alone, we have seen an expansion of technology unlike any other era before. Jobs are automating and many are quickly becoming obsolete.



https://www.researchgate.net/publication/322649699_The_Development_of_STEM_Literacy_Using_the_Learning_Process_of_Scientific_Imagineering_through_AR/figures

STEM is important because it pervades every part of our lives. Science is everywhere in the world around us. Technology is continuously expanding into every aspect of our lives. Engineering is the basic designs of roads and bridges, but also tackles the challenges of changing global weather and environmentally friendly changes to our home. Mathematics is in every occupation, every activity we do in our lives. By exposing students to STEM and giving them opportunities to explore STEM-related concepts, they will develop a passion for it and hopefully pursue a job in a STEM field.

STEM education typically including educational activities across all grade levels, from pre-school to post doctorate, and in both formal and informal classroom settings, [5]. The idea that education can remain unchanged amidst this shift is a dangerous fallacy and a disservice to students. Over the past eight years or so, educators have struggled to make sense of the many views and definitions of science, technology, engineering, and mathematics (STEM) education and what constitutes quality in STEM practices, [13].

A focus on connections, representations, and misconceptions can also aid teachers' pedagogy. The benefits of using an integrated STEM approach is that many of these practices lend themselves naturally to integrated STEM activities. Integrated STEM activities also allow teachers to focus on big ideas that are connected or interrelated between subjects. Berlin & White in [14] provide recommendations on how teachers should approach student knowledge:

- build on students' prior knowledge;
- organize knowledge around big ideas, concepts, or themes;
- develop student knowledge to involve interrelationships of concepts and processes;
- understand that knowledge is situation or context specific;
- enable knowledge to be advanced through social discourse;
- understand that knowledge is socially constructed over time.

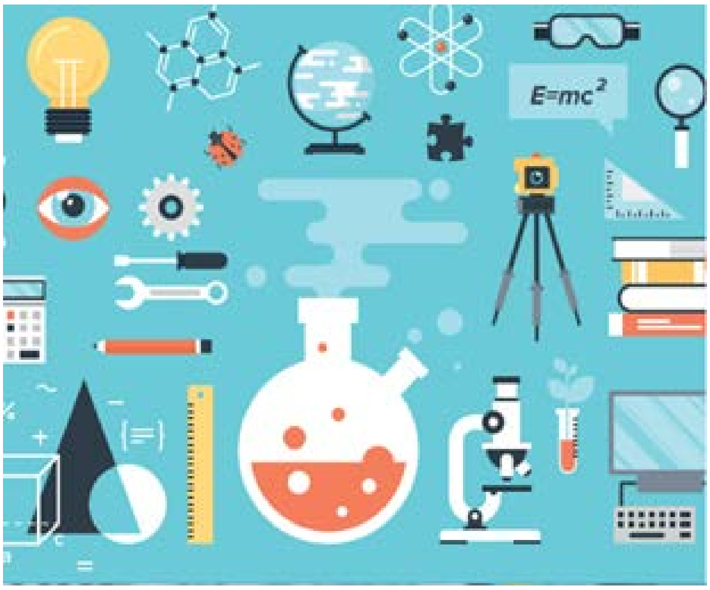
As it is described in [15], STEM education should begin while students are very young. In the elementary school, STEM education usually focuses on the introductory level STEM courses, as well as awareness of the STEM fields and occupations. This initial step provides standards-based structured inquiry-based and real-world problem-based learning, connecting all four of the STEM subjects. The goal is to pique students' interest into them wanting to pursue the courses, not because they have to. There is also an emphasis placed on bridging in-school and out-of-school STEM learning opportunities.

Programs outside of school can help children to see that STEM is more than a class to finish. Having activities that show real-life implication of STEM can pull together the ideas presented in school and help to show how they benefit our society and even our world as a whole. Children can see that what they are learning now is pertinent to their future and the future of the whole world, creating an interest often lacking when learning new concepts that do not seem to carry real-world application.

In the middle school, the courses become more rigorous and challenging. Student awareness of STEM fields and occupations is still pursued, as well as the academic requirements of such fields. Student exploration of STEM related careers begins at this level, particularly for underrepresented populations.

In the high school, the program of study focuses on the application of the subjects in a challenging and rigorous manner. Courses and pathways are now available in STEM fields and occupations, as well as preparation for post-secondary education and employment.

More emphasis is placed on bridging in-school and out-of -school STEM opportunities. STEM education and the application of technology empower every student to use their creativity and develop their critical thinking skills. STEM activities provide hands-on and minds-on lessons for the student. Making math and science both fun and interesting helps the student to do much more than just learn.



http://www.sfc.ac.uk/news/2019/news-70411.aspx (Downloaded: 31.05.2020)

The expertise of educators, whether in classrooms or in after-/out-of-school settings, is a key factor in determining whether the integration of STEM can be done well. At the most basic level, educator expertise combines knowledge of the subject matter with an understanding of effective approaches for teaching it to students with diverse learning styles. Such approaches include not only teaching strategies but also the skill with which educators plan lessons and work collaboratively to support student learning. STEM education has become an international topic of discussion over the past decade. This is driven by the changing global economy and workforce needs that indicate there will be a shortage of STEM prepared workers and educators around the world. STEM educational practices are intentional actions that schools, and educators take to create STEM learning environments that build student STEM capabilities and nurture STEM dispositions.

There are many benefits that have been connected with the use of integrated education, ‘‘Research indicates that using an interdisciplinary or integrated curriculum provides opportunities for more relevant, less fragmented, and more stimulating experiences for learners'' as is cited in [16]. Given the many views about the definition, benefits, and goals of STEM programs, it is not surprising that diverse methodologies exist for introducing these elements into educational systems, [13].

Over the last few decades, STEM education was focused on improving science and mathematics as isolated disciplines with little integration and attention given to technology or engineering as is cited in [17]. Becker and Park (2011) who are cited in [10] explored through a meta-analysis of 28 studies the effect of integrative approaches on student achievement. Results were in favor of interdisciplinary learning. There is a growing number of institutions that are partnering with schools to support STEM education. Tufts University has been working for over 15 years to integrate engineering into K-12 classrooms. They believe that engineering motivates students learning of the mathematics and science concepts that make technology possible. Professors, staff members, and students go into classrooms every week to assist teachers; they have monthly teacher support meetings, and training for teachers on technology resources according to the citations in [16].

Integrated STEM curriculum models can contain STEM content learning objectives primarily focused on one subject, but contexts can come from other STEM subjects, [17]. The science, technology, engineering, and mathematics (STEM) education model in most countries aims to teach science, mathematics, technology, and engineering in relation to primary, secondary, high school, and higher education, [8]. This was also a strategic approach adopted by scientists, technologists, engineers and mathematicians to create a stronger unified political voice. According to the citations in [8] an early outcome of this approach was the creation of the first Degree in STEM Education in 2005 by the Virginia Technology University as a way of highlighting the role of education in ensuring the delivery of appropriate STEM training.

How Will STEM Improve Student Learning?

Simply, STEM reflects real life. Jobs in the real world are interdisciplinary. We need to educate students in how subjects integrate and work together. They need to develop diverse skills sets and a passion for exploration and growth. We don't need students to memorize random facts anymore. We have so many facts at our fingertips now. In this high developed digital era, we need only seconds to have all the facts we need. Education is no longer about memorizing facts. Instead of that, it is about learning how to think critically and evaluate information. It is how to apply knowledge, research and skills to problem solve. Skills need to be taught in an applied way, as part of a greater whole, rather than the traditional approach of individual subject silos. As it is described in [18], STEM embraces the 4 C's identified as key in 21st Century education: Creativity, Collaboration, Critical Thinking, and Communication.

What separates STEM from the traditional science and math education is the blended learning environment and showing students how the scientific method can be applied to everyday life. It teaches students computational thinking and focuses on the real-world applications of problem solving. A curriculum that is STEM-based is necessary to include real-life situations to help the students learn. Thus, STEM helps to foster a love of learning. And the most important gift an education should give a student is a love of learning. The main characteristic of this learning system is practical training, during which students learn and work in a real way through experimentation. Teachers provide students with the necessary tools they need in order to obtain autonomy and use their own knowledge in a profitable and enriching way. To carry out this methodology, digital competence, teamwork and decision-making processes need to be integrated into each subject's curriculum. In addition, an important concept within the STEM model is to combine learning concepts with game-like practices.

Through STEM, students develop key skills including problem solving, creativity, critical analysis, teamwork, independent thinking, initiative, communication, digital literacy, etc. The improvement of STEM education will also contribute in increasing the results in different world rankings with math and science scores.

The importance of STEM education should be emphasized because it can improve the way how students comprehend and apply science. STEM education typically focuses on project-based learning in the classroom. The projects and activities incorporate technology to emphasize the application of science and prepare students for future classes.

STEM education classes provide useful information for teachers to be successful. The teaching category is the largest since content knowledge is the most important for teachers new to integrated STEM education. Teachers can build on the recommendations for effective teaching of integrated science and mathematics. While this study did not investigate self-efficacy, it is an important area for further study.

Much of the newest and most valuable knowledge involves more than one subject. Integrated STEM education can motivate students to careers in STEM fields and may improve their interest and performance in mathematics and science.

Effective STEM education is vital for the future success of students. The preparation and support of teachers of integrated STEM education is essential for achieving these goals. Future research can focus on the development of curricula materials and instructional models for STEM integration, connections between teacher education programs for integration and teachers' subsequent classroom teaching practices, and also ways in which teachers view STEM integration. High quality STEM education programs provide teachers with opportunities to collaborate with one another in unified efforts aimed at integrating the four subjects into one cohesive means of teaching and learning. It is when this objective is achieved that students gain access to meaningful curricular opportunities promoting critical thinking skills that can be applied to their academic as well as everyday lives.

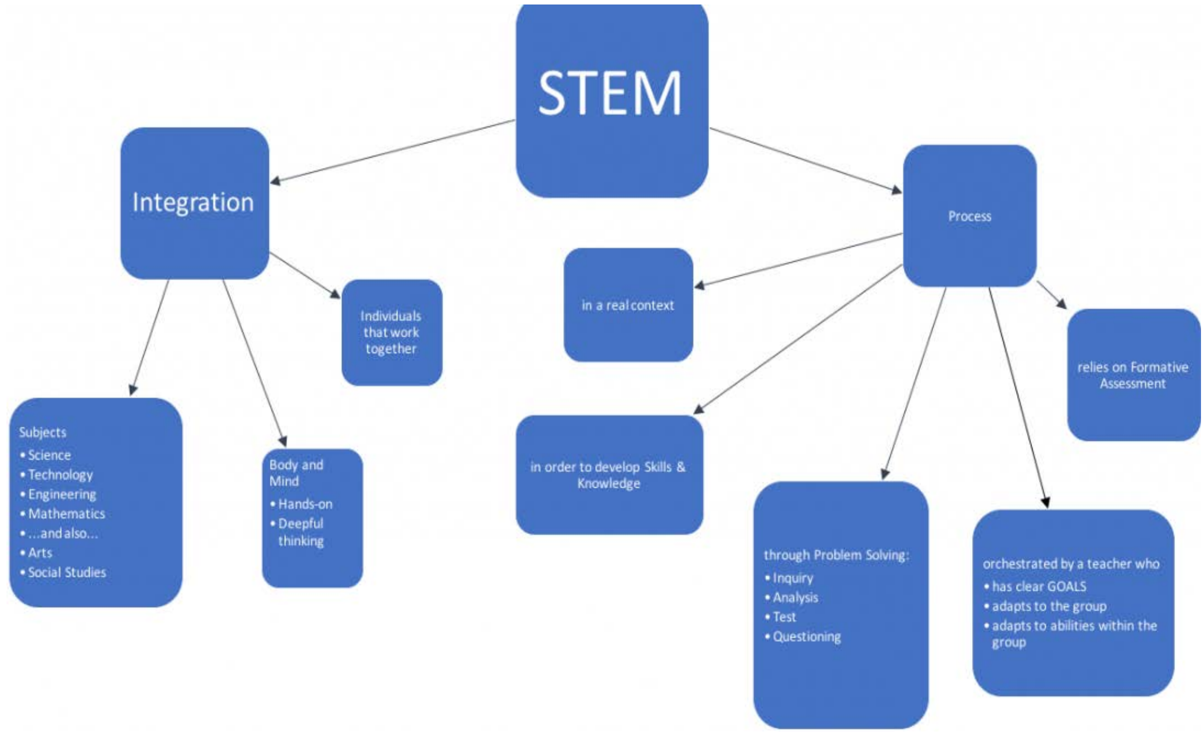
How to engage students in STEM programs?

"In the 21st century, scientific and technological innovations have become increasingly important as we face the benefits and challenges of both globalization and a knowledge-based economy. To succeed in this new information-based and highly technological society, students need to develop their capabilities in STEM to levels much beyond what was considered acceptable in the past." (National Science Foundation [19])

According to [20], STEM education has evolved into a meta-discipline, an integrated effort that removes the traditional barriers between these subjects, and instead focuses on innovation and the applied process of designing solutions to complex contextual problems using current tools and technologies. Engaging students in high quality STEM education requires programs to include rigorous curriculum, instruction, and assessment, integrate technology and engineering into the science and mathematics curriculum, and also promote scientific inquiry and the engineering design process.

The global economy is changing. Current jobs are disappearing due to automation and new jobs are emerging every day as a result of technological advances. According to a report in 2018, available on the website [21], as many as 2.4 million STEM jobs predicted to be unfilled by 2025, thus STEM talent challenge is global imperative. The continual advances in technology are changing the way students learn, connect and interact every day. Skills developed by students through STEM provide them with the foundation to succeed at school and beyond. STEM empowers individuals with the skills to succeed and adapt to this changing world.

Despite the increasing attention to STEM education worldwide, its stakeholders, in particular educational institution managers and classroom practitioners are still grappling to come into terms with what constitutes STEM education and how it can move to classroom settings, [22].



https://www.stem4math.eu/didactic

To do this, teachers must be able to offers standards-based STEM programs that use innovative instructional tools. That is to say that if teachers are prepared and have the tools, STEM can promote applied and collaborative learning. Technology has to be integrated into the culture, curriculum, teaching strategies and daily operations of classrooms to enhance learning and provide relevance. It is at this point that STEM becomes a meta-discipline and needs to be delivered to students in an interdisciplinary manner, within the constraints of the national/state course guidelines.

An important thing that was noticed regarding STEM education was a gender gap. Not so far, female students were significantly less likely to pursue a college major or career. Though this was nothing new, the gap was increasing at a significant rate. Male students were also more likely to pursue engineering and technology fields, while female students used to prefer science fields, like biology, chemistry, and similar. Overall, male students were three times more likely to be interested in pursuing a STEM career. So, the STEM curriculum should be aimed toward attracting underrepresented populations and encouraging female population to choose STEM as future carrier. Fortunately, nowadays STEM education breaks the traditional gender roles and there are more and more girls choosing STEM carrier.

Today's students are tomorrow's leaders. Occupations in STEM-related careers are some of the fastest growing and best paid of the 21st century, and they often have the greatest potential for job growth. It is important that each country remains competitive in fields of science, technology, medicine, and all of the other STEM fields. The best way to ensure future success and longevity it is to make sure that students are well versed in these subjects. Creating an appropriate good curriculum is the best way to ensure that students are exposed to math, science, and technology throughout their educational career.

What is important for high quality STEM education programs and curricula?

According to the authors in [20], high quality STEM education programs and curricula should reflect the following features:

- Include rigorous mathematics and science curriculum and instruction;
- At a minimum, (if separate STEM courses are not available in all areas) integrate technology and engineering into the science and mathematics curriculum;
- Promote engineering design and problem solving— (scientific/engineering) the process of identifying a problem, solution innovation, prototype, evaluation, redesign —as a way to develop a practical understanding the designed world;
- Promote inquiry—the process of asking questions and conducting investigations—as a way to develop a deep understanding of nature and the designed world (NSTA 2004);
- Be developed with grade-appropriate materials and encompass hands-on, minds-on, and collaborative approaches to learning;
- Address student outcomes and reflect the most current information and understandings in STEM fields;

- Provide opportunities to connect STEM educators and their students with the broader STEM community and workforce;
- Provide students with interdisciplinary, multicultural, and multiperspective viewpoints to demonstrate how STEM transcends national boundaries providing students a global perspective;
- Use appropriate technologies such as modeling, simulation, and distance learning to enhance STEM education learning experiences and investigations;
- Be presented through both formal and informal learning experiences;
- Present a balance of STEM by offering a relevant context for learning and integrating STEM core content knowledge through strategies such as project-based learning.

K–16 teachers of STEM, school and district leaders, community college and university STEM leaders, and other key stakeholders should embrace the following key points:

- Teachers of STEM should recognize the compelling and inherent opportunities of STEM to strengthen and support the teaching of STEM education, and, where possible, integrate STEM applications into the curriculum.
- Teachers of STEM should seek out and participate in quality professional development opportunities to enhance their knowledge of STEM and its application in meeting curricular requirements, and to gain exposure to practicing STEM professionals.
- Teachers of STEM should locate and use quality resources from STEM organizations to enhance and strengthen their curricula.
- School administrators and principals should support teachers in their efforts to integrate STEM within science and mathematics curricula.
- Collaborations among stakeholders in education, government, business, the community, and the media should be encouraged to coordinate the development and availability of STEM educational resources. STEM as an interdisciplinary discipline requires that pedagogical approaches must be altered from traditional approaches to support student learning.
- STEM Educators must implement instructional strategies that integrate the teaching of STEM in a way that challenges students to innovate and invent.
- STEM Educators must use problem-based and project-based learning with a set of specific learning outcomes to support student learning.

- STEM Educators must create meaningful learning opportunities provided context learning is delivered using applied and collaborative learning.
- STEM Educators must require students to demonstrate their understanding of these disciplines in an environment that models real world contexts for learning and work.
- STEM Educators must provide students with interdisciplinary, multicultural, and multi-perspective viewpoints to demonstrate how STEM transcends national boundaries providing students a global perspective that links students with a broader STEM community and workforce.

In [17] integrated STEM education is cited as “an effort to combine some or all of the four disciplines of science, technology, engineering, and mathematics into one class, unit, or lesson that is based on connections between the subjects and real-world problems”. Several benefits of STEM education include making students better problem solvers, innovators, inventors, self-reliant, logical thinkers, and technologically literate. From the analysis in [16], many studies have shown that integrating math and science has a positive impact on student attitudes and interest in school their motivation to learn, and achievement.

The research on teaching integrated mathematics and science provides a good basis for teaching integrated STEM education. Successful integration of science and mathematics depends largely on teachers’ understanding of the subject matter. Many teachers have holes in their own subject content knowledge and asking math and science teachers to teach another subject may create new knowledge gaps and challenges. What is known from research on effective practices in science and mathematics education provides insight into effective practices in STEM integration.

STEM lesson (STEM in schools)

Preparing today’s children to become the innovators and inventors of tomorrow begins with STEM education programs. With the rapidly developing technology, the labor force of the society has changed direction, and in the age of informatics, creative engineering applications have come to the forefront. Accordingly, the education levels of the labor force were also changed, [8]. For schools to include quality STEM education, it is important to understand teachers’ beliefs and perceptions related to STEM talent development. Teachers, as important persons within a student’s talent development, hold prior views and experiences that will influence their STEM instruction, [24]. Teachers who are included in these STEM education programs should be also prepared for innovation and modernization of their classes and lessons. According to NAE & NRC (2014), STEM literate means:

(1) awareness of the roles of science, technology, engineering, and mathematics in modern society;

(2) familiarity with at least some of the fundamental concepts from each area; and

(3) a basic level of application fluency (e.g., the ability to critically evaluate the science or engineering content in a news report, conduct basic troubleshooting of common technologies, and perform basic mathematical operations relevant to daily life).

Research has shown that teachers' content knowledge, experience, and pedagogical content knowledge have a large impact on self-efficacy. Over time, employing a student-centered approach to teaching with well-structured activities will allow for teachers to become more comfortable with the curriculum and for students to be successful. Supporting teachers in various ways and providing teachers with the necessary materials to do their job well may enable integrated STEM education teachers to be successful, [16].

According to the citations in [23], the integration of the individual disciplines is important for developing students' STEM literacy—the ability to understand and apply content from the STEM disciplines to solve real problems. The authors acknowledge that teaching STEM from the proposed approach is not possible in all circumstances and could limit the content taught from this approach. Some necessary knowledge in mathematics and sciences that are theoretically focused may not provide authentic engineering design applications as well as common STEM practices limited by current technology, [17]. To do so effectively requires that interdisciplinary approaches be employed in order to orient students towards problem-based learning.

There are a variety of ways in which teachers can introduce STEM into the classroom. The sample STEM lesson plan provides teachers with a step-by-step guide on how to properly introduce the concept to their students. It should involves a video lesson and then allows students to develop their own STEM lesson. Once teachers introduce STEM education to their students, they can analyze their classroom and determine how STEM projects and activities can be incorporated into it. Classroom set - up for STEM classes is key when it comes to implementing projects and activities. According to the authors in [25], implementation ideas include the following:

- Desks and other furniture should be arranged to allow students maximum flexibility when working on STEM projects and activities. This can mean large aisles between desks or creating workstations based on the particular project or activity.
- Collaboration is an important component of any STEM project or activity. The classroom should be arranged so that it enables students to work together, such as clusters of chairs.
- Visual supports, such as charts and posters, can be key in helping students remember key concepts.

https://www.govtech.com/budget-finance/Oklahoma-Schools-Recieve-Funding-for-STEM-Education.html Downloaded: 01.06.2020

STEM lesson should be similar to the science lessons and experiments. Actually, all the science results come from explorations, research and analysis. All experiences in the sciences are hand-on and inquiry based.

A typical STEM lessons usually involves four basic steps:

- Identify a real-world problem.
- Ask questions to explore the problem (and potentially solve the problem).
- Develop solutions.
- Explore a hands-on activity.

Students are extremely curious and impressionable, so instilling an interest at an early age could spark a lasting desire to pursue a career in any of these fields. By the time a student is ready to enter the workforce, they must have enough knowledge to make invaluable contributions to the STEM industries.

https://www.citizenschools.org/news/2017/5/15/dell-emc-supports-boston-after-school-stem-program
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There are some characteristics of the STEM lesson which should be considered by the teachers while they are preparing lesson in math and science.

1. STEM lesson should focus of the real-world problems in order to put the students in specific real position when they must analyze and think deeply because they will work on some real situation which is more attractive and interesting for them. They will feel that they are studying something productive and useful.
2. STEM lessons should include charts and engineering design processes. This engineering design process will help the students to move from the phase in which they identify the problem to the phase for creating and developing of the solution.
3. STEM lessons should enable the students to have hands-on inquiry and open-ended explorations. This means that teachers should provide additional materials for collaborative explorations in the groups where they can share the ideas, make decisions and find appropriate their solutions of the problem.
4. In STEM lessons the students should work in groups in order to make the students productive team workers.
5. STEM lessons should connect math classes and other science classes. Students can then begin to see that science and math are not isolated subjects but work together to find solutions of the problem.
6. STEM lesson should provide multiple right answers and approaches created from the students in order to solve the appointed problem. Teachers should admit failures of the students. In that way students will have opportunity to learn what is wrong, why is wrong and to try again some other approach.

Zemelman, Daniels & Hyde (2005) list ten best practices for teaching math and science, [16]:

- use manipulatives and hands-on learning;
- cooperative learning;
- discussion and inquiry;
- questioning and conjectures;
- use justification of thinking;
- writing for reflection and problem solving;
- use a problem-solving approach;
- integrate technology;
- teacher as a facilitator;
- use assessment as a part of instruction.

Problems and Challenges in STEM Education

By a review of the literature, with an emphasis on STEM pedagogical implementation, there seems to be a struggle in incorporating all four disciplines as there is still a constant focus on integrating just science and mathematics. It should be noted that interdisciplinary learning is a radical approach compared to most current teaching practices, [22]. The author in [9], says "STEM education refers to solving problems that draw on concepts and procedures from mathematics and science while incorporating the teamwork and design methodology of engineering and using appropriate technology".

One of the biggest concerns about STEM education is the lack of resources. Funding for the newest technology, training in how to use the new technology, plus the knowledge of how to use it effectively as a learning tool, are all areas where teachers struggle.

Another area where a lot of teachers struggle is a system that focuses on assessment and grades rather than a program that fosters innovation, creativity, critical thinking skills and problem solving skills. These achievements are not something that can be easily boxed up and assessed.

Furthermore, some teachers are simply not interested in learning how to teach STEM. They prefer to stay in the silos, keeping each of the stem subjects isolated into their own lessons.

Six categories of challenges and barriers to using STEM pedagogy in the classroom are described in [26]: pedagogical challenges, curricular challenges, structural challenges, student concerns, assessment concerns, and teacher supports.

Pedagogical challenges: Teachers perceive that STEM pedagogy requires some fundamental shifts in how they establish classroom environments and teach, and for some teachers these shifts are not always positive. Teachers mention STEM pedagogy requires a fundamental shift away from teacher-led instruction to student led-instruction. Teachers have to be able to step out of the director role and allow students to find their own way during the lesson, which might involve unexpected directions. Another similar concern is that teachers must have a view of instruction that aligns with the philosophy of the STEM curriculum authors. There must be a match between the teachers' pedagogy and the curriculum pedagogy. Teachers voiced a concern that they might incorrectly or inadvertently misinterpret the STEM developer's expectations. Teachers also expressed concerns about STEM pedagogy meeting the diverse needs of all learners, particularly those with disabilities and various cognitive abilities. One last pedagogy concern suggests that utilization of STEM could actually hinder direct instruction of science content.

Curriculum challenges: Some teachers, especially at the high school level, perceive the integrated nature of STEM curriculum is a challenge. Teachers had apprehension about following someone else's curriculum plan. Teachers were also concerned about integrating STEM curriculum into their existing curricula. District alignment and grade level standards can be inflexible, which prevents a smooth integration of STEM. In addition, teachers noted they felt STEM curriculum could be inflexible and the difficulty with combining two inflexible curricular plans. For some reasons, teachers had concerns about developing their own STEM-based curriculum with teachers from other subject areas. Teachers were also concerned about STEM curriculum's ability to impart meaningful learning. When implementing STEM curriculum, teachers were observed treating the inclusion of specific content as more of an afterthought.

Structural challenges: Teachers perceived typical school structures are barriers to the implementation of STEM education. Teachers felt the confines of class scheduling prohibited the interdisciplinary nature of STEM lessons, and various teachers teaching their own specific subjects were not conducive to interdisciplinary work either. This same scheduling prevented teachers in different subjects from planning together. The structure of student schedules, and lack of flexibility in them, was also mentioned as a barrier to STEM. Lack of control over pacing of curriculum and the sequence of instruction were also discussed as troublesome when teachers sought to integrate multiple disciplines for authentic STEM lessons. At the district level, teachers felt administrative and financial supports could be a challenge to STEM implementation. Another concern was a lack of technology resources available to students. Without student computers and other technology tools available, it was difficult to integrate the technology piece into STEM lessons. The last structural concern was the way education is organized and evaluated at the state level.

Student concerns: Teachers believe that students are unable or unwilling to be successful with STEM education or initiatives. Several studies (cited in [14]) reported teachers underestimate student abilities to solve STEM problems. Many of these teachers did not believe their students were competent enough in content areas to apply these skills to self-directed STEM problems. They felt these types of problems would be very difficult for their students and would cause their students to become unmotivated to learn. Teachers reported a need for instructional tools they could use to motivate students and get them interested in STEM subjects. In addition, rural teachers noted the challenges associated with modifying the curriculum in order to meet the needs of underperforming students. Teacher beliefs about their students may be related to their implementation fidelity of STEM curriculum.

Assessments, time, and knowledge: Teachers perceive that lack of quality assessment tools, planning time, and knowledge of STEM disciplines are challenges and barriers to STEM initiatives. In one study (cited in [26]), more than 40% of the teachers felt there was a lack of assessments for STEM programs. These teachers and many others felt there were not enough standardized classroom assessments to use with STEM lessons. This makes assessing student learning in a STEM curriculum very difficult. Teachers felt there were not enough formative assessments to discover what concepts students understood from other disciplines. Additionally, teachers were concerned about group grading. They felt unsure of how they would assess each member of the group individually to make sure they had mastery of the standards. Formative assessments help teachers know when re-teaching or remediating is necessary or when students already know the material. Teachers were concerned with the increased workload associated with STEM programming. They have to find more time to plan with other subject areas and to prepare the materials for students. Presenting the material and allowing for varying ability levels among students also required more time. This makes a lack of time one of the primary concerns teachers had when implementing STEM.

Teachers also believed they had a lack of subject matter knowledge concerning STEM content. Pre-service and in-service training was seen as inadequate in preparing teachers to implement STEM. Teachers felt they needed clarity about how the program was supposed to be implemented into existing programs. They did not feel fully prepared to integrate STEM subjects. Teachers also perceived a lack of instructional resources was a hurdle in their path to provide STEM opportunities for students. Although teachers deemed STEM education important and valuable, they were not comfortable with meeting the high teacher expectations they felt were associated with STEM. Feeling unsure about one's ability to teach STEM could lead teachers to a reduced confidence in their teaching efficacy.

Teacher supports: Five main areas were found in the research that addressed this need for support. They were in the areas of collaboration, curriculum, district support, prior experiences, and professional development. About *collaboration*, teachers believe that a culture of collaboration would increase the viability of STEM programs. It is important collaborating with other STEM teachers and university professionals in order to not only create an atmosphere that enhances preparation for STEM lessons, but also to model a team approach to students. STEM pedagogy required students to collaborate to solve challenges, so a teacher modeling the strength of a group approach is beneficial. Many teachers felt collaboration was the key to successful transdisciplinary teaching required for STEM lessons. Providing time and opportunities for collaborative planning and open communication between teachers may be critical to successful implementation. About *curriculum*, teachers believe that the availability of a quality curriculum would enhance the likelihood of success of STEM initiatives. It is important a flexible curriculum that is engineering - based. In order to be effective, the curriculum must be flexible enough to be used with various ability levels and educational environments while still being focused on the engineering design process. Teachers need specific, ready - made STEM problems they could use in their classrooms immediately. About *district support*, teachers perceive school district support, guidance, and flexibility were necessary for STEM initiatives. A supportive administrator or administrative team is important when teachers are implementing STEM pedagogy. It is necessary for their school districts to allow flexibility for the teacher to expand the curricula and instruction beyond national and state standards, so they were able to offer problems that meet student interests, talents, and academic needs. Teachers also felt that districts need to help parents and students understand course offerings and what STEM courses will teach them. About *prior experiences*, teachers perceive that previous experience using student - centered, inquiry models of instruction facilitate success in a

STEM initiative. Similarly, structured prior experiences by teachers were commonly seen as facilitators to STEM success. Teachers who had more science or math courses in college or had utilized similar instructional methods (i.e., problem-based learning, inquiry-based learning, questioning techniques, guided independent research studies) felt these experiences allowed them to promote the inductive and deductive reasoning across disciplines necessary for STEM. Confidence with STEM pedagogy increased because of these prior experiences. *Professional development* is also an important aspect. Teachers believe that well - organized and frequently available professional learning opportunities would facilitate successful STEM initiatives. The most often mentioned support that would increase the effectiveness of STEM education was learning opportunities for teachers to increase their ability to effectively integrate STEM content into their curriculum. Teachers at multiple stages in their careers reported significant increases in their confidence, knowledge, and efficacy to teach STEM after attending professional development programs. Effective professional development or continuing education needs to provide time and structure for teachers to explore how STEM can be integrated within their curriculum while focusing on increasing teacher's content knowledge and experiences with STEM.

Similar consideration and barrier about successful implementation of STEM education can be found in [27].

Extending STEM Education to Engineering Programs

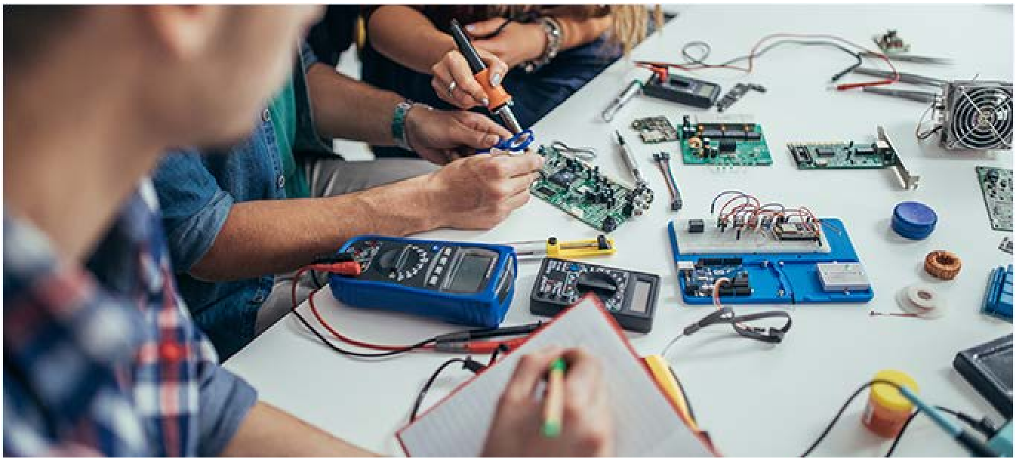
In the recent years many efforts have been done in order to integrate STEM programs in school programs at primary and secondary level. In [26] is given that comprehensive proposals are suggested to extend STEM education to be part of the Engineering programs at the undergraduate level, with focus on Electrical, Computer, and Systems Engineering. By default, most engineering programs are, some way or another, STEM - based, since they depend heavily on Math and natural sciences. Mathematics Education is one of the earliest disciplines to emerge in the structure of modern education and has one of the longest histories of being formally structured for learning. As is given in [28], mathematics is fundamental for many professions, especially science, technology, and engineering. Yet, mathematics is often perceived as difficult and many students leave disciplines in science, technology, engineering, and mathematics (STEM) as a result, closing doors to scientific, engineering, and technological careers. Although it is commonly acknowledged that mathematics is foundational to STEM, mathematics is being related to STEM education at a distance in practice and also in scholarship development. Holding the conception of mathematics as products does not support integrating mathematics with other STEM disciplines, as mathematics can be perceived simply as a set of tools for these disciplines. Mathematics has stronger linkages to creation and design than traditionally imagined. Therefore, its connections to engineering and technology could be much stronger. However, the deep-rooted conception of mathematics as products has traditionally discouraged students and teachers from considering and valuing design and design thinking. Conceiving mathematics as making sense should help promote conceptual changes in mathematical practice to value idea generation and design activity. Connections generated from such a shift will support teaching and learning not only in individual STEM disciplines, but also in integrated STEM education.

High percent of the students who enrolled at engineering programs at University have weak background in mathematics and sciences from their previous education. At University level as same as secondary level, Math is considered as difficult and complex subject by many students. Most of them think that in mathematics they must study abstract material which is useless. Because of the fast development of the digital and information technology, students rely on computer-aided tools to do the work for them. This is good if the students understand the basic mathematical concepts and use these additional digital tools as tool for faster calculations.

But the reality is something else. Most of the students use these educational technologies, only to solve the problem which is given by their teacher. The consequences are lack of deep understanding of basic mathematical concepts. Consequently, this leads to difficulties in understanding advanced topics that require difficult analytical skills, in contrast to the relatively easier numerical skills. Many mathematical software programs like MATLAB, Mathematica or other similar will surly extend the power of computing for many students; however, very often usage of such automated tools kills the need for deeper understanding of many mathematical and scientific subjects, [29]. These identified challenges show that the students should learn by doing, by design, and by applying the knowledge and skills, acquired through the educational program, in real world problems and realistic constraints. They can and they should use all the opportunities of the digital 21st century but they should use in the right way and in the right time.

Throughout the 21th century, each country should be focused on improving science, mathematics, and technology instruction, intending to not only increase literacy in those content areas but also expand existing workforces of scientists and engineers. The importance placed on the role of educational programs in preparing students to participate in the workforce and compete in the global economy was signaled by the continued participation in the early 21st century of many countries in the periodic international comparisons (TIMSS and PISA) of student knowledge and skills. Efforts were also being made to increase general awareness of STEM careers and to provide a deeper understanding of STEM content through application and problem-solving activities.

Engineers use problem-solving design processes that include analysis of the results and opportunities to improve those results based on the constraints and criteria specified. Engineers create technologies to solve problems, address and answer the needs. A broader perspective as found on a diverse team of engineers brings a better solution to a problem. The STEM education should provide an opportunity to create successful engineering. By explicitly teaching about engineering as a profession that relies on training and demonstrated abilities in not only math and science, but reading, writing, social studies from history to society, the arts, life skills such as productive collaboration, communication, persistence, creativity and ethics as well as local to global views, more students - and a more diverse group of students - will have the opportunity to pursue it as a career. This is an important global need, and it also should be a national imperative that impacts every sphere in humanity and society. The need for problem solvers has never been more critical to address some of the big challenges we face today.



https://www.ppic.org/blog/more-students-are-earning-stem-degrees/

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The educational institutions should be encouraged to seek methods that will lead toward the goal of graduating every student from high school with essential STEM knowledge and competencies, to succeed in future education and work.

While the goal is a prepared STEM workforce, the challenge is in determining the most-strategic expenditure that will result in the greatest impact on the preparation of students to have success in STEM fields. It is necessary, therefore, to determine the shortcomings of traditional programs to ensure that new STEM-focused initiatives are intentionally planned.

For that purposes and because the STEM education is very popular all over the world, we propose and we are starting to work on a project for developing a new methodology in order to support individuals in acquiring and developing basic skills and key competences and to construct open education and innovative practices in a digital era.

Project description

In order to achieve economic and industrial growth, the countries need highly trained STEM - Science, Technology, Engineering and Mathematics professionals. The European Union, as well as the rest of the world, is constantly in a lack of such profiles. On the other hand, mathematical courses have the main role in attracting students to STEM and reaching highly trained STEM professional, both by their curriculum and teaching methodologies. Although a lot of attention and resources were dedicated to the development of curriculums of different subjects including mathematics, teaching methodologies are not having the same attention. Most of the universities in teaching mathematics to STEM students still rely on the traditional nineteen-century approach based on blackboard and chalk. Videos and animation are poorly involved in teaching methodology. This approach is productive for a limited number of students, it is not interesting for students and does not follow the growing need for STEM professionals with high education and does not make use of the new technologies that are easily available nowadays. Therefore, the main objectives of this project are the design of new interactive methods and practices for teaching mathematics in STEM context in the spirit of active and collaborative learning; implementation of these methods in selected real-life courses, their improvement, and dissemination. These methods will both, attract students to STEM and contribute to the development of highly skilled STEM professionals. Additionally, in the past decades, we are witnessing the rapid development of new technologies that can be exploited in the educational process. So, the development of the e-learning platform for the new interactive methods will also be one of the objectives. This project is carried transnational because as stated above the need for modern and efficient methodologies for teaching mathematics in a STEM context and attracting students to STEM are global and do not recognize national boundaries.

The project aims to develop new methods and practices for teaching mathematics to STEM students that will be based on collaborative learning and will make use of the modern technologies that are easily available nowadays. The new methods and practices will be far more effective in learning and bringing mathematics closer to the STEM students and in attracting students to STEM, then the traditional still dominant approach based on blackboard and chalk. Also, this project is complementary of projects on developing curriculum's for STEM studies (there were lots of such TEMPUS projects) and is a continuation of projects focusing on mathematics on primary and high school.

Expected Results

During the realization of the project we expect:

- To justify that active and collaborative learning is the most appropriate approach for teaching mathematics in STEM context that will bring mathematics closer and make it adaptable for a wider range of students that will further lead to well-trained STEM professionals and attract students to STEM.
- To have new interactive methods and practices for teaching mathematics in a STEM context in the spirit of active and collaborative learning using new and modern technologies. This will include producing lecturing and learning material that will make use of new technologies for representative topics in mathematics, applying them in real life courses, analysis of the experiences and improvement of the methods.
- Students to gain a positive attitude towards mathematics. The motivation for learning mathematics is expected to be increased, and this will gain better achievements of the students not only in mathematics but also in other STEM areas.
- University teachers to gain better competence. They will look at the teaching process from the point of view of the students and have a better understanding of it. This is the way they are going to upgrade their teaching skills.
- All of the participants will gain better linguistic and communication skills.

- Through the work on this project, meeting different people and collaborate with them, they will appreciate the differences and become more open and tolerant of the changes that have to occur in the process of education.

After projects completion we expect that the mathematical knowledge of the undergraduate STEM students will be improved, their motivation to carry on with STEM studies will be bigger, and the general attitude toward STEM will be on a higher level.

Project methodology

This new teaching methodology which is developed, implemented and used in the framework of this project will enhance STEM students' knowledge, creativity, and skills in mathematics within the STEM context. Using active collaborative learning and ICT methodology for learning will lead to enlarge students' skills to solve practical problems and improve their entrepreneurial thinking and key competencies. Students will be an active part of the educational process using this new methodology. The active learning of the proposed new methodology will enlarge their motivation to study, and the results are unavoidable. Students gained knowledge and skills in math can be applied for success in today's world, such as critical thinking, problem-solving, communication and collaboration. A project of this kind is an excellent opportunity for making arguments between minds, criticizing different opinions on some topic, all of it with one goal: achieving excellent skills of math of students to be used in everyday life. The new method of active and collaborative learning will help students to work effectively in diverse teams, respect cultural differences, and respond open-mindedly to different solution ideas and values. The level of knowledge of mathematics for the STEM students will be increased. All of this gives us sustainable investment, performance, and efficiency.

Math education is STEM education (Mathematics in STEM education)

Math is literally everywhere as an integral part of all fields of study. Mathematics is a common thread shared by all these disciplines, and the Arts too, wherever patterns, volume, shapes and representation are used to delight the senses. Mathematics is everywhere. Teaching mathematics well is an important component of a comprehensive STEM program. The mathematics that students learn in school includes content and thinking that can be used as tools for tackling integrative STEM problems, [30]. There is more to mathematics, however, than being part of STEM, [30]. But, because of the quality of teaching, students find mathematics very abstract and fear mathematics. Much can be gained in support of the teaching and learning of mathematics through connecting and integrating science, technology, and engineering with mathematics, both in mathematics classes and in STEM activities.

STEM education could be seen as a means that support a constructivist approach in learning as teachers facilitate and scaffold students' meaningful learning, as is cited in [10]. Mathematics is often mentioned as underpinning the other disciplines of STEM because it serves as a language for science, engineering, and technology. Is that enough acknowledgement of the potential role of mathematics in STEM, [9]? The implementation of an integrated STEM education raises many challenges for the teaching and learning of mathematics but "transforming the current educational paradigm toward a STEM education perspective" has the potential to "foster the connectedness that reflects the way the world works outside of school and assist students to develop the knowledge and ability to deal with change and challenge in sensible ways", [9].

NCTM (National Council of Teachers of Mathematics) supports connections between curricula, technology integration in education, and critical thinking, but not supports any restriction of mathematics learning. NCTM have recognized the importance of addressing STEM fields in education and affirm the essential role of a strong foundation in mathematics as the center of any STEM education program. In addition to integrative experiences connecting the disciplines of STEM, students need a strong mathematics foundation to succeed in STEM fields and to make sense of STEM - related topics in their daily lives. In order to be made some integration of technology in support of mathematics learning and as well as to be made some connections between curricula both within mathematics and to contexts outside of mathematics, have been published the guidelines *Principles and Standards for School Mathematics* (NCTM 2000) and then reinforced in *Principles to Actions* (NCTM 2014).

According to NTCM, the mathematical practices outlined in the standards of many states have much in common with the scientific and engineering practices. This kind of practices emphasize the importance of understanding problems, developing and using models to solve problems, constructing viable arguments based on evidence, and critiquing the reasoning of others. When we engage students in the standards for mathematical practice, we are making connections to and supporting science education. Such practices and examples give the opportunity for mathematics teachers to make meaningful connections to science (and other disciplines) in support of STEM educational goals while maintaining the integrity of mathematics learning standards.

If in the “STEM program” the teaching and learning mathematics is not taught conceptually but rather as a procedural tool to solve various disjointed applications, or if the mathematics is not developed within a coherent mathematical learning progression, then the “STEM program” fails. Thus, any STEM education program (including out-of-school activities) should support and enhance a school’s mathematics program.

Mathematics is very important course in STEM education. All STEM programs’ creators have primary positive attitude towards mathematics’ importance. They encourage the deep learning of mathematical concepts and put the mathematics in STEM curriculum as very important and necessary subject for appropriate STEM education. Because of all changes in the education on all levels: primary, secondary and university level in STEM context, teaching and learning mathematics must remain part of every curriculum during an era that emphasizes STEM preparation.

Teaching mathematics well is an important component of a comprehensive STEM program. There is more to mathematics, however, than being part of STEM. Although STEM education as a commonly recognized field does not have a long history, its rapid development can help introduce ideas for exploring how mathematics can be taught and learned. The mathematics that students learn in school includes content and thinking that can be used as tools for tackling integrative STEM problems. But it also includes content that might be considered "just math" or might be connected to non-STEM disciplines. Mathematics goes beyond serving as a tool for science, engineering, and technology to develop content unique to mathematics and apply content in relevant applications outside of STEM fields (NCTM).

Professional recommendations for the teaching and learning of mathematics include offering students challenging, engaging, and relevant problems consistent with STEM recommendations from the public and private sector. Teaching mathematics and science well, according to these recommendations, can help students develop creativity, reasoning, and problem-solving skills that align with the goals of STEM programs.

The program of any course of mathematics of an effective STEM program should be designed to develop the content and practices that characterize effective mathematics programs while maintaining the integrity of the mathematics. Other, important principles, such curricular connections and the appropriate integration of technology, are only additional tool to ensure students learn important mathematics at a deep level and are confident in their ability to use mathematics in their everyday lives.

Implementing effective STEM education requires dedicated, organized, and knowledgeable individuals. It is important to have teachers that are committed to being long-term PLTW teachers and not just waiting for a math, science, or other job to become available. Teacher turnover can have negative effects for schools in terms of school cohesion, teaching effectiveness, and students' achievement. While teachers are developing their content knowledge of integrated STEM education, they can focus on quality strategies for teaching.

A growing number of institutions are offering integrated programs that lead to licensure in both math and science, particularly at the middle school level that might serve to lessen the effect of this issue. Since teachers may have different licensures and backgrounds, it is important for schools to provide support and time for collaboration.

According to NCTM [31], mathematics teachers and teachers of STEM should:

- Teach according to professional recommendations
- Whenever mathematics is included in a STEM activity, make sure that the mathematics addresses academic standards appropriate for the grade level and that it is taught in ways that support the development of mathematical thinking and quantitative reasoning.
- To support STEM education within the mathematics program, look for opportunities to integrate science, technology, and engineering in meaningful ways as students tackle problems involving mathematics in relevant settings.
- Whether teaching STEM or teaching mathematics, recognize whether one discipline is the primary emphasis of an activity and maintain the integrity of the discipline in terms of content, nature of thinking, and assessment.

Program/curriculum developers should:

- When developing programs and materials for mathematics, look for opportunities to integrate science, technology, and engineering in meaningful ways as applications for mathematics in solving problems in relevant settings.

- Whenever STEM activities might not fully address grade-level appropriate standards in mathematics, look for ways that the activities can support the overall development of problem solving, critical thinking, questioning, and academic curiosity.
- When assessing STEM learning, recognize the unique nature of integrative STEM activities and use or develop authentic assessment tools that look at connections and address problems integrating the STEM disciplines.

mathSTEM method

Stating that mathematics is a basis for the other disciplines sets mathematics up in a supporting role in integrative STEM education contexts. Ideally, mathematics should be given more standing and be considered an enabler or imperative for the advancement of understanding of concepts in other disciplines. But, from other hand high percent of students since their primary school shows mathematical anxiety during the learning process. There are many researches in which are considered and analyzed many methods that are applied in the process of teaching and learning mathematics in order to overcome the students' mathematical anxiety. In [32] is shown that one way to assist teachers in overcoming students' mathematical anxiety during the learning process is learning based on Science, Technology, Engineering, and Mathematics (STEM). By applying the STEM approach in learning it is expected that it can lead to meaningful learning for students. In this STEM context in [32] is shown that the mathematical modelling is playing an important role in STEM integration and education. Mathematical modeling is the process of analyzing real-life situation or realistic situation using mathematical methods. Mathematical modeling involves a complex process in which a problem state encountered in real life is formulated mathematically and solved with the help of mathematical models, and the solution is interpreted and evaluated in the real world. Mathematics is used to represent, analyze, predict, or otherwise make sense of real-world situations, [33], [34]. The idea that mathematical modeling cycles should be used in STEM education at all levels from primary to tertiary education has gained importance in recent years, since it increases the students' motivation towards the lesson and they learn better by concentrating their attention, [8]. But it also includes content that might be considered "just math" or might be connected to non-STEM disciplines. Problems involving mathematical models of finance might or might not connect to science or engineering and might or might not involve in-depth uses of technology. Likewise, art might be integrated into a mathematics lesson that does not involve either science or engineering. Mathematics goes beyond serving as a tool for science, engineering, and technology to develop content unique to mathematics and apply content in relevant applications outside of STEM fields, [30].

It is seen that technological developments are taken into consideration in the conceptualization of the mathematical modeling process. Considering the modeling used in the mathematical modeling process, the emergence of different frameworks and approaches reveals the complex structure of the process. For this reason, it is seen that the studies related to the mathematical modeling process, taking into account the different effects of technology, are combined with STEM, and this leads to the emergence of richer cognitive and metacognitive processes.

As a result of the importance of STEM activities in solving mathematical modeling and real-life problems in different disciplines, STEM activities continue to be integrated into schools. As a result, it can be said that the teaching done by using mathematical modeling together with STEM increases the students' motivation toward the lesson; they learn better by concentrating their attention on the subject, leaving a positive effect on them; and the students' success and attitudes toward the lesson increase. Solving real-life problems in the future through STEM and mathematical modeling will continue to play an important role in providing innovative and creative problem-solving perspectives in the cultural and economic development of the countries.

Nowadays STEM is common in the formal and informal learning processes in most countries. All approaches of learning are based on various frameworks of what STEM actually means. Based on these frameworks a vast literature is being compiled of the successes and pitfalls of STEM education. In recent years STEM has suffered various critiques [35]. It remains difficult to obtain a full integration of the various disciplines partly because the nature of disciplines differ fundamentally, disciplines like science and technology were over-represented with often less attention on mathematics and engineering. But, the practice has shown that STEAM education without deep understanding of mathematics could not reach its goals.

What kind of topics are to be studied within STEAM? Most authors state that open-ended, real-world problems are to be considered as they most likely offer the best possibilities for a true integration of the various disciplines. Using realworld problems raises the motivation of the learners when they are related to their everyday-live experiences or offer the prospective of a meaningful solution of a valuable problem.

Some of the most effective strategies being used to improve Math teaching outcomes are aligned with the STEAM Education methodologies, [33]. Stating that mathematics underpins the other disciplines sets mathematics up in a supporting role in integrative STEM education contexts. Ideally, mathematics should be given more standing and be considered an enabler or imperative for the advancement of understanding of concepts in other disciplines, [9]. These include framing a math problem in a multidisciplinary context, something real world that children can relate to. This helps increase engagement and promotes deeper thinking and problem solving, and encourages other important soft skills key to math mastery, like persistence, [33]. The whole idea with STEM is that students need to figure out solutions to problems. They need to be the ones who are using critical-thinking skills and innovation. And, as teachers, we really have to give students the opportunity to be innovative. It's not something that can be delivered to a student standing in front of the classroom, [8]. Math is also an important element in teaching kids to code, another engaging modality to help introduce math in a multidisciplinary setting like creating a simple computer game or animation sequence. Basically, a blended approach is a great approach, with plenty of options for educators, [33].

It is a tendency in educational circles that learning should be interesting and fun. STEAM is presented almost exclusively in the form of problems to the learners. The discipline is problem-centered which means that subject-matter content and attitudes are learned through the solution of real-world problems. As we have mentioned above, STEAM is a transdisciplinary interaction between science, technology, engineering, mathematics, arts, humanities and ecological awareness. The content is delivered in a performing problem method, were students and teachers co-operate as co-researchers. It is the role of the teacher to select those problems that arouse interest by the students, meet the set goals of the educational process and are within reach of the capabilities of the learners. It is the teacher who makes the difference in the learning process, who in the background regulates the process through successful selection of the problems, feedback and feed forward during the problem finding and solving process and careful and adequate evaluation afterwards [36].

In [35] there are studies about STEAM education and STEM methodologies, together with the target group, research sample and country where the research took place. As a structure for the discussion there are used four categories often considered in discussing curriculum reform: the student, the teachers, assessment and the context. The category assessment includes the development of evaluation criteria and instruments, while the category context can include factors inside (e.g. school policy) and outside the school (e.g. governmental or regional policy). Within the category student in those studies the authors distinguish explicitly between pre-service teachers (i.e. students in teacher education) and other students in elementary, secondary of higher education. In the last group the focus lies on the acquisition of STEAM related skills and competences, while in the first subgroup the emphasis lies on their future profession as a teacher and their ability and attitudes towards the implementation of STEAM as an educational vision. Interpersonal relations and collaboration are important factors for learning and were explicitly used in this context. However it was concluded that there have to be done some steps towards changing and introducing new pedagogical forms that better prepare students for professional technical education, for higher education and the labor market. As educational methods, project-based learning and collaborative technologies can assist in understanding the STEAM principles.

Because of that in the framework of the project the new mathSTEM method is developed. MathSTEM method is a new interactive teaching method. This new method is based on the tasks for the modern so-called 21st-century learning. It is prepared within computer-supported collaborative learning, based on constructiveness, enabling students to create their own way of learning. In particular, the process of mathematical modeling in a view of new technology is applied in mathematics courses for STEM students. The suitable software is used to support the needs and specifications of the project. The mathSTEM method enable students to collaborate between them, share their opinions and increasing their knowledge on math topics. The new teaching method involves communication between the students and besides math skills also develop their literacy skills, ICT skills, critical thinking skills, cultural skills and make social inclusion. By using of this new teaching methodology, the STEM students will be better motivated to learn mathematical courses. This will lead not only to better results on the exams, but students will be also better prepared for increasingly complex life and work environments in today’s world. This includes increasing students’ creativity and innovation, critical thinking and problem solving, communication and collaboration.

Why mathematical software in mathematical courses?

The history of contemporary mathematical education in schools and universities, among other things, is the history of a struggle against computers and IT. Not so long ago the use of calculators in math classes was forbidden. Later mathematical software for the solution of more complex tasks (Mathematica, Maple, Matlab, Mathcad, SMath, Derive, etc.) was not recommended for use by students, (and of course was extremely expensive for researchers), [37].

Teachers who forbade the use of such computer tools claimed that deep understanding the process of solving a given problem is the most important aspect of learning mathematics. They thought that while studying, the final result is less useful and less interesting than the method by which it is obtained. All of these teachers who used the traditional method of teaching by using of blackboard and chalk said that when some student is trying to solve a given problem they must think and do mental effort, which is actually the biggest advantage for the students who learn mathematics.

From teachers' point of view, the using of the mathematical software bring absence of the students' thinking and the mental efforts. By using of the computer and digital technologies the students become capable only for clicking to obtain the result which is calculated by computer. But today's high school and college students do not understand this point of view and consider it outdated and unsustainable. Moreover, they cannot imagine school without a computer. Authors in many researches have concluded that in nowadays most of the students have better results in mathematical exams if they use some mathematical software (Mathematica, Matlab, Mathcad, SMath etc.), [37], [38]. The mentioned computer programs are quite expensive. They are not always affordable for schools and universities. Additionally, such programs need to be on personal computers (tablet PCs, smartphones) for pupils and students to do their homework. However, most companies that develop math programs give substantial discounts to educational institutions and in some cases, they share the program for free.

Modern computer methods must not be ignored at school, especially at the universities where the students of engineering faculties faced up with complex mathematical problems. The computer methods must be included as additional and helpful tool, for simplifying of the complicated mathematical calculations. The fact that usually in mathematics most of the teacher in schools require from their students to memorize a set of rules and theorems must not prevent the students from tackling more complex nonstandard tasks, where the emphasis should be done on "mathematics, physics, chemistry, etc."

Usually at most of the technical faculties, in the first-year students have a few mathematical subjects. The teachers have presentation at their classes and usually use the traditional method for teaching. But is this being what exactly students needed and what they can learn with this approach from their professors and assistants? They are not afraid of computers because they had long since mastered them at their work. But they are very frightened of mathematical analysis: limits, derivatives, integrals, and so on. If their math teacher wants to help his students not just learn the basics of Mathematics and Computer Science, but also get pleasure from it, he has to make efforts to combine the traditional method and modern approach by using computers. This can be done with Mathematical Software and Teaching so they will get not only the knowledge and skills. They also will get joy and satisfaction from study, as well as from any other difficult but fruitful work. In [16] is shown that the Computer algebra systems (CAS's) such as MAPLE, MATLAB, MUPAD and MATHEMATICA can be used as a powerful assistant to perform the symbol manipulations and computations in algebra as well as calculus. It has been suggested that these systems will benefit undergraduates and postgraduates in mathematics, engineering and physics by keeping track of the details in complicated manipulations.

Which are advantages of using mathematical software in teaching mathematics at university level?

The authors in [37] noted the following:

1. Advanced mathematical computer programs allow using a fresh approach to the teaching of mathematics in schools and universities, taking into account the attraction of pupils and students to computers.
2. By means of graphics and animation, one can significantly increase the understanding of pupils and students of the basic concepts and theorems of mathematics.
3. Modern information technologies can transform and change the traditional solutions of mathematical problems.
4. In order to make progress in the influence of the computer to the process of learning mathematics the teacher must exercise direct guidance.

But besides these, such important advantages of the using of mathematical software at the mathematical classes, many studies have shown several obstacles that teachers experience. In [39] and [24] the authors found a number of barriers for the integration of ICT into lessons:

(1) lack of confidence among teachers during integration,

(2) lack of access to resources,

(3) lack of time for the integration,

(4) lack of effective training,

(5) facing technical problems while the software is in use,

(6) lack of personal access during lesson preparation and

(7) the age of the teachers.

But all of these obstacles can be avoided if the teachers want to modernize their work and to be open minded and have desire for improvement of the teaching process. Every teacher should try to give more and more from himself for the students. Students should learn modern approaches during the class, which will teach them how to use the knowledge in practice in real life situations. Teachers cannot use modern approaches if they do not want to change the curriculum of the course. All contents of the course should be adequate to the method.

Because of that, in the framework of this mathSTEM project is developed a new teaching method which can help of all mathematics professors to have modern approach at the classes with the students of technical sciences. There are given lesson plans which can be used for better organization of the classes and for increasing of students' interest and motivation for learning mathematics. Mathematics is the queen of the sciences, but also mathematics is powerful tool for solving problems in many sciences. Let mathematics become interesting, favorite and applicable subject for the students. Let apply mathematical knowledge in other technical sciences. And the most important, let allow the computer software to do the calculations and to help us to solve the practical problem in easier and more productive way.

Lessons plans

Before designing STEM education on some education level, the teachers who will participate in the STEM team must be able to determine the approach to implementing STEM learning to be used. There are three approaches in implementing STEM namely silo, embedded, and integrated, [40]. The learning process which will be implemented on the classes must be determined before. According to [41], the learning planning process is carried out to answer four questions, including; 1) what are the objectives to be accomplished as the results of instruction? 2) What topics will we cover? 3) What procedures are best for directing the learning? and 4) How do we evaluate instruction? In [42], the four questions are translated in the form of a learning planning component consisting of: defining objectives, compiling contents, selecting strategy and making media, and designing evaluation.

First, the planning process begins with defining of STEM education objectives. The competencies of a science material that students must understand, must be analyzed. The competency then becomes the basis for making competency achievement indicators that are used as a reference to determine the achievement of a competency that has been written previously. Second, the teacher analysis of STEM material is based on predetermined learning objectives. The teacher must analyze the material from the point of view of the four component materials. Third, the selection of learning strategies and making media is based on the needs of the teacher in making the product, the availability of time, and the characteristics of the material being taught. STEM education in science learning is directed to provide space for students to make products. However, teachers are recommended to use three approaches, including project-based learning, 5E instructional model, and project-based learning.

To support effective mathematics instruction, we propose the use of the mathSTEM method for the students at tertiary level and postgraduate level, especially for the students at engineering faculties. By using of this method, the students will study mathematical concepts in a different point of view. Instead of the traditional way of teaching, when the teachers actually teach the basic definitions and theorems for some mathematical concept, and then students do some given examples, by using of this mathSTEM method they started with consideration of some real-life problem in order to show the students the importance of knowing of that mathematical concept. In this way, engineering students will be more motivated to study mathematics in order to applicate that knowledge in the other subject which are essential for their future profession.

The teacher should choose adequate real-life problems which are nearly connected to the main goal of the study program. After that the teacher must include (if is possible) some mathematical software for visualization or in order to simplify calculations while solving some mathematical problem. By using of mathematical software in order to solve a given real-life problem, students will not be bored on the mathematical classes. Their motivation and interest will be increased, because in that way they will have a sense that mathematics can be learned in order to be applied in some other subject, not only as an abstract formula and concept.

The teacher must be prepared with adequate lesson plan for easier realization of the process of teaching mathematics. For that purpose and according to make easier implementation of the mathSTEM method at the mathematical classes, we are giving some lesson plans which coverage some mathematical content. All the lesson plans are given in one template.

Taking into consideration the nature of the mathematical concept for which they relate to, we can classify them in few categories:

- Mathematical analysis. Function theory;
- Complex Analysis;
- Linear algebra;
- Geometry and Analytical Geometry;
- Differential equations;
- Numerical methods and programming;
- Statistics;
- Game theory;
- Algorithms and Discrete structures;
- Mathematical modelling.

In every lesson plan, there are given the most important things for successful class for the teachers and for the students at the same time. In the template are given:

- Learning objectives and outcomes;
- Teaching methods/ strategies / techniques:
- Materials/ equipment;
- Previous knowledge assumed;
- Short description of the content;
- Outline of lesson (summary of tasks/activities);
- Extension activities for students who are progressing faster/slower;
- Assessment and
- References.

In every of these lesson plan for particular mathematical topic practical examples from real world are analyzed. Also, in every lesson plan is used one or more mathematical software like Wolfram Mathematica, Maple, GeoGebra, Desmos graphing calculator, Acrobat reader for visualization, MegaStat, STATA 14, SymboLab etc. In this way we are giving helpful material for the mathematics teachers to use them on their classes and also to gain an idea for other similar activities at their classes in the mathematical courses. These lessons plans are basis for the e-platform on which the mathSTEM method will be given in details. The lesson plans are given in annex which follows.

Conclusion

The collaborative work of the participants in this project resulted with the development of a new methodology and creation of innovative ways and approaches for teaching and learning mathematics in STEM (**s**cience, **t**echnology, **e**ngineering and **m**athematics) context using modern technologies.

STEM is an interdisciplinary approach to learning where academic concepts are coupled with real-world lessons. Students apply science, technology, engineering, and mathematics in contexts that make connections between the classroom and the world around them.

Mathematics is an important part of the integrated STEM education. Studying mathematics in this context should be done in a way that real problems should be concerned. Mathematical problem solving can be difficult for teachers to teach as well, thus they need a starting point like this Guidebook is, and appropriate lesson plans for STEM classes. However, helping students to become good problem solvers is a very important goal, and challenging and exciting at the same time. The teacher's job is to develop the natural ability of students as problem solvers to its maximum extent. Teachers do not have to put attention on the pure math procedures, but to teach students how to apply knowledge in the real problem context, and that is STEM context. Through teaching mathematics in STEM context, students develop key skills including problem solving, creativity, critical thinking, collaboration and teamwork, independent thinking, initiative, communication, digital literacy, etc, known as 21st century skills. These skills help in students' better adopting in the changing real world, during the studies and after their graduation. Thus, STEM helps to foster a love of learning. And the most important gift an education should give a student is a love of learning. Creating an appropriate good curriculum is the best way to enrich such goal. Teacher's approach has also significant impact in it.

For that purposes and because the STEM education is very popular all over the world, in the frame of this project a new method in teaching mathematics in STEM context, and that is mathSTEM method, is developed. It is an interactive method developed in order to support individuals in acquiring and developing basic skills and key competences and to construct open education and innovative practices in a digital era. The methodology involves the new digital technologies that are easily available nowadays and can contribute in better understanding and easily acquiring knowledge, and its better application in practice.

References

[1] M. Sanders, STEM, STEM education, STEMmania, The Technology Teacher, Dec./Jan.- 2009, 20-26.

[2] ”STEAM Education for a Reality Oriented and Fun Learning Environment” [https://www.teacheracademy.eu/course/steam-education/]

[3] Butz, W. P., Kelly, T. K., Adamson, D. M., Bloom, G. A., Fossum, D., & Gross, M. E. (2004). Will the scientific and technology workforce meet the requirements of the federal government? Pittsburgh, PA: RAND.

[4] https://www.nctm.org/Standards-and-Positions/Position-Statements/Building-STEM-Education-on-a-Sound-Mathematical-Foundation/

[5] T. J. Kennedy, M.R.L.Odell, *“Engaging Students In STEM Education”*, Science Education International Vol. 25, Issue 3, 2014, 246-258. [http://www.icaseonline.net/sei/september2014/p1.pdf]

[6] https://talkstem.org/what-is-stem/

[7] Steve Murphy, Amy MacDonald, Lena Danaia, Cen Wang, *“An analysis of Australian STEM education strategies”*. Policy Futures in Education 2019, Vol. 17(2) 122–139. [https://researchoutput.csu.edu.au/ws/portalfiles/portal/31360521/21391151_Published_article.pdf]

[8] https://www.intechopen.com/books/theorizing-stem-education-in-the-21st-century/the-role-of-mathematical-modeling-in-stem-integration-and-education

[9] Noleine Fitzallen, *“STEM Education: What does mathematics have to offer?” (*2014) [https://www.researchgate.net/publication/279750165_STEM_Education_What_does_mathematics_have_to_offer]

[10] Mahmut Kertil, Cem Gurel, *“Mathematical Modeling: A Bridge to STEM Education”.* Volume 4, Number 1, 2016. [https://files.eric.ed.gov/fulltext/EJ1086722.pdf]

[11] Bashman JD, Israel M, Maynard K. An ecological model of STEM education: Operationalizing STEM FOR ALL. Journal of Special Education Technology. 2010;23(3):10-19

[12] Vongai Mpofu, A Theoretical Framework for Implementing STEM Education

[13] Lynn Barakos, Vanessa Lujan, Craig Strang,*"Science, technology, Engineeringe, Mathematics (STEM) catalyzing change amid the confusion,* Lawrence Hall of Science at the University of California, Berkeley. [https://files.eric.ed.gov/fulltext/ED534119.pdf]

[14] D.F.Berlin, A.L. White, "Using technology in assessing integrated science and mathematics learning", Journal of Science Education and Technology, (4), (1995), 47-56;

[15] https://www.livescience.com/43296-what-is-stem-education.html

[16] Micah Stohlmann, Tamara J. Moore, Gillian H. Roehrig, *"Journal of Pre-College Engineering Education Research (J-PEER)"* Volume2, 2012. [https://docs.lib.purdue.edu/cgi/viewcontent.cgi?article=1054&context=jpeer

[17] Todd R. Kelley and J. Geoff Knowles, *"A conceptual framework for integrated STEM education"* Kelley and Knowles International Journal of STEM Education (2016). [https://core.ac.uk/download/pdf/81882354.pdf]

[18] https://www.steampoweredfamily.com/education/what-is-stem/

[19] A National Action Plan for Addressing the Critical Needs of the U.S. Science, Technology, Engineering and Mathematics Education System; National Science Foundation, October 2007 https://www.nsf.gov/pubs/2007/nsb07114/nsb07114.pdf

[20] Engaging Students in STEM Education T. J. KENNEDY*, M. R.L. ODELL†, Science Education International Vol. 25, Issue 3, 2014, 246-258

[21] https://www.stemconnector.com/stemconnector-releases-state-of-stem-report/

[22] Holmes K, Gore J, Smith M, Lloyd A. An integrated analysis of school students' aspirations for STEM careers: Which student and school factors are Most predictive? International Journal of Science and Mathematics Education. 2018;16(4):655-675

[23] *"The Role of Mathematics in interdisciplinary STEM education"* [https://link.springer.com/article/10.1007/s11858-019-01100-5#Sec2]

[24] Holmlund TD, Lesseig K, David S. Making sense of "STEM education" in K-12 contexts. International Journal of STEM Education. 2018;5(32):3-18

[25] https://study.com/teach/stem-education.html

[26] Kelly C. Margot, Todd Kettler ,Teachers' perception of STEM integration and education: a systematic literature review, *International Journal of STEM Education* **volume 6**, Article number: 2 (2019)

[27] J.A.Ejiwale, Barriers to successful implementation of STEM education, Journal of education and learning, (2013), Vol 7(2), p. 63-74.

[28] "*STEAM Education: an overview of creating a model of integrative education"* [https://www.researchgate.net/publication/327351326_STEAM_Education_an_overview_of_creating_a_model_of_integrative_education]

[29] Ashraf A. Zaher, Issam W. Damaj, Extending STEM Education to Engineering Programs at the Undergraduate College Level, International Journal of Engineering Pedagogy, Vol 8, No 3, (2018)

[30] *"Building STEM Education on a Sound Mathematical Foundation",* A joint position statement on STEM from the National Council of Supervisors of Mathematics and the National Council of Teachers of Mathematics

[31] www.nctm.org

[32] Noleine Fitzallen, STEM Education: What does mathematics have to offer? Conference: 38th annual conference of the Mathematics Education Research Group of Australasia, At: Sunshine Coast, June 2015

[33] https://www.steamcraftedu.com/better-mathematics-with-steam-education/

[34] M.Tezer, The role of mathematical modeling in STEM integration and education, Theorizing STEM Education in the 21st Century, (2019)

[35] Stefan Haesen, Erwin Van de Put, "STEM Education in Europe: A Comparative Analysis Report", 2018.

[36] John Hattie, "Visible learning for teachers: Maximizing impact on learning", Routledge/Taylor & Francis group, 2012.

[37] Bogomolova, E.P., Maksimova, O.V., 2014. "Analysis of the methods of mathematical statistics influence of Internet - support for learning mathematics in technical colleges". In: Information Technology of the educational process of a modern university [electronic resource]: Sat of reports. Intern. Internet Conf., Minsk, Nov 1-30. 2013 - Minsk: BSU, 2014. - p. 273-286. Available via: http://elib.bsu.by/handle/1 23456789/ 89685, accessed August 25, 2014.

[38] https://www.researchgate.net/publication/232415430_Teaching_and_Learning_using_Mathematics_Software_The_New_Challenge

[39] Jones, A. (2004). A review of the research literature on barriers to the uptake of ICT by teachers. UK: Becta

[40] Roberts A and Cantu D 2012 Applying STEM Instructional Strategies to Design and Technology Curriculum Technology Education in the 21st Century 73 111-118

[41] Peter F O and William G 2013 Developing the Curriculum (London: Pearson)

[42] https://www.researchgate.net/publication/336920650_Designing_lesson_plan_of_Science_Technology_Engineering_Mathematics_STEM_education_in_science_learning

ANNEX

Lesson plans

Mathematical analysis. Function theory

LESSON PLAN MathSTEM project No.: 2019-1-HR01-KA203-060804		
Partner Organization	South East European University	
Course/Year	**Calculus/**First Year	
Topic	**Integrals**	
Lesson title	Application of definite integral	
Lesson duration:	2 hours	Date:

Learning objectives and outcomes
By the end of the lesson all students should be able to: 1. To understand the geometrical meaning of the definite integral. 2. To apply the definite integral in some simple problems using the regular graphs on decart's coordinate system. 3. To apply the definite integral in different problems for finding the length, perimeter and area of some elementary figures. By the end of the lesson some students should be able to: 4. To find the area of irregular figures in a plane using the definite integration 5. To find the area between two curves 6. To find the area of rotational solids 7. To find the volume of rotational solids. 8. To distinct the cases and the formulas obtained when the curve is rotated around x-axis and when it is rotated around the y-axis.
Teaching methods/ strategies / techniques:
• brainstorming • discussion • interactivity • collaborative learning • visualization • real life problems • individual work
Materials/ equipment
The course materials including books, Internet connection and student computers, Mathematical software for visualization, online resources on Google, etc.

Previous knowledge assumed:
Functions, the graph of a function, limit of a function, derivative of a function, Integral of a function, definite and indefinite integration.
Short description of the content:
In this lecture students will learn concerning some applications of definite integral in the case of geometrical calculations. At the beginning they will learn about the geometrical meaning of the definite integral. Then, they will learn about the way on how to calculate the length of the curve by using the definite integration, how to calculate the area of the figures in the plane and how to calculate the volume of the rotational figures on the space. All these aspects will be followed by the corresponding

examples as well as illustrations by using the appropriate mathematical software. The different real-life problems will be taken into the consideration during this discussion.

Outline of lesson: (summary of tasks/activities)
• Joint activity (5 min) The teacher starts the class by asking some questions concerning the calculation of definite integral as well as the areas of some geometrical figures bounded by straight lines. Students respond to the asked questions and there is a discussion concerning these facts in order to make connection with the new topic. • Teacher activity (10 min) Teacher explains how the area of triangle can be found by expressing it as the sum or the difference of the areas of right angled triangles. From this, teacher continues to explain that the same logic can be used to find the area of any figure bounded by straight lines, by dividing it up into triangles. • Student activity (10 min) Students are requiring to divide some figure into a triangles and to give their opinion on how the area can be calculated. • Teacher activity (15 min) Teacher explains how to get the formula for calculation of the area under the curves. Explanation is followed by the frontal discussion with students. • Student activity (10 min) Students individually solve few examples with calculation of the area of some figures under the given curve. • Student activity (10 min) Students use online software (symbolab, integral calculator etc.) to have a view concerning the obtained figure for which they are calculating the area and at the same time they can compare the result. After comparison they continue by solving other examples in order to be aware about the importance of applying the mathematical software in such situations, because sometimes even if they are able to solve and to make conclusions by their selves, applying software is much faster, easier and much clear. • Joint activity (5 min) The teacher gives an example about the area between two curves. Students can have the view at the figure using the software and then they give comments how the area can be found. They can see the result by using the software. • Teacher activity (10 min) Teacher gives the formula for calculating the length of a curve as well as the formula for the volume and area of the rotational solids. This is followed by some graphical illustrations using online software. • Student activity (10 min) Students individually solve few examples with calculation of the length of the curve and the volume of rotational solids. They use the online software to have the look at the figure and then compare their results. • Joint activity / teacher-students (5 min) At the end of the class there is a general overview and a summary of gained knowledge about the different geometrical applications of definite integration.

Extension activities for students who are progressing faster/slower
• For students who do not have sufficient progress, a special group will be formed and will be worked on a basic level with additional support from the teacher and with the support of corresponding online software. • For students who fully understand the objectives of the lesson, they will engage with more advanced exercises where either the integration is more complicated, or the obtained figure is more complicated. • Advanced students will be committed to helping students with lower knowledge.
Assessment:
1. Students engage in homework assignments concerning the mentioned topics. 2. Students are committed to solving their homework through different software which are appropriate for them. 3. We will form the Google meet group through Google classroom platform for any kind of additional discussion concerning the discussed issues.

References:

1. Introduction to integration, Part 2: The definite integral, Mary Barnes, Mathematics learning center, University of Sidney, 1999.
2. Calculus, volume 1, Edwin Herman and Gilbert Strang, openstax, 2018, Rice University.
3. Calculus 1, Paul Dawkins, pdf, online book, 2007.

LESSON PLAN MathSTEM project No.: 2019-1-HR01-KA203-060804		
Partner Organization	South East European University	
Course/Year	**Calculus**/First Year	
Topic	**Series**	
Lesson title	Numerical Series	
Lesson duration:	2 hours	Date:

Learning objectives and outcomes
By the end of the lesson all students should be able to: 1. Understand the concept of numerical series 2. Understand the partial sums and the sum of the numerical series 3. Calculate the sum of a geometric series and some basic series 4. Understand and use series convergence tests By the end of the lesson some students should be able to: 5. Describe a strategy to prove the convergence of a given series. 6. Apply numerical series in the field of economics
Teaching methods/ strategies / techniques:
• brainstorming • discussion • interactivity • collaborative learning • visualization • real life problems
Materials/ equipment
The course materials including books, Internet connection and student computers, Mathematica software for visualization, Symbolab online software, and online resources in Google.

Previous knowledge assumed:
Numerical sequences, arithmetic and geometric sequences, limits of numerical sequences, finite sums and their properties.
Short description of the content:
In this lecture students will learn about numerical series definition, partial sums, the sum of numerical series, geometric series, algebraic properties of convergent series, convergent and divergent series, criteria of convergent series.

Outline of lesson: (summary of tasks/activities)
• Teacher activity (5 min) The teacher starts the class by asking some questions to the students about numerical sequences, arithmetic and geometric sequences, convergence and divergence of sequences, finite sums and their properties.

- Student activity (10 min)

Students provide answers and debate about the arithmetic and geometric sequences, convergence and divergence of sequences, and properties of finite sums.

- Teacher activity (15 min)

The teacher introduces the definitions of numerical series, partial sums and the sum of the series. Examines an example of a numerical series for finding the partial sums and the series sum on the whiteboard and with the software Wolfram Mathematica.

- Student activity (15 min)

Students in smaller cooperative groups solve three exercises calculating the sum of numerical series in parallel they use Wolfram Mathematica to solve the exercises. Once they have completed the exercises, they compare the results with the other groups and conduct a social and respectful debate about the progress of the solution and the obtained results.

- Teacher activity (20 min)

The teacher explains the geometric series, solves an example, and stimulates students for active participation in solving examples on the whiteboard and with Wolfram Mathematica software. Next the teacher introduces the concepts of convergence and divergence of series as well as the criteria of convergence of numerical series.

- Student activity (10 min)

Students in groups solve practice exercises calculating the sum of geometric series, convergence and divergent series, and apply the criteria of convergence. They also solve the same exercises with the Wolfram Mathematica software, compare solutions and develop constructive debate in a positive environment.

- Joint activity / teacher-students (10 min)

The teacher presents an applied problem in the field of economics, finds the mathematical model related to geometric series, then the students solve it, and finally is discussed the solution of this practical problem.

- Joint activity / teacher-students (5 min)

At the end of the class there is a general overview and a summary of gained knowledge on numerical series and their convergence, use of software, analysis of solutions as well as application in solving practical problems in various fields.

Extension activities for students who are progressing faster/slower
• For students who do not have sufficient progress, a special group will be formed and will be worked on a basic level with additional support from the teacher and with the support of Wolfram Mathematica and Symbolab online software. • For students who fully understand the objectives of the lesson, they will engage with more advanced and applied exercises from various real-life areas. • Advanced students will be committed to helping students with lower knowledge.

Assessment:

1. Students engage in homework assignments regarding the calculation of the sum of numerical series, geometric series, examination of the convergence and divergence of series, and criteria of convergence.

2. Students are committed to solving their homework through mathematical software such as wolfram mathematica and Symbolab software.

3. Student groups are formed, where each group has a leader who coordinates collaboration when solving practice problems by communicating through online classroom platform.

4. The teacher communicates with group leaders and gives them additional tasks and professional support regarding the lesson and advanced exercises.

References:

1. Calculus, Volume 2, Edwin Herman & Gilbert Strang
   Second Edition, 2017, OpenStax – Rice University, (page, 450 – 495).
2. Kalkulus, Tanush Shaska
   Aulonna Press, 2018, Second Edition (page, 229-246).
3. Wolfram Mathematica software (https://www.wolfram.com/mathematica/online/)
4. Symbolab online software (https://www.symbolab.com/solver)

Complex Analysis

LESSON PLAN TEMPLATE mathSTEM project No.: 2019-1-HR01-KA203-060804		
Partner Organization	University of Zagreb Faculty of Electrical Engineering and Computing	
Course/Year	**Complex Analysis** / 2020	
Topic	**Möbius transformation**	
Lesson title	Möbius transformation	
Lesson duration	1 hour	Date:

Learning objectives and outcomes
By the end of the lesson all students should be able to: 1. Recognize and define Möbius transformation 2. State main properties of Möbius transformation 3. Apply elementary geometric transformations (translation, rotation, dilatation) By the end of the lesson some students should be able to: 4. Explain and apply inversion 5. Use general Möbius transformation to map various subsets in complex plane
Teaching methods/ strategies / techniques:
• discussion • visualization • application in engineering
Materials/ equipment
• The course materials including student computers, Internet connection, online Geogebra software for visualization, youtube videos

Previous knowledge assumed:
Complex functions, Analytic functions, Conformal mappings, Stereographic projection
Short description of the content:
Möbius transformation (or bilinear mapping) is a rational linear function which is composition of elementary geometric transformations: translation, rotation, dilatation and inversion. It is one of the most important conformal mappings in complex analysis which has many applications in computing and electrical engineering.

Outline of lesson: (summary of tasks/activities)
1. Teacher introduces topic of the lesson through elementary geometric transformations, using online mathematical software Geogebra for visualization. Student should be able to write mathematical notations of this transformations by themselves. Duration: 15 minutes 2. Teacher explains inverse transformation through Geogebra and shows youtube video “Möbius transformation revealed”. This should be achieved through high level of interactivity. Duration: 10 minutes 3. With high level of student participation, teacher explains main properties of Möbius transformation (conformal property; mapping circle to circle; mapping through only three points). Student should be able to derive these properties by themselves. Duration: 10 minutes 4. Teachers gives 2 exercises for mapping various subsets in complex plane. Students solve them in small groups. Duration: 15 minutes 5. Teachers gives relevant examples of application of Möbius transformation in technical engineering studies showing them online through google and youtube. Duration: 10 minutes

Extension activities for students who are progressing faster/slower
• If some students do not meet the lesson objectives, the teacher will work directly with them in a small group or individually. • For students who completely understand the objectives, have them exercises prepared in advance and encourage solving problems through Geogebra • Those who cope with tasks more quickly help others and practice collaborative learning.

Assessment:
Students will be given written exercises to evaluate their learning progress and accquired knowledge.

References:
1. D. Wunsch, *Complex variables with Applications*, Addison-Wesley, 1994. 2. T. Needham, *Visual complex analysis*, Oxford, 2000.

Linear algebra

LESSON PLAN MathSTEM project No.: 2019-1-HR01-KA203-060804		
Partner Organization	South East European University	
Course/Year	**Linear Algebra**/First Year	
Topic	**Systems linear equations**	
Lesson title	Discussing the solutions of the system of linear equations	
Lesson duration:	2 hours	Date:

Learning objectives and outcomes
By the end of the lesson all students should be able to: 1. Know the general form of the systems of the linear equations 2. Distinct between the square system of linear equations and the no square system of linear equations 3. Distinct between homogeneous and non homogeneous systems of linear equations 4. understand when two systems of linear equations are equivalent 5. give answer when there can be applied the Cramer's rule and when the Gaussian rule for solving system of linear equations By the end of the lesson some students should be able to: 6. understand when a given system of linear equations is impossible (has no solution) 7. apply elementary transformations 8. Recognize when a system is consistent and when it is inconsistent by using the concept of teh rank of a matrix. 9. To understand when a consistent system has just one and when it has infinite number of solutions. 10. Use different mathematical software for discussion the solutions of system of linear equations.
Teaching methods/ strategies / techniques:
• brainstorming • discussion • interactivity • collaborative learning • visualization • individual work
Materials/ equipment
The course materials including books, Internet connection and student computers, Mathematica software for visualization, online resources on Google, etc.

Previous knowledge assumed:
Linear equations, matrix theory, detemrinants.
Short description of the content:
In this lecture students will learn about the cases when a system of linear equations has a solution and when there are no solutions. If there is a solution also they will be able to know when it has just

one and where there are infinite number of solutions. For this purpose they must be able to find the rank of a matrix which is applied for this purpose. Also they will learn to discuss about the solutions in the case of homogeneous systems of linear equations. They will also learn to apply the elementary transformations which are very important for such applications. So, simply saying they will be prepared to the next step which is solving the systems of linear equations.

Outline of lesson:
(summary of tasks/activities)

- Joint activity (5 min)

The teacher starts the class by asking some questions concerning the meaning of linear equation and continues by matrices and determinants. Students respond to the asked questions and there is a discussion concerning these facts in order to make connection with the new topic.

- Teacher activity (10 min)

Teacher explains what is the meaning of the system of linear equations by defining the homogeneous and nonhomogeneous case as well as the square and non square system of linear equations. Then teacher gives the matrix form of the system of linear equations .

- Student activity (15 min)

Students are requiring to transform some given systems into the corresponding matrix form, as well as finding the rank of a matrix and the rank of extended matrix.

- Student activity (10 min)

Students compare their results concerning the rank of a matrix by the results from applying the special software as symbolab or even they can use Matlab. If there is mismatching they have to check their previous work.

- Teacher activity (10 min)

The teacher explains the conditions when the system of linear equations is consistent and when it is inconsistent. And also gives explanation that consistent system of linear equations may have just one and infinite number of solutions.

- Student activity (10 min)

Students individually solve few examples by just making conclusion concerning the number of solutions without trying to find the solutions.

- Student activity (10 min)

Students compare their results doing the same discussion. After comparison they continue by solving other examples in order to be aware about the importance of applying the mathematical software in such situations, because sometimes even if they are able to solve and to make conclusions by their selves, applying software is much faster.

- Teacher activity (5 min)

The teacher explains the triviality of homogeneous system of linear equations and gives the condition when such a systems have nontrivial solutions.

- Student activity (10 min)

Every student individually tries to make discussion about homogeneous systems and compares the conclusions by applying corresponding software.

- Joint activity / teacher-students (5 min)

At the end of the class there is a general overview and a summary of gained knowledge about the solutions of systems of linear equations.

Extension activities for students who are progressing faster/slower

- For students who do not have sufficient progress, a special group will be formed and will be worked on a basic level with additional support from the teacher and with the support of corresponding online software.
- For students who fully understand the objectives of the lesson, they will engage with more advanced exercises which contain parameters and other form of discussions.
- Advanced students will be committed to helping students with lower knowledge.

Assessment:

1. Students engage in homework assignments concerning the discussion of solutions of systems of linear equations.
2. Students are committed to solving their homework through different software which are appropriate for them.
3. We will form the Google meet group through Google classroom platform for any kind of additional discussion concerning the discussed issues.

References:

1. Linear Algebra, Richard Bronston & Gabrial B. Costa,
   Second Edition, 2007, Academic Press – Elsevier, (page, 31 – 48).
2. Linear Algebra with Applications, Jim Defranza & Daniel Gagliardi
   Published by McGraw-Hill, 2009, (page, 1-25, 48-53, 62-68).
3. Linear Algebra, K.R. Matthews,
   Departments of Mathematics, University of Queensland, 2013, (page, 1 – 21).

LESSON PLAN MathSTEM project No.: 2019-1-HR01-KA203-060804		
Partner Organization	South East European University	
Course/Year	**Linear Algebra**/First Year	
Topic	**Systems of linear equations**	
Lesson title	Solving of systems linear equations	
Lesson duration:	2 hours	Date:

Learning objectives and outcomes
By the end of the lesson all students should be able to: 1. Know the general form of the systems of the linear equations 2. Solve systems by Cramer's formulas and by the Gauss method 3. Solve systems using a mathematical software By the end of the lesson some students should be able to: 4. Solve systems of linear equations by different methods 5. Recognize systems that have no solution or an infinite number of solutions 6. Solve applied problems using the Gauss method
Teaching methods/ strategies / techniques:
• brainstorming • discussion • interactivity • collaborative learning • visualization • real life problems
Materials/ equipment
The course materials including books, Internet connection and student computers, Mathematica software for visualization, and online software Symbolab on Google.

Previous knowledge assumed:
Linear equations, the method of substitution, the method of elimination, determinants, inverse matrices.
Short description of the content:
In this lecture students will learn about some methods of solving systems of linear equations (determinant method, inverse matrix method, Gauss method). Students will learn to distinguish between methods, which method is more suitable for solving systems of linear equations. Students will also learn how to apply these methods to solve practical problems in different areas and interpret the results.

Outline of lesson: (summary of tasks/activities)

- Teacher activity (5 min)

The teacher starts the class by asking some questions to the students about determinants and inverse matrices that are related to the methods of solving systems of linear equations.

- Student activity (10 min)

Students provide answers and debate about the determinants and inverse matrices as well as the use of inverse matrices and determinants in solving systems of linear equations.

- Teacher activity (15 min)

The teacher explains the methods of solving the systems of linear equations, the Cramer's method and the inverse matrix method. Solve an example with both methods on the whiteboard and with Symbolab online software, then compare these two methods.

- Student activity (15 min)

Students in smaller cooperative groups solve two exercises (systems of linear equations) with Cramer's method and inverse matrix method, in parallel they use Symbolab software or Wolfram Mathematica to solve the exercises. Once they have completed the exercises, they compare the results with the other groups and conduct a social and respectful debate about the progress of the solution and the results obtained.

- Teacher activity (15 min)

The teacher explains the Gaussian method of solving linear equation systems, solves an example, and stimulates students for active participation in solving examples on the whiteboard and with Wolfram Mathematica software. Finally, an analysis is made for detecting when a system of equations does not have a solution, and detecting when a system of equations has infinitely many solutions.

- Student activity (15 min)

Students in groups solve practice problems using the Gaussian method; they also solve the same exercise with the Wolfram Mathematica software, compare solutions and develop constructive debate in a positive essence.

- Joint activity / teacher-students (10 min)

The teacher presents an applied problem in the field of business, finds the mathematical model as a system of linear equations then the students solve it with the Gaussian method, and finally the solution of the system is the solution of the practical problem.

- Joint activity / teacher-students (5 min)

At the end of the class there is a general overview and a summary of gained knowledge on systems solving methods, use of software, analysis of solutions as well as application of systems in solving practical problems in various fields

Extension activities for students who are progressing faster/slower

- For students who do not have sufficient progress, a special group will be formed and will be worked on a basic level with additional support from the teacher and with the support of Wolfram Mathematica and Symbolab online software.
- For students who fully understand the objectives of the lesson, they will engage with more advanced and applied exercises from various real-life areas.
- Advanced students will be committed to helping students with lower knowledge.

Assessment:

1. Students engage in homework assignments regarding the Cramer's method, the inverse matrix method, and the Gauss method.

2. Students are committed to solving their homework through mathematical software such as Wolfram Mathematica and Symbolab software.

3. Student groups are formed, where each group has a leader who coordinates collaboration when solving practice problems by communicating through online classroom platform.

4. The teacher communicates with group leaders and gives them additional tasks and professional support regarding the lesson and advanced exercises.

References:

1. Linear Algebra, Richard Bronston & Gabrial B. Costa,
   Second Edition, 2007, Academic Press – Elsevier, (page, 31 – 48).
2. Linear Algebra with Applications, Jim Defranza & Daniel Gagliardi
   Published by McGraw-Hill, 2009, (page, 1-25, 48-53, 62-68).
3. Linear Algebra, K.R. Matthews,
   Departments of Mathematics, University of Queensland, 2013, (page, 1 – 21).
4. Wolfram Mathematica software (https://www.wolfram.com/mathematica/online/)
5. Symbolab online software (https://www.symbolab.com/solver)

Geometry and Analytical Geometry

LESSON PLAN TEMPLATE mathSTEM project No.: 2019-1-HR01-KA203-060804		
Partner Organization	University of Split / Faculty of Civil Engineering, Architecture and Geodesy	
Course/Year	**Descriptive Geometry,** 1st year of Undergraduate Professional Study of Civil Engineering	
Topic	**Orthogonal projections in the Monge method**	
Lesson title	Orthogonal projections of plane figures and the method of revolution	
Lesson duration:	45 minutes	Date: April, 2020

Learning objectives and outcomes
By the end of the lesson all students should be able to: 1. comprehend the idea of the true dimension of a plane figure 2. carry out the standard procedure of a point / plane figure revolution into the planes of projections 3. recognize the difference between the true dimension of a plane figure and its orthogonal projections, both presented simultaneously on 2D-surface By the end of the lesson some students should be able to: 4. predict the behavior of orthogonal projections when changing the position of the plane of the figure 5. analyze visibility of plane figures depending on their 3D-space quadrant belonging 6. manage to add the height and construct a solid whose basis is actually the constructed plane figure
Teaching methods/ strategies / techniques:
• discussion • construction • collaborative learning • visualization • detecting the analogous situations with plane figures / solids included in real life environment
Materials/ equipment
The course materials include • internet connection with online resources (S. Gorjanc, E. Jurkin, I. Kodrnja, H. Koncul: Descriptive geometry, web textbook, Faculty of Civil Engineering, Zagreb, 2019) • optional: Flatland-A Journey of Many Dimensions, animated film (2007), after the novel E.A. Abbott: Flatland-A Romance of Many Dimensions; www.flatladthemovie.com • V. Szirovicza, E. Jurkin: Descriptive geometry, CD textbook, Croatian Society for Geometry and Graphics & Faculty of Civil Engineering, Zagreb (2005) • student computers with GeoGebra software installed and with prepared templates for construction and visualization of the specific tasks

Previous knowledge assumed:
Orthogonal projections in the Monge method on the planes of projections for the top, front and side views, depending on the 3D-space quadrants to which the following objects belong: points, lines, line segments, plane figures. Lines in a plane: properties. Lines orthogonal to a plane: properties. The true dimension of a line segment. The true dimension of a plane figure that is parallel to the plane of projection or is perpendicular to it. Perspective affinity.
Short description of the content:
Gaspard Monge (1746-1818) was a French mathematician and inventor of descriptive geometry, the one after whom the main method of orthogonal projections onto pair of mutually orthogonal planes was named. 3D-space is thus divided into four quadrants and object in them are observed by the top and the front view in the horizontal and vertical plane, respectively. The projections of object are deformed comparing to their true shape and dimensions in 3D-space. The idea is to control the stage of deformation in order to manage constructions of the objects alone or combined with more complex tasks, in particular when considering plane figures. The method of revolution provides us with the original shape and dimensions of a plane figure which it possesses in 3D-space. For a plane in a general position we can obtain the true dimension of any figure laid in it, in the same 2D-surface where we construct their orthogonal projections by the top and the front view.

Outline of lesson:
(summary of tasks/activities)

1. Introduction (approx. duration: 5 minutes)

At the very beginning of the lesson students are motivated for the contents that follows. The teacher presents visualizations on the screen and these include various solids in general positions toward the planes of projections (prisms, pyramids, coni and cylinders) in the final construction stage. The Power Point presentation can serve as an appropriate tool since it is of interest for students to experience the final orthogonal projections of a solid, with and without top and front view visibility included. The aimed purpose is that the students ask themselves about the single steps of a complex construction they see in both projections simultaneously: how is the basis of a solid constructed or how is the height of a solid obtained. Bases of solids are either polygons or circles in a general position in 3D-space and the current lesson deals with their orthogonal projections in the planes of projections. Thus it also gives an answer to the main question about the solids.

Orthogonal projections do not reflect the true dimensions of the objects in 3D-space, actually they are almost always deformed comparing to their spatial shape. First of all, students need to become aware of the fact that polygons and circles are plane figures and as such they are subordinated to all demands that the plane position in 3D-space assigns.

The teacher then reminds the group of the process of determination of the true length of:

1. a line segment (90^o rotation of its projecting trapezium onto the plane of projections) which thus becomes visible in its true length on the same 2D-surface as its projections themselves;
2. a plane figure that is parallel to a plane of projections: if the figure is parallel *e.g.* to the horizontal plane of projections, it reflects in the top view its true dimension which coincides with its top view projection;
3. a plane figure that is perpendicular to a plane of projection: if the figure is perpendicular *e.g.* to the horizontal plane of projections, its top view projection is a line segment on the first trace of its perpendicular plane and the procedure for its true dimension is then similar to that of a line segment: 90^o rotation onto the horizontal plane of projections.

The teacher can support all of these cases by examples in Power point presentation in combination with the blackboard, as a finale of the introductory part of the lesson.

2. The method of revolution: spatial explanation (approx. duration: 5 minutes)

When a plane figure belongs to a plane in a general position toward the planes of projections, *i.e.* to a plane for which the inclination angle differs from 90^o, then a more sophisticated method is required. The teacher continues with the presentation in Power Point. A spatial position of a general plane is given and one point in the plane is emphasized. Its revolution onto the plane of projections around the corresponding trace of its plane is explained in detail. Using the presentation and the blackboard, the teacher puts an accent on the steepest line through the chosen point whose orbit around the trace of the plane describes the orbit of the point. There are two possible solutions of the process of the revolution of the point. Radius of revolution as well as the center of revolution need to be emphasized, too, for the sake of the later construction.

3. Task 1: revolution of a point in a general plane (approx. duration: 10 minutes)

Given two traces of a plane in a general position and given one orthogonal projection of a point in that plane, construct the revolution of the point onto the horizontal plane of projections, around the first trace of the given plane.

The formerly explained task now needs to be done in orthogonal projections of the Monge method. Students are instructed to open the GeoGebra program and use the prepared templates with the given elements for the task. Each student works at their own computer, but they also collaborate among themselves. Step by step construction, with the instructions and the help of the teacher is the following:

1. finding the other orthogonal projection of the point (by means of a line in the plane);
2. determination of the true length of the radius of revolution;
3. recognizing the angle of revolution (the inclination angle of the plane or its supplemental angle);
4. revolution of the point back on the steepest line, according to the angles (two solutions);
5. recognizing the existence of spatial perspective affinity between the plane in a general position and the chosen plane of projections together with its elements: fixed axis, rays, orthogonal projection of perspective affinity itself onto the plane of projections.

4. Task 2: construction of the orthogonal projections of a square in a general plane (approx. duration: 20 minutes)

Given two traces of a plane in a general position and given one orthogonal projection of a side of a square which is laid in that plane, construct the orthogonal projections of the square (the top and the front view).

Since the second task is more complex than the first one, those students who had problems accomplishing the first task are now organized to sit at the same computer with ones who managed the task in time and successfully. Students open the new prepared template and similarly as before, under the supervising ad with the help of the teacher do the task which is divided into the following steps:

1. find the other orthogonal projection of the side of the square, by means of the line(s) in the plane;
2. apply the method of revolution to one of two given square vertices (a vertex is rotated into the plane of projections around the corresponding trace of its general plane);
3. determine the position of another vertex after the revolution, this time not using the whole procedure of revolution, but making use of perspective affinity;
4. construct the square in its true dimension, laid onto the plane of projections;
5. bring two vertices of the square back into the corresponding orthogonal projection (anti-revolution), around the trace of the plane, depending on the choice of the revolution: use perspective affinity for the first vertex, and the last one can be achieved by simple parallelism property;
6. bring two vertices into the second orthogonal projection, by means of the lines of the plane

and parallelism;

7. check the parallelism of the opposite sides of the square in both orthogonal projections and the visibility of the square in the top and in the front view projection;

8. move the elements of the construction in GeoGebra: the traces of the plane or the position of the square side and then see how other elements change, reflecting the different positions in 3D-space, belongings to adjoining quadrants and change of visibility according to that. Try to recognize and make conclusions to which quadrants the vertices belong and how the plane of the square is laid toward the planes of projections.

The advantage of GeoGebra is that students can experience more than one position of the object in the Monge method by solving one task constructively.

3. Task 3 (additional): rising the height of a square pyramid (approx. duration: 10 minutes)

In continuation of the previous task, construct the center of the square, rise the line orthogonal to the plane of the square and apply the given length of the height of a square pyramid, from its basis center.

This task overlaps in time with the previous one, since it is organized for those students who managed to do the previous task successfully and in time and are capable of more sophisticated one: construction of a pyramid with the square basis. Steps are described in the text of the task and corresponding steps have been already separately solved during the former lectures. Furthermore, the teacher can make a sketch with an explanation for pyramid visibility in two orthogonal projections (the top and the front view) before the constructions of its edges in GeoGebra.

4. Task 4 (alternative to Task 3): comprehending the idea of dimensions through the animated film "Flatland-A Journey of Many Dimensions"

For those students who are not capable of the additional Task 3 during the current lesson, an insert of the animated film "Flatland-A Journey of Many Dimensions" about geometric characters living in a two-dimensional world can serve as a motivation for further comprehending of the idea of dimensions in 3D-space and more. Since the film is given on the internet, all students can watch it later on or even read the novel by E. A. Abbott from the 19th century, after which the film was made.

Extension activities for students who are progressing faster/slower

- Students who do not cope with the tasks within the lesson are going to do the first (simple) task individually, in collaboration with other colleagues or with the teacher's active assistance and in that case the second task will do eventually;
- Students who completely understand the objectives of both tasks get an extra task, to upgrade the second task by raising the height of a solid (a square pyramid) whose basis is the square with the constructed orthogonal projections;
- Moreover, students who cope with both tasks successfully are encouraged to help those who do not in a way of collaborative learning and team work.

Assessment:

Since solid bases are polygons or circles, it is of interest to solve a similar task to Task 2 (and Task 3) which includes a circle. The teacher gives an assessment with the same blank template in Geogebra which students are supposed to do at home, where the text of Task 2 (Task 3) is slightly altered:

instead of the square side, diameter of a circle is given.

Task (homework):

1. Given two traces of a plane in a general position and given one orthogonal projection of a diameter of a circle which is laid in that plane, construct the orthogonal projections of the circle (the top and the front view).

2. In continuation of the previous task, rise the line orthogonal to the plane of the circle and apply the given length of the height of a cylinder, from its basis center. Construct its upper basis and determine the visibility of the cylinder in the top and the front view.

References:

- S. Gorjanc, E. Jurkin, I. Kodrnja, H. Koncul: Descriptive geometry, web textbook, Faculty of Civil Engineering, Zagreb, (2019)
- V. Szirovicza, E. Jurkin: Descriptive geometry, CD textbook, Croatian Society for Geometry and Graphics & Faculty of Civil Engineering, Zagreb (2005)

LESSON PLAN TEMPLATE mathSTEM project No.: 2019-1-HR01-KA203-060804		
Partner Organization	University of Split / Faculty of Civil Engineering, Architecture and Geodesy	
Course/Year	**Principle of Projections 1**, 1^{st} year of Undergraduate University Study of Architecture and Urban Planning	
Topic	**Orthogonal axonometry method**	
Lesson title	Orthogonal axonometry of domes	
Lesson duration:	45 minutes	Date: December, 2020

Learning objectives and outcomes

By the end of the lesson all students should be able to:

1. construct the orthogonal axonometric image of the square basis of an object which is vaulted by the spherical dome
2. determine the main elements of the contour sphere and of the ellipses as circles' projections in vertical planes
3. draw all existing lines (arcs) of the spherical dome according to their visibility in 3D-orthogonal axonometric image

By the end of the lesson some students should be able to:

4. comprehend the procedure of construction of pendentive dome, by means of the constructed spherical dome and the construction of a half-sphere
5. predict the embedding of pendentives as transitional elements of the pendentive dome
6. have an idea for construction of the bohemian dome over the rectangular basis of an object, instead of the quadratic one

Teaching methods/ strategies / techniques:

- discussion
- construction
- collaborative learning
- visualization
- geometric analysis of architectural structures-domes
- recognizing the types of domes in architecture using geometric knowledge

Materials/ equipment

The course materials include

- internet connection with online resources: S. Gorjanc, E. Jurkin, I. Kodrnja, H. Koncul: Descriptive geometry, web textbook, Faculty of Civil Engineering, Zagreb, (2019)
- V. Szirovicza, E. Jurkin: Descriptive geometry, CD textbook, Croatian Society for Geometry and Graphics & Faculty of Civil Engineering, Zagreb (2005)
- Slides with images of different types of famous domes (online resources) or examples from our country / city (photos)
- student computers with AutoCad software installed and with prepared templates for construction and visualization of the specific tasks

Previous knowledge assumed:

Orthogonal projections in the Monge method on the planes of projections for the top, front and side views, basics of orthogonal axonometry method, construction of the orthogonal axonometric image of a circle leaned in any plane of projections and construction of a half-sphere over the circle section.

Short description of the content:
Axonometric methods are often used in representation of architectural structures and are complementary to the Monge method in descriptive geometry as well as to perspective. Namely, often orthogonal projection onto two planes of projections (the Monge method) does not offer spatial visibility that would suffice in architectural practice. On the other hand, axonometry provides a multitude information on the observed object and is often combined with perspective method of reality simulation. Thus, these three methods can continuously be combined and mutually supplemented. Among various axonometric methods, orthogonal axonometry provides the best opportunity to construct and to visualize spherical elements of objects, and these are of a special interest when presenting vaults that are parts of spheres. Orthogonal axonometry as a method is based on orthogonal projection of the coordinate system with its 3 axes (x, y and z) onto the chosen surface - plane of projections. The xy- plane, yz-plane and xz-plane thus leave their traces in the plane of projections which form the *trace triangle*, the basis for further construction of the orthogonal projection of the coordinate x, y, z – system. With this coordinate system on the plane of projections involved, all of the classical methods for determining true dimensions of objects, needed for their final axonometric projection, can be applied. Taking into account that the contour circle of the sphere stays with the same radius in orthogonal axonometry as in its true length, it is much easier to construct all of the spherical elements and also the transitional ones when leaning the various types of dome vaults down onto the quadratic (rectangular) basis. Three main types of domes in architecture are spherical, bohemian and pendentive domes.

Outline of lesson:
(summary of tasks/activities)

1. Introduction (approx. duration: 5 minutes)

Before the main task of the lesson, students are firstly prepared through the presentation of the main spots in the previous lesson on orthogonal axonometry: trace triangle, orthogonal axonometric image of a polygon, pyramid, then of the circle and a sphere. A special accent is on the easier construction of the major axis and minor axis of an ellipse as a projection of a circle, than it is case with other axonometric methods. Also the contour circle of a sphere does not change its true length of the radius in this sort of projection. The construction of the sphere and its circular sections are needed in the sequel as well.

2. On three main types of domes in architecture (approx. duration: 5 minutes)

Using slides with images of various architectural objects vaulted by domes, not only world famous, but also the examples from their own city / country, the teacher explains the elements of domes which are built over the square or rectangular bases of objects. Three main types of domes are spherical dome (over the square basis), pendentive dome (spherical dome upgraded by a half-sphere supported by pendentives or spherical triangles) and bohemian dome (over the rectangular basis). Three types of domes are simultaneously presented by their pairs of orthogonal projections where the geometric elements of their architectural concept can be recognized properly by students. This presentation serves as a starting point in the following task.

3. Task 1: construction of a spherical dome in orthogonal axonometry (approx. duration: 25 minutes)

> *Given a trace triangle and two orthogonal projections of a spherically vaulted object with a square basis, construct the orthogonal axonometric image of the corresponding spherical dome.*

Students are instructed to open the AutoCad program and use the prepared templates with the given elements for the task. Each student works at their own computer, but they also collaborate among themselves. Step by step construction, with the instructions and the help of the teacher is the following:

1. rotation around the first trace of the trace triangle in the plane of projections and determining the true dimension of the basis square;
2. according to the rules of orthogonal axonometric projection, carrying out the projection of the basis square, as a parallelogram;
3. drawing the contour circle of the dome with the true length of the given sphere radius;
4. construction of ellipses as projections of circles in the vertical planes (parallel in pairs): major axis, minor axis, the highest point and two points on the square for each (half)-ellipse in the vertical plane;
5. visualization and drawing the arcs of the spherical dome, according to its spatial visibility.

4. Task 2 (additional): construction of a pendentive dome in orthogonal axonometry (approx. duration: 10 minutes)

In continuation of the previous task, i.e. upgrading the spherical dome, construct the orthogonal axonometric image of the pendentive dome.

Now students who managed to finish the first task get copied instructions with a scheme of how to upgrade the new type of dome out of the constructed one, in the same template in AutoCad:

1. put the center of the horizontal circle section of the half-sphere on the z-axis;
2. determine the true length of the radius of the half-sphere;
3. construct the half-sphere as before;
4. put accent on the common points of the half-sphere and the spherical dome, i.e. on the constructed spherical triangles-pendentives and determine the visibility of the constructed pendentive dome.

Extension activities for students who are progressing faster/slower

- Students who do not cope with the tasks within the lesson are going to do the first part of the task (orthogonal axonometry of the square basis of the object) individually, in collaboration with other colleagues or with the teacher’s active assistance and also the contour sphere and at least one circle projected as an ellipse in one of four vertical planes. The rest of the task will be finished eventually;
- Students who completely understand the objectives of the task will get an extra task, to upgrade the second task by raising the half-sphere on the horizontal circular section of the spherical dome and thus construct the pendentive dome with the specific spherical triangles-pendentives as transitional elements;
- Moreover, students who cope with both tasks successfully are encouraged to help those who do not in a way of collaborative learning and team work.

Assessment:

The objects with rectangular bases demand another type of vaulting by a sphere: corresponding dome is named bohemian dome and differs from the spherical dome in its appearance. Students should notice what geometrical causes for these differences are. The teacher gives an assessment with the blank template in AutoCad which students are supposed to do at home, where the previous task is slightly altered: instead of the square basis, a rectangular one is given, together with the top and the front view of the object in its orthogonal projections.

Task (homework):

Given a trace triangle and two orthogonal projections of a spherically vaulted object with a rectangular basis, construct the orthogonal axonometric image of the corresponding bohemian dome.

References:

- S. Gorjanc, E. Jurkin, I. Kodrnja, H. Koncul: Descriptive geometry, web textbook, Faculty of Civil Engineering, Zagreb, (2019)
- V. Szirovicza, E. Jurkin: Descriptive geometry, CD textbook, Croatian Society for Geometry and Graphics & Faculty of Civil Engineering, Zagreb (2005)
- KoG, Scientific and Professional Journal of Croatian Society for Geometry and Graphics, No. 7, Zagreb, (2003)

LESSON PLAN TEMPLATE mathSTEM project No.: 2019-1-HR01-KA203-060804		
Partner Organization	University of Split / Faculty of Civil Engineering, Architecture and Geodesy	
Course/Year	**Computer Geometry** / 1st year of Undergraduate University Study of Geodesy and Geoinformatics 2020.	
Topic	**Planar transformation**	
Lesson title	Perspective collineation and affinity	
Lesson duration:	45 min	Date: March, 2020

Learning objectives and outcomes

By the end of the lesson all students should be able to:

1. draw collinear image of polygon
2. draw affine image of polygon
3. draw and solve elementary constructive tasks using computer program of dynamic geometry (GeoGebra)

By the end of the lesson some students should be able to:

1. draw collinear image of circle
2. draw affine image of circle

Teaching methods/ strategies / techniques:

- discussion with students about possible solutions of problems
- collaborative learning between students (working in group)
- visualization
- finding corresponding real life problems

Materials/ equipment

The course materials include

- internet connection with online resources (S. Gorjanc, E. Jurkin, I. Kodrnja, H. Koncul: Descriptive geometry, web textbook, Faculty of Civil Engineering, Zagreb, 2019);
- V. Szirovicza, E. Jurkin: Descriptive geometry, CD textbook, Croatian Society for Geometry and Graphics & Faculty of Civil Engineering , Zagreb (2005)
- student computers with GeoGebra software installed and with prepared templates for construction and visualization of the specific tasks

Previous knowledge assumed:

Students are assumed to have previous basic knowledge on the geometry of the projective plane, such as knowing basic elements (points and lines) and relations between them (the incidence structure of points and lines), as well as the system of axsioms. Students are already familiar with few planar transformations: translation, rotation, reflection, central symmetry. It would be useful to remind themselves of the properties of those transformations.

Short description of the content:

Planar transformation is every bijective transformation which maps a plane onto itself, meaning that each point of the plane is mapped onto one and only one point of the same plane, and vice versa, each point of the plane is the image of only one point in the same plane.

If the planar transformation preserves collinearity of the points (the incidence structure of points and lines) that it is called a planar collineation.
We will study two planar collineations - perspective collineation and perspective affinity. These transformations are of crucial importance for the planar and spatial objects constructions, as well as for the better understanding of some spatial relations.

Outline of lesson: (summary of tasks/activities)
INTRODUCTION: Planar transformation (duration approximately 5min) – teacher introduces topic of the lesson using Power Point presentation PERSPECTIVE COLLINEATION • Basic properties (duration approximately 5min) – teacher explains using Power Point presentation and interacts with students enabling them to predict which properties are preserved in this method of transformation • Perspective collinear image of a point and a straight line (duration approximately 5min) – teacher presents using computer program of dynamic geometry GeoGebra • Perspective collinear image of a polygon (duration approximately 10min) – students solve problem and draw on computers using GeoGebra. Learning is collaborative, high level of interactivity and student participation are achieved. PERSPECTIVE AFFINITY • Basic properties (duration approximately 5min) – teacher explains using Power Point presentation and interacts with students enabling them to derive properties of this method of transformation • Affine image of a point and a straight line (duration approximately 5min) – teacher presents using Geogebra • Affine image of a polygons (duration approximately 10min) – students solve problem and draw using GeoGebra. Learning is collaborative, high level of interactivity and student participation are achieved.

Extension activities for students who are progressing faster/slower
• If some students do not meet the lesson objectives, the teacher will work directly with them in a small group or individually. • For students who completely understand the objectives, have them exercises prepared in advance in GeoGebra. They will be encouraged to investigate and give reasons for their solution process. • Those who cope with tasks more quickly help others and practice collaborative learning.

Assessment:
Two exercises will be given to students in order to better understand and adopt lesson objectives, as well as to evaluate themselves.

References:

- S. Gorjanc, E. Jurkin, I. Kodrnja, H. Koncul: Descriptive geometry, web textbook, Faculty of Civil Engineering, Zagreb, (2019)
- V. Szirovicza, E. Jurkin: Descriptive geometry, CD textbook, Croatian Society for Geometry and Graphics & Faculty of Civil Engineering, Zagreb (2005)

LESSON PLAN TEMPLATE mathSTEM project No.: 2019-1-HR01-KA203-060804		
Partner Organization	University of Zagreb, Faculty of Food Technology and biotechnology	
Course/Year	**Mathematics 2** / 2020	
Topic	**Analytical Geometry of Three Dimensions**	
Lesson title	Analytical Geometry of Three Dimensions	
Lesson duration	1 hour	Date: 25.02.2020.

Learning objectives and outcomes
By the end of the lesson all students should be able to: 6. Parametric forms of a line and of a plane in the 3D 7. Implicit form of a plane in the 3D 8. Mutual position of a plane and a line in 3D 9. Orthogonal projection of the line to the plane 10. Finding the symmetric line of another line with regard to a plane 11. Finding the volume of a parallelepiped
Teaching methods/ strategies / techniques:
• discussion • visualization • application in engineering
Materials/ equipment
• The course materials including student computers, internet connection, acrobat reader for visualization

Previous knowledge assumed:
Determinants, binary operation on two vectors: dot product and cross product.
Short description of the content:
Analytical Geometry of Three Dimensions is treated trough application of three dimensional vectors. Its basic objects, a plane and a line are firstly described by radial vectors (parametric form). Implicit form of a plane is further described via dot and cross product. Definition of the mixed product and its interpretation as volume of a parallelepiped.

Outline of lesson: (summary of tasks/activities)
1. Teacher introduces topic of the lesson through applications of vectors, using prepared interactive pdf files. Student should be able to visualize and detect defining object of lines, planes and parallelepipeds. Duration: 15 minutes 2. Teacher derive parametric form of a line and a plane, and then derives implicit form of a plane. Duration: 15 minutes 3. Teacher discuss, by examples, various cases to find the line and the plane. This will be achieved through high level of interactivity with 3D visualization (acrobat reader). Duration: 10 minutes 4. Teacher introduce notion of mixed product of vectors, discuss on the associativity property and then gives geometric interpretation as volume of parallelepiped. Duration: 10 minutes 5. Teacher gives exercises tasks on the mixed product and then discuss results using interactive 3D models. Duration: 10 minutes

Extension activities for students who are progressing faster/slower
• For students who completely understand the objectives, teacher introduce the problem/solution for minimal distance between skew lines

Assessment:
Students will be given written exam.

References:
1. J. Stewart, *Calculus,* Brooks/Cole, Publishing Company 7th edition, 2010. 2. Internal script: G. Dražić, J. Jakšetić, M. Praljak, I. Perić, A. Vukelić, *Matematika 2*

Differential Equations

LESSON PLAN TEMPLATE mathSTEM project No.: 2019-1-HR01-KA203-060804		
Partner Organization	Hacettepe University	
Course/Year	**Differential Equations**/3. year	
Topic	**First order differential equation**	
Lesson title	A mathematical model eliciting activity: An application of linear differential equation	
Lesson duration:	50 min	Date :

Learning objectives and outcomes
By the end of the lesson all students should be able to: 1. Understands the applications of mathematical concepts in other fields of science. 2. Makes the solution of linear differential equations for some special values. 3. Interpret the solution of linear differential equations in accordance with the real-life problem. By the end of the lesson some students should be able to: 4. Express a daily life problem with mathematical concepts such as rate of change, integral factor, linear differential equations. 5. Have the ability to create mathematical models for real life problems.
Teaching methods/ strategies / techniques:
Modeling activity is presented on a real life problem by using the following teaching methods. • Brainstorming • Discussion • Debates During the modeling activity • collaborative learning students are divided into two groups and asked to discuss the problem. With the help of computer for the solution of the problem • visualization and the model eliciting activity is discussed.
Materials/ equipment
The course materials including • books of differential equations, • real life problems • Internet connection and student computers for modeling eliciting, • Maple and Excel software for visualization, • online resources on Google, etc.

Previous knowledge assumed:
We recommend that students should be familiar with the following notions: Increasing and decreasing functions, Derivative, Rate of change, Integral techniques, Linear differential equation. It is preferred that students have used the software program Maple and Excel before.

Short description of the content:
A daily life problem will be presented to students in the context of mathematical modeling activity. Preparatory questions and class discussions will be held to understand the problem. Then students will be divided into two groups and will try to model the problem mathematically. In this process, students' suggestions for solution will be evaluated, software such as Excel, Maple will be discussed by creating visualizations for solution suggestions of groups and guidance for solution will be provided. At the end of the process, mathematical models created by the groups for the solution of the problem will be discussed in the classroom.

Outline of lesson:
(summary of tasks/activities)

The beginning of the model eliciting activity (10 min)

The necessary equipment for the event (powerpoint presentation of the problem, necessary graphics-tables in Excel and Maple program for visualization) will be available.

Students will be informed that they will work individually during the activity process, then group work, and finally group solutions will be shared with the class.

Later, students will be presented with a modeling activity titled “Çıldır Lake Yellow Carp Fish Under Danger of Destruction”. The problem is given below.

Application Problem:

ÇILDIR LAKE YELLOW CARP FISH IS IN DANGER OF DESTRUCTION

Çıldır Lake, which of %60 in Ardahan and of %40 in Kars, is one of Turkey’s largest lake. The main income sources of the people living in the region of this lake with an area of $123\ km^2$ are fishing and agriculture. The most famous fish of Çıldır Lake, which hosts various fish species, is “Yellow Carp”. Yellow Carp lay eggs in spring. However, the reasons such as drought in recent years, the hunting of fish species such as seagulls of fish eggs, the wrong hunting activities of people, invasive species derived in the lake, such as freshwater lobster, attack fish eggs and greenery in the lake have threatened the extinction of the Yellow Carp. As an employee of the team working to prevent the extinction of the Yellow Carp extinction, you are asked to report when the Yellow Carp in Çıldır Lake will disappear and help your team.

After the modeling activity is presented, it is foreseen that students will be given 5-10 minutes to read and understand the activity carefully. Then, the group work will be started and it will be emphasized that every member of the group should actively participate in the process.

Model eliciting activity preparation process (10 dk)

In order to make sure that the activity is fully understood at the beginning of the group work, the students will be asked some questions such as below and it will be checked if they fully understand the problem situation in the activity.

1. How many Yellow Carp are there in Çıldır Lake now?

2. What is the number of Yellow Carp that grows in the farm environment (without any predator threats or natural threats) in a year?

3. How many Yellow Carp fish do the hunter threat and natural threats destroy in a year?

These questions will be supported by graphics and tables in Excel and Maple software.

Model eliciting activity process (20 min)
In this process, students will not be directed to a certain solution in order to enable students to make their own decisions about the solution. If there are students who do not participate actively in group works, these students will be asked to share their thoughts about the solution of the problem with their friends. It will not be expected to find a single solution planned in advance, students will be encouraged to find the correct answer. Different ways of thinking and solution will be encouraged and the thoughts and methods of all students will be listened to. Towards the end of the activity, students will be told that they should be prepared for the presentation of the solutions they have produced for the problem situation.

End of model eliciting activity (10 dk)

A student who will present the solution will be determined in each group. The order of starting presentations will be determined by taking into account the solution approaches. While a group is presenting, others will be listened carefully. An appropriate discussion environment will be created where students will evaluate the solutions of the groups, ask questions and make comments.
Concepts such as rate of change in the activity, linear differential equations, integral multiplier will be dealt with formally and its relationship with the problem will be revealed.

Extension activities for students who are progressing faster/slower

- For students who do not understand the effectiveness, a discussion will be made on the list of necessary variables and assumptions in the problem.
- Suggestions will be given to students who understand the problem in establishing a mathematical model.
- Those who cope with tasks more quickly help others and practice collaborative learning.

Assessment:

• Homework about writing model eliciting activity problem is given.

• Teacher takes notes about student learning progress and assess their knowledge and skills by asking specific questions.

References:

Berry, J. & Houston, K. (1995), Mathematical modeling, Bristol, UK.
Dost, Ş. (Ed.) (2019). Matematik Eğitiminde Modelleme Etkinlikleri, Pegem Akademi: Ankara.
Saaty, T. L., & Alexander, J.M. (1981). Thinking with models: mathematical models in the physical, biological, and social sciences. Pergamon Press, Oxford.

LESSON PLAN TEMPLATE mathSTEM project No.: 2019-1-HR01-KA203-060804		
Partner Organization	Hacettepe University	
Course/Year	**Differential Equations**/3. year	
Topic	**First order differential equation**	
Lesson title	Linear differential equation	
Lesson duration:	50 min	Date :

Learning objectives and outcomes
By the end of the lesson all students should be able to: 1. Recognize first order linear differential equations. 2. Shows whether a given function is a general solution or a particular solution of a given linear differential equation. 3. Finds integrating factor of a linear differential equation. 4. Makes geometric interpretation of solutions of the first order linear differential equations and search singular solutions. By the end of the lesson some students should be able to: 5. Explains the ralationship between a linear differential equation and a one parameter family of curves. 6. Finds singular solution of a linear differential equation. 7. Makes applications of the first order linear differential equations on oral problems.
Teaching methods/ strategies / techniques:
• discussion • debates • Pre/post question • Word problems
Materials/ equipment
The course materials including • books of differantial equations • Maple and Excel software for visualization • Power point presentations • Personal computer and a projector

Previous knowledge assumed:
• We recommend that students should be familiar with the following notions: Derivative, rate of change, integral techniques, exact differential equations. • It is preferred that students have used the software program Maple before.
Short description of the content:
Generally various differential equations types are used to express mathematical models explaining real life problems on the disciplines such as physics, engineering, astronomy, economy and statistics. While these mathematical models are generated variables changing depending on each other and rate of change of one quantity according to the other quantity (derivative) are needed. Besides it is easier solving the connection between derivatives using differential equation models and interpreting counterparts of these on real life. In this course, the concept of first-order linear differential equation will be introduced. By integrating factor, the algorithm of the equation and solution will be presented. On the other hand, applications of the first order differential equation in verbal problems will be made.

Outline of lesson:
(summary of tasks/activities)

Motivation: (5 min)

- Recall the concept of derivative and rate of change
- Mention to students that the importance of differential equations in mathematical models of real life problems

(10 min)

- The definition of the first order linear differential equation is given.
- It is mentioned that the equations of this kind are not generally complete.

Example: Examine the properties of the differential equations given below and show that they are not complete.

(1) $y' + xy = x^2$

(2) $xy' - 2y = \sin x$

(15 min)

- The integral factor of the linear differential equation is calculated.
- Solution algorithm of linear differential equation is obtained.

Example:

(1) Find the general solution of the differential equation $y' + \frac{1}{x}y = x^2$.

(2) Find the solution of the initial-value problem $y' - 3y = e^{2x}$, $y(0) = -1$.

(20 min)

- First order differential equations are applied in verbal problems.

Example 1: (Newton Cooling law) Let T be the temperature of an object at time t and the constant temperature of the environment where T_0 is located such that

$$\frac{dT}{dt} = \alpha(T - T_0), \quad (\alpha\ constant)$$

In a hotel room with a room temperature of 16, the body temperature at the time of a dead person is 20. It is observed that the body temperature of this person drops to 15 after 2 hours.

Accordingly, calculate how many hours before the moment the person was killed.

Example 2: The population of a country is increasing in proportion to itself. The population is 20 million in 1990 and 22 million in 2010. Accordingly, calculate the population in 2040.□□□□□□□□

Extension activities for students who are progressing faster/slower
• For students who do not reach the learning outcomes of the course, a general review is made. Individual work is carried out between the teacher and the student. • Students who reach the learning outcomes of the course are presented with an advanced real life problem and asked to create solutions through group work.

Assessment:
• Assignments related to first order differential equation solutions are given. • Writing a real-life problem containing a differential equation is given. • The teacher takes notes about the students' progress based on observations for the class and evaluates their knowledge and skills by asking specific questions.

References:
Braun M., Differential Equations ad their Applications, Springer-Verlag, 1993. Churchill, R.V.E., Modern Operational Mathemations in Engineering, Mc. Graw Hill, 1994. Shaums Series, Differential Equations, Mc. Graw Hill, 1995. Thomas, G., Calculus, Tenth Edition, Addison Wesley, 2001.

Numerical methods and programming

LESSON PLAN TEMPLATE mathSTEM project No.: 2019-1-HR01-KA203-060804		
Partner Organization	University of Zagreb, Faculty of Food Technology and biotechnology	
Course/Year	**Numerical methods and programming**/2020 – winter semester	
Topic	**Interpolation and approximation of functions**	
Lesson title	Interpolation and approximation of functions	
Lesson duration:	45 min	Date:

Learning objectives and outcomes
By the end of the lesson all students should be able to: 1. analyze problem and choose the appropriate analytical or numerical method for solving it 2. comparing alternative methods for numerical solutions of a given problem 3. implement analytical and numerical methods by using a computer By the end of the lesson some students should be able to: 4. analyze the importance and accuracy of the results obtained by calculating 5. use computer tools to help in mathematical processes and to acquire the new information
Teaching methods/ strategies / techniques:
• discussion • visualization • application in engineering
Materials/ equipment
• The course materials including Internet connection, student computers and *Wolfram Mathematica software* for visualization.

Previous knowledge assumed:
• Matemathics 1 • Matemathics 2 • Basic Informatics
Short description of the content:
Within this lesson, we will deal with polynomial and trigonometric interpolation and approximation. We will calculated the value of some elementary functions by using the Taylor's polynomial. In addition, we will use least squares method - discrete case, continuous case and trigonometric polynomial (Fourier polynomial) for approximation of functions. Interpolation is used for constructing new data points within the range of a discrete set of known data points. In engineering and science, one often has a number of data points, obtained by sampling or experimentation, which represent the values of a function for a limited number of values of the independent variable. It is also often required to interpolate, i.e., estimate the value of that function for an intermediate value of the independent variable. A closely related problem is the approximation of a complicated function by a simple function. Suppose the formula for some given function is known, but too complicated to evaluate efficiently. One example would be interpolation of residence time distribution experimental curves in reactors and bioreactors performed by tracer test experimental method.

Outline of lesson: (summary of tasks/activities)
1. Teacher gives a brief overview of the theory covered in previous lessons, which are related to this topic. Duration: 5 minutes 2. The teacher demonstrates and explains the computer implementation of the problem of approximation and interpolation of functions with the help of the *Wolfram Mathematica* software package. Student should be able to write mathematical

notations. Duration: 20 minutes

3. Students and teacher together compare different methods for numerical solutions to a given problem. Duration: 10 min.
4. Teachers gives relevant examples in technical engineering studies. Duration: 10 minutes

Extension activities for students who are progressing faster/slower
• If some students do not meet the lesson objectives, the teacher will work directly with them in a small group or individually. • For students who completely understand the objectives, have them do a real-world application problem prepared in advance (with problem solving). They should be encouraged to research and write independently programs in *Wolfram Mathematica* to help them in solving the problems of approximation and interpolation of functions. • Those who cope with tasks more quickly help others and practice collaborative learning.

Assessment:
Students will be given exam consists of two parts: written and practical.

References:
1. Internal script: I. Perić, A. Vukelić, *Numeričke metode i programiranje* 2. I. Ivanšić*, Numerička matematika* 3. S. Wolfram, (2003) *The Mathematica book*, Fifth Edition, Wolfram media & Cambridge University Press, Champaign, Cambridge

Statistics

LESSON PLAN TEMPLATE mathSTEM project No.: 2019-1-HR01-KA203-060804		
Partner Organization	University of Zagreb, Faculty of Textile Technology	
Course/Year	**Statistics**, 1. year	
Topic	**Discrete Statistics**	
Lesson title	Parameters of statistical data sets and measures of dispersion	
Lesson duration:	90 minutes	Date

Learning objectives and outcomes
By the end of the lesson all students should be able to: 1. Calculate average, median and parameters of dispersion 2. Use Excel functions to calculate the above parameters 3. Comment the values of calculated parameters By the end of the lesson some students should be able to: 4. Write their own function in Excel to obtain the above parameters 5. Compare the above parameters
Teaching methods/ strategies / techniques:
• discussion • visualization • real life problems
Materials/ equipment
• The course materials including books, Internet connection and student computers, Excel

Previous knowledge assumed:
Students have to know basics of Excel
Short description of the content:
Content of lecture is parameters of statistical data sets (average, median) and measures of dispersion (sample variance, sample standard deviation, standard error, coefficient of variation)

Outline of lesson: (summary of tasks/activities)
1. We start our lesson with example. What is the average grade if grades are 4,5,5,3,2,4,5. Students have experience of calculation average grade, so the results of this example is simple. Then we write the formula for calculating the average that is arithmetical mean. Second example have this numbers: 5,6,7,5,3,5,30. We again calculate the average and we give characteristics of average. Now we have data in table so that every data has frequency that is how many times it occurs. We again calculate the average. Duration: 15 minutes 2. Then we introduce another parameter: median. Median is the number in the middle of a set of numbers that is half the numbers have values that are greater than the median and half the numbers have values that are less than the median. Students have to notice that only the numbers in the middle of a set of numbers have effect on median. Extreme values don't effect the value of median. We give one example how to calculate median of a set of numbers.

Duraton: 10 minutes

3. We also give examples in Excel. Student learn the functions in Excel, average and median. Duration:15 minutes
4. Now we continue with parameters of dispersion. We start with example of two sets of numbers with equal average and different span and we introduce the first parameter of dispersion, sample variance which measures how far a set of random numbers are spread from their average value.

   We give formula for sample variance and calculate its value for both sets of numbers. Students have to observe that if span of numbers is greater that also the sample variance is greater. In Excel we use formula var.s for calculating sample variance and var.p for calculating variance for all population.

   Duration:15 minutes
5. We continue with the second parameter, sample standard deviation and the formula for its calculation. Next we introduce coefficient of variation and standard error. Duration: 10 minutes
6. Next we continue with Excel. We introduce formulas in Excel for the above parameters. Time needed for this activity is 20. Duration: 10 minutes
7. Now we are ready for examples from real life. One example is grade in Mathematics for 35 students. They have to calculate average, median and measures of dispersion using Excel. Another example are paycheck in one factory and they have to calculate the above parameters and also comment their value. Duration: 10 minutes
8. At the end of lesson we demonstrate Descriptive Statistics. There we have all the parameters above and few more. The students get the task to translate all the parameters in Croatian and by using Excel Help or Internet to learn more about new functions that are given in Descriptive Statistics. Duration: 15 minutes

Extension activities for students who are progressing faster/slower

- If some students do not meet the lesson objectives, the teacher will work directly with them in a small group or individually.
- For students who completely understand the objectives, have them do a real-world application problem prepared in advance (with problem solving). They should be encouraged to investigate and give reasons for their solution process.

Assessment:

All the student should be able to calculate average, median and sample variance for row data by hand and in Excel.

References:

1. K. Krulić Himmelreich, K. Smoljak, *Statistika, Fakultetski udžbenik u e- obliku*, Zagreb, 2013.
2. M. Papić, *Primijenjena statistika u MS Excelu za ekonomiste, znanstvenike i neznalice*, Zoro d.o.o., Zagreb, 2005.

LESSON PLAN TEMPLATE mathSTEM project No.: 2019-1-HR01-KA203-060804		
Partner Organization	University of Zagreb, Faculty of Food Technology and biotechnology	
Course/Year	**Statistics** / 2020	
Topic	**Statistical hypothesis testing - Two-sample Student's *t*-test**	
Lesson title	Two-sample *t*-test	
Lesson duration:	1 hour	Date :

Learning objectives and outcomes
By the end of the lesson all students should be able to: 1. Recognize situations where the two sample *t*-test is used 2. Apply the appropriate, two-tailed or one-tailed, version of the test 3. Calculate the test statistics and determine its critical values 4. Draw conclusions and interpret results of the test By the end of the lesson some students should be able to: 5. The need for modifications and/or extensions of the test under different model assumptions 6. Interpret the *p*-value of the test
Teaching methods/ strategies / techniques:
• discussion • visualization • application in real life
Materials/ equipment
• The course materials including the Internet connection, Acrobat Reader for visualization, YouTube video

Previous knowledge assumed:
Probability theory, Statistical hypothesis testing, Continuous distributions
Short description of the content:
Measurement and comparison of the same variable in two different populations naturally arises in many real life situations and problems. Hence, statistical tests which, based on the samples from the two populations, try to conclude whether there is a difference between the expected, mean values of the variable are widely used. Based on the underlying assumptions on the model, various versions of the test exist and the two sample Student's *t*-test occurs when the variable in question is normally distributed with unknown variance.

Outline of lesson: (summary of tasks/activities)
1. The teacher introduces the topic by giving several examples where samples from two populations are drawn and compared. The students are encouraged to come up with other possible examples of such two-sample measurements. Duration: 10 minutes 2. The teacher introduces notation, explains the model and derives the appropriate test statistics and its distribution. The students are led to discuss potential critical values of the test statistics and, thus, to distinguish between the left-tailed, right-tailed and two-tailed version of the test. Duration: 15 minutes 3. An example with concrete data is described on which the two sample *t*-test is performed step-by-step. The students actively participate and discuss each step of the procedure. Duration: 20 minutes 4. Other relevant examples and datasets are presented which students discuss and solve. Some potential model drawbacks are discussed together with possible modifications and remedies. Duration: 15 minutes

Extension activities for students who are progressing faster/slower
1. If some students do not meet the lesson objectives, the teacher will work directly with them in a small group or individually. 2. Those who cope with tasks more quickly help others and practice collaborative learning

Assessment:
Students will be given a written exam.

References:
4. Internal script: I. Franjić, *Statistika* 5. M. R. Spiegel, L. J. Stephens, *Statistics*, 4th Edition, Schaum's Outline Series, McGrow-Hill, New York, 2008.

LESSON PLAN MathSTEM project No.: 2019-1-HR01-KA203-060804		
Partner Organization	South East European University	
Course/Year	**Statistics**/Second Year	
Topic	**Regression Analysis**	
Lesson title	Linear regression with one regressor	
Lesson duration:	2 hours	Date:

Learning objectives and outcomes
By the end of the lesson all students should be able to: 1. Know the general form of the linear regression model with one regressor (simple linear regression model) 2. Estimate the coefficients of the simple linear regression model using the OLS estimator 3. Predict the effect on Y of a one unit change in X – using a sample of data on these two variables 4. Interpret the obtained results By the end of the lesson some students should be able to: 5. Test one sided and two sided hypothesis 6. Recognize the least squares assumptions 7. Identify how well the OLS regression line fits the data 8. Use of the several statistical software packages to estimate the coefficients of the regression models and display the scatter plot.
Teaching methods/ strategies / techniques:
• Brainstorming • discussion • interactivity • collaborative learning • visualization • real life problems
Materials/ equipment
The course materials include: books, Internet connection and student computers, MegaStat software package, Symbolab online software, and online resources in Google.

Previous knowledge assumed:
Data, sources and types, Random variables and probability distributions, Expected values, mean and variance, The normal distribution, The student *t* distribution.
Short description of the content:
This lecture shows how a hypothetical linear relationship between two variables can be quantified using appropriate data. The principle of least squares regression analysis is explained and expressions for the coefficients are derived.

Outline of lesson: (summary of tasks/activities)
• Teacher activity (5 min) The teacher starts the class by asking some questions to the students about random variables and types of random variables as well as questions related to the dependent and independent variables. • Student activity (10 min) Students answer and make a debate about the types of random variables with concrete examples. Define and elaborate the dependent variable and independent variable and their relationship for different examples. • Teacher activity (20 min)

The teacher gives the general form of the simple linear regression model. Explains the variables, the intercept and the slope as well as the error term and its role. Also, the model is elaborated through the respective scatter plot. Next the teacher solves an example with both methods on the whiteboard and with MegaStat. Finally, the least square assumptions are explained.

- Student activity (10 min)

Students in smaller cooperative groups estimate the coefficients of the simple linear regression model by formulas and using MegaStat or STATA software. Once they have completed the exercise, they compare the results with the other groups and conduct a social and respectful debate about the progress of the solution and the obtained results.

- Teacher activity (15 min)

The teacher explains the testing of hypothesis about the regression coefficients, solves an example, and stimulates students for active participation in interpreting the results. Also, an analysis is made for one sided hypothesis and two sided hypothesis. The examples are demonstrated through MegaStat software.

- Student activity (15 min)

Students in groups solve a practice problem. They solve the same exercise with the formulas and MegaStat software, compare solutions and develop constructive debate in a positive ambiance.

- Joint activity / teacher-students (10 min)

The teacher presents an applied problem in the field of business, and then estimates the regression model, the scatter plot and the goodness of fit of data through Mega Stat. The interpretation of the results is made simultaneously with students.

- Joint activity / teacher-students (5 min)

At the end of the class there is a general overview and a summary of gained knowledge on simple linear regression model, use of software, analysis of solutions as well as application of simple linear regression for solving practical problems in various fields.

Extension activities for students who are progressing faster/slower

- For students who do not have sufficient progress, a special group will be formed and will be worked on a basic level with additional support from the teacher and with the usage of software.
- For students who fully understand the objectives of the lesson, they will engage with more advanced and applied exercises from various real-life areas.
- Advanced students will be committed to helping students with lower knowledge.

Assessment:

1. Students engage in homework assignments regarding the simple linear regression model.
2. Students are committed to solving their homework through the statistical software MegaStat and STATA 14, and Wolfram Mathematica.
3. Student groups'are formed, where each group has a leader who coordinates collaboration when solving practice problems by communicating through online google classroom platform.
4. The teacher communicates with group leaders and gives them additional tasks and professional support regarding the lesson and advanced exercises.

References:

1. Statistics for Business and Economics, David R. Anderson; Dennis J. Sweeney; Thomas A. Williams. Edition 13, 2017, Cengage Learning
2. Statistics for Business and Economics, Jim McClave, Terry Sincich, Edition 13, 2017, Pearson
3. MegaStat https://megastat.software.informer.com/
4. Symbolab online software (https://www.symbolab.com/solver)
5. Wolfram Mathematica software (https://www.wolfram.com/mathematica/online/)

Game Theory

LESSON PLAN TEMPLATE mathSTEM project No.: 2019-1-HR01-KA203-060804		
Partner Organization	Hacettepe University	
Course/Year	**Introduction to Game Theory**/ 4. year	
Topic	**Introduction to Game Theory, Nash Equilibrium**	
Lesson title	Nash Equilibrium	
Lesson duration:	50 minutes	Date :

Learning objectives and outcomes
By the end of the lesson all students should be able to: 1. Define the basic concepts of a game: Players, strategies… 2. Represents game in normal form 3. Recognize the classic example of the game theory, Prisoner's Dilemma 4. Defines strategy, dominant strategy, best response 5. Find the dominant strategies of a given game 6. Describe Nash Equilibrium By the end of the lesson some students should be able to: 7. Find the Nash Equilibrium of a game 8. Appraise the application of Prisoner's Dilemma to a different real-word problems 9. Identify Nash Equilibrium in various daily life problems 10. Recognize the economic theory exemplified by the Prisoner's Dilemma
Teaching methods/ strategies / techniques:
• brainstorming • discussion • debates • collaborative learning • visualization • real life problems • Pedagogical approaches (Think-pair-share) method • Pre/post question • Active learning
Materials/ equipment
• Computer and projector • Internet connection • Online resources on Google • Power point presentations • Homework questions • Books

Previous knowledge assumed:
We recommend that students are familiar with the following terms: Games, Payoffs, Strategies. These notions will be recalled very shortly in the introduction. Warm-up activity (worksheet 1) will be a single activity then the results will be discuss with the whole class
Short description of the content:

In this lesson we will recall some important concepts in the theory of games, starting with strategy, payoffs, payoffs matrix. Beside them this lesson covers the two most basic concepts of the games, dominance and best response. Finally, the most popular solution concepts in all of game theory called Nash Equilibrium will be introduced with some very well-known examples.

Outline of lesson:
(summary of tasks/activities)

Introduction (5 minutes)
ASK: What is a game?
Players, strategies and what we need more? PAYOFFS

WARM-UP: Extra point question (10 minutes)
a) Worksheet 1 is given to the students
b) Discuss on the result
c) ASK: Is it important for players to think about each other's strategy chocies?

EXAMPLE: Prisoner' Dilemma (5 minutes)
a) Worksheet 2 is given to the students
b) This example will be discussed with the following new definitions.

EXPLAIN: Basic notions in game theory will be introduced to students (20 minutes)
- Payoffs
- Payoffs matrix
- In payoffs matrix first number shows payoff for person in row
- In payoffs matrix second number shows payoff for person in column

ACTİVİTY: Describe the payoffs matrix for Prisoner's Dilemma
- Dominant strategy
- Dominant strategy equilibrium
- Best Response
- Nash Equilibrium

ACTİVİTY: Revisiting Prisoner's Dilemma (5 minutes)
- Using the payoff matrix determine what the dominant strategy for each player is.
- Identify the Nash Equilibrium
- **DİSCUSS:** Is this the best outcome for both sides? Why?

CONCLUSION: (5 minutes)
- Every dominant strategy equilibrium is a Nash Equilibrium
- Not every Nash equilibrium is a dominant strategy equilibrium

HOMEWORK: Find some examples which the Nash Equilibrium can be used in everday life.

WORKSHEET 1: Extra Point Question

If everyone chooses collude, all students get 10 bonus points in final exam.
If everyone chooses to collude, but one person defect, that person defecting gets 50 bonus points and no other student get any points.
If more than 1 person chooses to defect, no student get any points.

WORKSHEET 2: Prisoner's Dilemma

Two suspects of arrested for a crime, but the district attorney only has enough evidence to convict them on 2 year sentences each. So she separates the two suspects into different rooms, and has the police make an offer to each suspect: "Look we both know that we only have enough evidence to convict you for two years, so I'm going to make you a deal. If you confess to the crime and your partner's involvement, but your partner remains silent, we'll reduce your sentence to 1 year while your partner will get 4 year. Of course if both of you confess then we will have enough evidence to convict you both but as a favor I'll reduce your sentence to 3 years instead of 4. The suspects know that the same deal was offered to each of the, and the suspects are selfish so they care only about reducing their sentence as much as possible. So what would the suspects do?

HOMEWORK

Problem1. Suppose that two people decide to form a partnership firm. The revenue of the firm depends on the amount of effort expended on the job by each person and is given by:

$$R(e_1,e_2)=a_1e_1+a_2e_2$$

Where e_1 is the effort level of person 1 and e_2 is the effort level of person 2. The number a_1 and a_2 are positive constants. The contract that was signed by the partners stipulates that person 1 receives a fraction t(between 0 and 1) of the firms revenue, and person 2 receives a 1-t fraction. That is, person 1 receives the amount $tr(e_1,e_2)$ and person 2 receives $(1-t)r(e_1,e_2)$. Each person dislikes effort, which is measured by a personal costs of e_1^2 for person 1 and e_2^2 for person 2. Person I's utility in this endeavor is the amount of revenue that this person receives , minus the effort costs e_i^2. The effort levels are chosen by the people simultaneously and independently.

1. Define the normal form of this game
2. Using dominance compute the strategies that the players rationally select
3. Suppose that you could set t before the players interact. How would you set t to maximize the revenue of the firm?

Problem 2:Splitting the Dollar(s) : Players 1 and 2 are bargaining over how to split \$10. Each player i names an amount, s_i, between 0 and 10 for herself. These numbers do not have to be in whole dollar units. The choices are made simultaneously. Each player's payoff is equal to her own money payoff . We will consider this game under two different rules. In both cases, if $s_1 + s_2 \leq 10$ then the players get the amounts that they named (and the remainder, if any, is destroyed).

(a) In the first case, if $s_1 + s_2 > 10$ then both players get zero and the money is destroyed. What are the (pure strategy) Nash Equilibria of this game?

(b) In the second case, if $s_1 + s_2 > 10$ and the amounts named are different, then the person who names the smaller amount gets that amount and the other person gets the remaining money. If $s_1 + s_2 > 10$ and $s = s_2$, then both players get \$5. What are the (pure strategy) Nash Equilibria of this game?

(c) Now suppose these two games are played with the extra rule that the named amounts have to be in whole dollar units. Does this change the (pure strategy) Nash Equilibria in either case?

Extension activities for students who are progressing faster/slower
• If some students do not meet the lesson objectives, the teacher will work directly with them in a small group or individually. • For students who completely understand the objectives, have them do a real-world application problem prepared in advance (with problem solving). They should be encouraged to investigate and give reasons for their solution process. • Those who cope with tasks more quickly help others and practice collaborative learning.

Assessment:
• The homework sheet is given to the student to understand the Nash Equilibrium concept and evaluate their understanding in a real life related situation. • Teacher takes notes about student learning progress and assess their knowledge and skills by asking specific questions.

References:
• Joel Watson, Strategy: An introduction to Game theory, W.W. Norton & Company, 2002

LESSON PLAN TEMPLATE mathSTEM project No.: 2019-1-HR01-KA203-060804		
Partner Organization	Hacettepe University	
Course/Year	**Introduction to Game theory** /4. year	
Topic	**Mixed Nash Equilibrium**	
Lesson title	An application of partial derivative: Mixed Nash Equilibrium	
Lesson duration:	50 minutes	Date :

Learning objectives and outcomes

By the end of the lesson all students should be able to:

1. Defines and understand the mixed strategies
2. Recognize the example of matching pennies, battle of sexes
3. Determine the mixed Nash Equilibrium algebraically

By the end of the lesson some students should be able to:

4. Determine the mixed strategy not only by algebraically but also by partial derivatives
5. Find the Nash Equilibrium of a game (mixed or pure)
6. Describe the difference between pure and mixed strategies

Teaching methods/ strategies / techniques:

- discussion
- debates
- collaborative learning
- visualization
- real life problems
- Pre/post question
- Active learning

Materials/ equipment

- Internet connection,
- Personal computer and projector
- Online resources on Google
- Power point presentations
- Books

Previous knowledge assumed:

We recommend that students are familiar with the following terms: Games, Payoffs, Strategies, dominant strategies, Nash Equilibrium. And students must know the partial derivative of several variable functions.

Short description of the content:

In this lesson we will deal with the games which do not have Nash Equilibrium. The idea behind this topic is to randomize your pure strategies called mixed strategies.

So the other player cannot be sure of what I will do. This is the main idea of mixed strategy Nash Equilibrium. The Nash Equilibrium of some games will be found by using partial derivative and algebraically.

Outline of lesson:
(summary of tasks/activities)

Introduction: (2 minutes)
So far we have only dealt with pure strategy Nash equilibrium. Now

- Recall the pure strategy Nash equilibrium and best response
- Give the students the following example to introduce the concept of mixed strategy Nash Equilibrium
- The students are divided by couples and play the following game.
- Who wins this game?

Example: (Matching pennies) (8 minutes)
In the game of matching pennies each player has a penny and must secretly turn the penny to either heads or tails. They then reveal their choices simultaneously. If the pennies match—both heads or both tails—player A keeps both pennies, so he wins one from player B. If the pennies do not match—one heads and one tails-B keeps both pennies, so she wins one from A.

ASK: The students discuss the following questions by recalling the previous lesson.

a) Write the game matching pennies by normal form.
b) What are the payoffs?
c) Find the Nash equilibrium.

EXPLANATİON: (10 minutes)
Since there is no any pure strategy Nash equilibrium in the game matching pennies, the teacher will introduce the following new concepts.

- Mixed strategy
- Expected payoffs
- Mixed Nash Equilibrium

EXAMPLE: (10 minutes)

- Revisiting matching pennies to find the Mixed Nash Equilibrium
- Solve the problem by algebraically
- Solve the problem by partial differentiation

ASK: Ask the students whether there is an example with two or three Nash Equilibrium

ACTİVİTY: Real life problem (15 minutes)

- The problem -battle of sexes – are introduced to students.
- The teacher gives to the student worksheet 1.
- Make sure all the students understand the given example.
- The students will be divided into two groups

ASK: 1. Explain the payoff matrix.
2. Find the followings:
   a) Pure strategy Nash Equilibrium
   b) Mixed strategy Nash Equilibrium
3. At the end of the activity each group present their solutions.
4. Try all the students in the groups participate actively this solution process

CONCLUSİON: (5 minutes)

- Pure strategy Nash Equilibrium doesn't always exist
- Mixed strategy Nash Equilibrium always exists
- Give the students homework.

WORKSHEET 1

Battle of sexes
A husband (player 1) and wife (player 2) would like to go for an evening out. They can either go to ballet or to the basketball game. They prefer to be together than apart. Conditional on being together though, the husband prefers to go to the basketball game and the wife prefers to go to the ballet. We can represent the game in the following way:

Wife

Husband

	Ballet	Basketball
Ballet	1,2	0,0
Basketball	0,0	2,1

HOMEWORK

Penalty Shotsgame

Player 1 has to take a soccer penalty shot to decide the game. She can shoot Left, Middle, or Right. Player 2 is the goalie. He can dive to the left, middle, or right. Actions are chosen simultaneously. The payoffs (which here are the probabilities in tenths of winning) are as follows.

Player 2

	l	m	r
L	4,6	7,3	9,1
M	6,4	3,7	6,4
R	9,1	7,3	4,6

Player 1

(a) For each player, is any strategy dominated by another (pure) strategy?

(b) For what beliefs about player 1's strategy is m a best response for player 2? For what beliefs about player 2's strategy is M a best response for player 1? [Hint: you do not need to draw a 3-dimensional picture!].

(c) Suppose player 2 puts himself in player 1's shoes and assumes that player 1, what ever is her belief, will always choose a best-response to that belief. Should player 2 ever choose m?

(d) Show that this game does not have a (pure-strategy) Nash Equilibrium?

Extension activities for students who are progressing faster/slower
• If some students do not meet the lesson objectives, the teacher will work directly with them in a small group or individually. • For students who completely understand the objectives, have them do a real-world application problem prepared in advance (with problem solving). They should be encouraged to investigate and give reasons for their solution process. • Those who cope with tasks more quickly help others and practice collaborative learning.

Assessment:
• The homework sheet is given to the student to understand the mixed Nash Equilibrium concept and evaluate their understanding in a real life related situation. • Teacher takes notes about student learning progress and assess their knowledge and skills by asking specific questions.

References:
• Joel Watson, Strategy: An introduction to Game theory, W.W. Norton & Company, 2002 • Prajit K. Dutta, Strategies and games: Theory and practice, MIT press, 1999

Algorithms and Discrete Structures

LESSON PLAN MathSTEM project No.: 2019-1-HR01-KA203-060804		
Partner Organization	South East European University	
Course/Year	**Discrete Structures**/Second Year	
Topic	**Graph theory**	
Lesson title	Basic concepts of graph theory	
Lesson duration:	2 hours	Date:

Learning objectives and outcomes
By the end of the lesson all students should be able to: 1. Understand the concept of graph. 2. Understand and make distinction between different types of graphs . 3. Understand what is a multigraph. 4. Define the undirected graphs. 5. Use the theorem which gives the relation between the edges and the degree (Handshaking theorem). 6. Define and sketch the sub graph of the given graph. By the end of the lesson some students should be able to: 7. Understand the concept of graph isomorphism. 8. Find the chromatic number of graph. 9. Find the sum and the product of two graphs. 10. Understand and apply the concept of Euler path and Euler circuit.
Teaching methods/ strategies / techniques:
• brainstorming • discussion • interactivity • collaborative learning • visualization • real life problems • individual work
Materials/ equipment
The course materials including books, Internet connection and student computers, Mathematical software for visualization, online resources on Google, etc.

Previous knowledge assumed:
Set theory, Relations, properties of relations.
Short description of the content:
In this lecture students will learn concerning some basic concepts of graph theory. At the beginning they will learn about the concept of undirected graphs, how it is defined, the different forms of graphs, the basic properties etc. They will learn about the theorem which gives the relation between the

number of edges and the number of vertices in the regular graph. The isomorphism of the graphs will be defined, the chromatic number of the graph and application of coloring theory as well as the operations with graphs will be learned. At the end they will learn about the Euler path and Euler circuit and application of this theory.

Outline of lesson: (summary of tasks/activities)
• Joint activity (5 min) The teacher starts the class by asking some questions concerning relations and graphical representation of relations. Students respond to the asked questions and there is a discussion concerning these facts in order to make connection with the new topic. • Teacher activity (10 min) Teacher gives the definition of simple graph, multigraph, undirected graph. Then he continues explaining what is the degree of vertex, what is the regular graph, what is a isolated, pendant vertex etc. • Student activity (10 min) Students are requiring to sketch few graphs and to discuss about the type of each sketched graph. To determine if they are either regular or not regular finding the degree of vertices. • Student activity (10 min) Students use the appropriate software to sketch the graphs and to make the same discussion by the graphs sketched directly in front of them. If it's possible they should make comparison with previous discussed graphs. They can also see different forms of the same graph by using online software like the software “Graph online” (https://graphonline.ru/en/), "dreampuf" (https://dreampuf.github.io/GraphvizOnline/), est. • Teacher activity (10 min) Teacher gives the Handshaking theorem and corollary of that theorem and then defines the special forms of graphs such are complete graph, cycle and wheel. • Student activity (10 min) Students individually solve few examples using Handshaking theorem and its corollary and sketch special cases of complete graphs, cycles and wheels. Students use online software to have a view concerning the obtained graphs and also confirmation for the cases where the Handshaking theorem doesn't allow to sketch the graphs which do not fulfill the condition of the theorem. • Joint activity (5 min) The online software is used to define the subgraph and the partial graph of the given graph. It is done by frontal discussion using online sketched graphs. • Teacher activity (10 min) Teacher gives the definition for graph isomorphism and chromatic number of graph. Teacher also explains the Euler path and Euler circuit. He uses online software for discussion concerning these topics. • Student activity (10 min) Students individually find the chromatic number of a graph demonstrating by using the online software and also determine either two different graphs are isomorphic or not. They use the online software for having clear picture. • Student activity (5 min) Students are requiring to solve an applicational problem using the theorem of Euler circuit.

- Joint activity / teacher-students (5 min)

At the end of the class there is a general overview and a summary of gained knowledge about the basic concepts of graph theory.

Extension activities for students who are progressing faster/slower

- For students who do not have sufficient progress, a special group will be formed and will be worked on a basic level with additional support from the teacher and with the support of corresponding online software.
- For students who fully understand the objectives of the lesson, they will engage with more advanced exercises where the applied problems using coloring theory and the theorem of Euler circuit will be considered.
- Advanced students will be committed to helping students with lower knowledge.

Assessment:

1. Students engage in homework assignments concerning the mentioned topics.
2. Students are committed to solving their homework through different software which are appropriate for them.
3. We will form the Google meet group through Google classroom platform for any kind of additional discussion concerning the discussed issues.

References:

1. Discrete Mathematics, Keneth A. Ross and Charles R.B. Wright, McGraw-Hill, 2003.
2. Discrete Mathematics and its Applications, Keneth H. Rosen, fourth eddition, WCB McGraw-Hill, 2008.
3. Mathematics for Computer Science, Graph theory II, Strini Devadas and Eric Lehman, Lecture notes, pdf, online material, 2005.

LESSON PLAN TEMPLATE mathSTEM project No.: 2019-1-HR01-KA203-060804		
Partner Organization	University of Zagreb, Faculty of Electrical Engineering and Computing	
Course/Year	**Discrete Mathematics** / 2020	
Topic	**Euler phi function**	
Lesson title	Euler phi function	
Lesson duration	1 hour	Date:

Learning objectives and outcomes
By the end of the lesson all students should be able to: 1. recognize and define Euler phi function 2. recognize and use an explicit formula for Euler phi function 3. state main properties of Euler phi function, state Euler and little Fermat theorem 4. apply an explicit formula for Euler phi function in finding ''inverse'' i.e. solving equation of the form phi(*n*)=*k*, where *k* is a fixed positive integer By the end of the lesson some students should be able to: 5. explain and prove main properties of Euler phi function 6. apply Euler function and Euler theorem in solving more complicated number theory problems
Teaching methods/ strategies / techniques:
• discussion • visualization • application in number theory and cryptography
Materials/ equipment
• the course materials including student computers, internet connection, online calculator for Euler phi function, inverse Euler phi function and modular arithmetic

Previous knowledge assumed:
Basic concepts in elementary number theory, especially divisibility, prime numbers and congruences
Short description of the content:
In number theory, Euler phi function counts the positive integers up to a given integer *n* that are relatively prime to *n*. It is one of the most important arithmetic functions which has many applications in number theory and cryptography.

Outline of lesson: (summary of tasks/activities)
1. Teacher introduces the topic of the lesson by defining the Euler phi function and listing some particular values via online presentation. By using definition, students determine Euler function for some particular values and check their solutions via online calculator. Duration: 5 minutes 2. Teacher explains and proves multiplicativity property of Euler phi function. The corresponding proof should be established through high level of interactivity. Duration: 10 minutes 3. By virtue of multiplicativity, teacher derives an explicit formula for Euler function when an argument is given in its canonical representation. The corresponding formula should be established through high level of interactivity. Moreover, students should be able to derive the formula by themselves. After the proof, students utilize the formula in calculating Euler phi function for some particular values and check their solutions via online calculator. Duration: 10 minutes 4. Teacher gives several exercises about finding the inverse of Euler function i.e. solving equation of the form phi(*n*)=*k*, where *k* is a fixed positive integer. Students solve them in small

groups. Their solutions are checked via online calculator for finding inverse of the Euler phi function. Duration: 15 minutes

5. With high level of student participation, teacher explains some additional properties of Euler phi function. In particular, teacher states and proves Euler and little Fermat theorem. Students should be able to establish the corresponding proofs by themselves. Duration: 15 minutes
6. Teacher gives an easy number theory problem which relies on Euler phi function and Euler theorem. Students solve the exercise on their own and check their solutions via online calculator for modular arithmetic. After checking solution, teacher announces the next lesson-application of Euler and little Fermat theorem in modular arithmetic and public key cryptography. Duration 5 minutes

Extension activities for students who are progressing faster/slower

- If some students do not meet the lesson objectives, the teacher will work directly with them in a small group or individually.
- For students who completely understand the objectives, the teacher had prepared more complicated number theory problems involving Euler phi function.
- Those who cope with tasks more quickly help others and practice collaborative learning.

Assessment:

Students will be given written exercises to evaluate their learning progress and accquired knowledge.

References:

1. A. Dujella, *Teorija brojeva*, Školska knjiga, Zagreb, 2019. (in Croatian)
2. N. Koblitz, *Course in Number Theory and Cryptography*, Springer, 1994.

LESSON PLAN TEMPLATE mathSTEM project No.: 2019-1-HR01-KA203-060804		
Partner Organization	University of Split / Faculty of Science	
Course/Year	**Complexity of Algorithms**, 2[st] year of Graduate Study	
Topic	**Minimum Spanning Tree**	
Lesson title	Minimum Spanning Tree	
Lesson duration:	45 minutes	Date: December, 2020

Learning objectives and outcomes
By the end of the lesson all students should be able to: 1. construct connected graph for classical problems in theory of algorithms 2. recognize which classical problems can be solved using minimum spanning tree 3. apply Prim's algorithm for solving minimum spanning tree problem 4. explain difference between Prim's and Kruskal's algorithm 5. state and prove lemma which enables construction of Prim's and Kruskal's algorithm
Teaching methods/ strategies / techniques:
• discussion • construction • finding corresponding real life problems • visualization • pre/post question
Materials/ equipment
The course materials include • Computer and projector • Internet connection • Beamer presentation

Previous knowledge assumed:
Graph Theory, Turing machines, greedy algorithms, time and space complexity, mathematical programming tools
Short description of the content:
Some real-life problems can be solved by representing the problem with connected graph and then using algorithms for finding minimum spanning tree. Two classical algorithms are Prim's and Kruskal's. Both algorithms belong to the class of greedy algorithms but have big difference. We will explain what is the difference between those two algorithms and then prove the result which allows construction of these algorithms.

Outline of lesson: (summary of tasks/activities)

1. Before talking about minimum spanning tree we recall basic elements for constructing greedy algorithms and repeat basic elements of graphs and connected graphs. Duration: 5 minutes

2. Using slides we introduce one problem from real-life (bulding roads between cities at minimal cost) and explain how can it be represented by concepts of connected graph and minimum spanning tree. Duration: 5 minutes

3. Teacher explains two algorithms for solving minimum spanning tree problem: Prim's and Kruskal's. It is emphasized that both algorithms are greedy algorithms but have important difference. Using visualization on slides construction of these algorithms is presented with emphasizing the difference. Duration: 15 minutes

4. Students are asked why these two algorithms gives us correct answer to minimum spanning tree problem. After short discussion we state the result which answers to that question. At the end of the lesson we prove that statement and give visual explanation. Duration: 15 minutes

Extension activities for students who are progressing faster/slower

Students who completely understand the objectives of the task will be asked following questions: what if graph is not connected (how can algorithms be changed to give some results in this case and what results)? what could be proper representation of graph in some of the programming tools (SciLab, Maxima)?

Assessment:

Task (homework):

Students will be given task to solve minimum spanning tree problem with Prim's algorithm for connected graph with ten nodes.

References:

- David M. Mount, *Design and Analysis of Computer Algorithms,* lecture notes, Dept. of Computer Science, University of Maryland, College Park, MD, 20742.
- M. Sipser, *Introduction to the Theory of Computation*, PWS Publishing Company, 1996.

LESSON PLAN TEMPLATE mathSTEM project No.: 2019-1-HR01-KA203-060804		
Partner Organization	University of Split / Faculty of Science	
Course/Year	**Complexity of Algorithms**, 2nd year of Graduate Study	
Topic	**Dynamic Programming - Introduction**	
Lesson title	Dynamic Programming - Introduction	
Lesson duration:	45 minutes	Date: December, 2020

Learning objectives and outcomes
By the end of the lesson all students should be able to: 1. recognize problems which are solvable by method of dynamic programming 2. state two main qualities dynamic programming rely on, optimal substructure and overlapping subproblems 3. understand two ways of how a solution is constructed (top-down, bottom-up) 4. interpret concept of memorization 5. explain basic concepts of dynamic programming on the problem of Fibonacci's numbers 6. explain difference between brut-force and dynamic programming in solving problem of Fibonacci's numbers
Teaching methods/ strategies / techniques:
• discussion • construction • finding corresponding real life problems • visualization • pre/post question
Materials/ equipment
The course materials include • Computer and projector • Internet connection • Beamer presentation

Previous knowledge assumed:
Graph Theory, Turing machines, greedy algorithms, time and space complexity, mathematical programming tools
Short description of the content:
One of the most popular methods for designing algorithms today is dynamic programming. They are usually applicable when searching for the solution which is optimal in some way and can be interpreted as a result of some sequence of decisions. Dynamic programming solutions rely on two qualities, optimal substructure (or principle of optimality) and overlapping subproblems. Overlapping subproblems lead to concept of memoization, where we don't have to repeat recursive calls many times, instead we remember it in some table of values. We explain all this concepts on two simple examples, adding of integers and problem of Fibonacci's numbers.

Outline of lesson: (summary of tasks/activities)
1. Dynamic programming is introduced as one of the most popular methods for constructing algorithms, especially for optimization problems (shortest path, minimal cost, …). Duration: 5 minutes **2.** Basic concepts of dynamic programming are introduced and explained. We mention similarity to "divide and conquer" method (both methods solve large problem by solving smaller subproblems) and big difference between these methods (in dynamic programming subproblems must overlap). Principle of optimality (that is quality of optimal substructure) is introduced and stated. We introduce concept of memoization (related to overlapping subproblems) and bottom-up and top-down ways of constructing solution. Duration: 30 minutes **3.** Students are asked about some example of dynamic programming. After short discussion we explain method on Fibonacci's numbers. Using visualization on slides we emphasize concept of memoization and difference between brut-force method. Duration: 10 minutes

Extension activities for students who are progressing faster/slower
1. Students who completely understand the objectives of the task will be asked to prepare task for homework in advance 2. Students who completely understand the objectives of the task will be asked to think of some examples of problems which could be solved using method of dynamic programming

Assessment:
Task (homework): *Students will be given task to implement both algorithms for Fibonacci's numbers (by dynamic programming and brut-force) in some of the programming tools (SciLab, Maxima) and see the difference in time and space complexity between two*

References:
• David M. Mount, *Design and Analysis of Computer Algorithms,* lecture notes, Dept. of Computer Science, University of Maryland, College Park, MD, 20742. • M. Sipser, *Introduction to the Theory of Computation*, PWS Publishing Company, 1996.

LESSON PLAN TEMPLATE mathSTEM project No.: 2019-1-HR01-KA203-060804		
Partner Organization	University of Split / Faculty of Science	
Course/Year	**Complexity of Algorithms,** 2st year of Graduate Study	
Topic	**Dynamic Programming - Examples**	
Lesson title	Dynamic Programming - Examples	
Lesson duration:	45 minutes	Date: December, 2020

Learning objectives and outcomes
By the end of the lesson all students should be able to: 1. recognize problems which are solvable by method of dynamic programming 2. state two main qualities dynamic programming rely on, optimal substructure and overlapping subproblems 3. analyze problem and choose optimal substructure for dynamic programming 4. compare two ways of how a solution is constructed (top-down, bottom-up) and decide which one to use 5. use mathematical programming tools to implement algorithm constructed by dynamic programming
Teaching methods/ strategies / techniques:
• discussion • construction • finding corresponding real life problems • visualization • pre/post question
Materials/ equipment
The course materials include • Computer and projector • Internet connection • Beamer presentation

Previous knowledge assumed:
Graph Theory, Turing machines, greedy algorithms, time and space complexity, mathematical programming tools
Short description of the content:
One of the most popular methods for designing algorithms today is dynamic programming, especially for optimization problems. We consider number of problem which are solved by dynamic programming (0-1 knapsack, weighted interval scheduling, shortest path). On one example elements of dynamic programming are explained (optimal substructure, overlapping subproblems, dynamic programming algorithm).

Outline of lesson: (summary of tasks/activities)

1. Brief overview of previous lesson is given (introduction to dynamic programming). Duration: 5 minutes

2. Large class of real-life problems can be solved by method of dynamic programming. We explain four problems that we will solve with this method (shortest path, integer knapsack, 0-1 knapsack, longest common subsequence). Duration: 10 minutes

3. Problem of shortest path in connected graph covered. Using this example we introduce elements of dynamic programming and for every element we visualization on slides is used. We emphasize the concept of memoization (related to overlapping subproblems) and using slides we show how memorization table is constructed "step by step". At the end of example we explain what other results from algorithm can interest us. Duration: 30 minutes

Extension activities for students who are progressing faster/slower

- Students who completely understand the objectives of the task will be asked to prepare task for homework in advance
- Students who completely understand the objectives of the task will be asked to think of some examples of problems which could be solved using method of dynamic programming

Assessment:

Task (homework):

Students will be given task to implement algorithm for matrix multiplication gained by dynamic programming in some of the programming tools (SciLab, Maxima) and see the difference in time and space complexity between this algorithm and algorithm gained implementing standard matrix multiplication

References:

- David M. Mount, *Design and Analysis of Computer Algorithms,* lecture notes, Dept. of Computer Science, University of Maryland, College Park, MD, 20742.
- M. Sipser, *Introduction to the Theory of Computation*, PWS Publishing Company, 1996.

Mathematical modeling

LESSON PLAN TEMPLATE mathSTEM project No.: 2019-1-HR01-KA203-060804		
Partner Organization	Hacettepe University	
Course/Year	**Mathematical modeling** /4. year	
Topic	**Logistic function**	
Lesson title	Logistic function and logistic population model	
Lesson duration:	50	Date :

Learning objectives and outcomes
By the end of the lesson all students should be able to: 1. Describe the logistic function 2. Find the output value of a logistic function for a given input value 3. Graph the logistic function by using some software programs (desmos) 4. Determine whether a relation described graphically or symbolically represents a logistic function 5. Describe the logistic population model By the end of the lesson some students should be able to: 6. Identify whether a growth curve describes exponential, logarithmic or logistic population model 7. Solve an equation or inequality involving a logistic function 8. Construct of logistic population model
Teaching methods/ strategies / techniques:
• brainstorming • discussion • debates • collaborative learning • visualization • real life problems • Pedagogical approaches (Think-pair-share) method • Pre/post question • Active learning
Materials/ equipment
• Davis, Mary Ellen and C. Henry Edwards, Elementary Mathematical Modelling: Functions and Graphs, Prentice Hall, 2007 • “Exponential&LogisticGrowth” Khan Academy, www.khanacademy.org/science/biology/ecology/population-growth-and-regulation/a/exponential-logistic-growth • Desmos graphing calculator, www.desmos.com • Power point presentations • Personal computer and a projector • Internet connecton

Previous knowledge assumed:
We recommend that students are familiar with the following terms: Exponential function, logarithm function, derivative, integral techniques. These notions may need to be recalled before the lesson. It is preferred that students have used the software program (desmos) before.
Short description of the content:
Students will know that many different growth curves exist for populations in our daily lives. In exponential growth populations per individual growth rate stays the same regardless of population size and makes the population grow faster and faster as it gets larger. In nature, populations grow exponentially for some period but at the end they will be limited by source. The students will understand that modeling helps to describe and also predict population growth over time. Specially if the population has limited resources we will talk about logistic function and logistic population model. In logistic growth, population's per individual growth rate gets smaller and smaller as population size approach a maximum imposed by limited resources. Exponential growth produces J-shaped curve, while logistic growth produces S-shaped curve, because of the limited resources.

Outline of lesson:
(summary of tasks/activities)

Details about the class: The class consists of 10 students.

Extension activities:
Introduction: (15 minutes)
The introduction part consists of the type of growth (decline) models that the students had been learned so far. Some of them which are natural growth models and exponential, logarithmic growth models will be recalled and prepare students for the new model called logistic population model.

- Recall the natural growth model by telling, in equal units of time they grow by the same percentage.
- Ask students to give some examples.
- Give the students the following example:

  Example: Suppose you deposit 1000 Euro in saving account that draws %10 interest compounded annually. How much money you get back after 1 year?

- Recall the exponential and logarithmic growth models.
- Ask students about the increases of these functions.
- Suppose the interest is compounded semiannually and even more frequently and continuously.
- Give the students the following example.

  Example: Suppose you deposit 1000 Euro in saving account that draws %10 interest. What will your account be after 1 year

- **compounded semiannually**
- **compounded continuously**

- Provide students with the following "warm-up" problem. (See worksheet 1)
- Have students think on the real-life problem. Specially the increases amount and ask them the following question:

Question: Why this population doesn’t continue to grow? What is the reason?

- Then let the students share their answers.
- **Graph:** Model how can we graph the values? Is the function similar to exponential function or logarithmic function?
- Introduce the topic by connecting the warm-up problem to the lesson objectives.

Activity (25 minutes)

- Introduce the students with a new type of mathematical function that can serve to model a limited population.
  Transition: The power point presentation is opened and the definition of the logistic function and the variables are introduced to students.
- According to the learning objectives the students can find the output value of a logistic function for a given input value.
  Transition: The worksheet 2 is given to the students for an exploration.

Explore:

- Students work together in two groups and write the function
- Students create a table of values of the logistic function and discuss the leveled off point.
- Students graph the function by computer.
  Transition: The worksheet 3 for an exploration is given.

Explore:

- Students work together in two groups and write the function
- Students create a table of values of the logistic function and discuss the leveled off point.
- Students graph the function by computer.

Explain and discussing

- In the worksheets 2 and 3 students discuss the question that “Where do these functions approach as x gets larger?”
- Students give their answers geometrically by using the graphs
- Students give their answers by limiting the functions.
- Students discuss the influence of carrying capacity
- Students create their own examples of limited population

Activity: (Construction of the logistic population model) (8 minutes)

Transition: The slides are shown on the board to give the construction of the logistic growth model.

Explain:

- Teacher explain the construction of the logistic growth population model.
- Students extends their understanding by following each step to obtain the logistic growth model
- Students discuss the carrying capacity in the given growth model

Evaluate (2 minutes)

A general lesson conclusion is done by summarizing the concepts of logistic function, logistic population model.

- The homework sheet is given to the student to solve the questions and evaluate their understanding in a new real life related situation.

- Teacher takes notes about student learning progress and assess their knowledge and skills by asking specific questions.

Worksheet 1

A forest habitat was initially stocked with 10 rabbits. Over the next three years the rabbit population in this forest is

t-months	P rabbits
0	10
3	21
6	40
9	62
12	80
15	91
18	96
21	98
24	99
27	100
30	100
33	100
36	100

Worksheet 2

Let a=3 b=1 and c=12. Find the logistic function. Then graph the function by computer software program.

Worksheet 3

Let a=39 b=0.3 and c=100. Find the logistic function. Then graph the function by computer software program.

Homework sheet

1. Let a=4 b=0.05 and c=300. Find the logistic function. Then graph the function by computer software program.

2. Find the formula for the logistic population model for the given limiting population M, the initial population P_0 and additional population conditions.
   $M=1000$ $P_0=200$ and $P(25)=400$

Extension activities for students who are progressing faster/slower
• If some students do not meet the lesson objectives, the teacher will work directly with them in a small group or individually. • For students who completely understand the objectives, have them do a real-world application problem prepared in advance (with problem solving). They should be encouraged to investigate and give reasons for their solution process. • Those who cope with tasks more quickly help others and practice collaborative learning.

Assessment:
• The home work sheet is given to the student to solve the questions and evaluate their understanding in a new real life related situation. • Teacher takes notes about student learning progress and assess their knowledge and skills by asking specific questions.

References:
• Davis, Mary Ellen and C. Henry Edwards, Elementary Mathematical Modelling: Functions and Graphs, Prentice Hall, 2007 • "Exponential&Logistic Growth" Khan Academy, www.khanacademy.org/science/biology/ecology/population-growth-and-regulation/a/exponential-logistic-growth