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Maximal regularity for evolution equations and application to the Stefan problem

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Maximal regularity is a useful tool for solving abstract parabolic evolution equations. A variant of the one-phase quasistationary Stefan problem can be reduced to a single evolution equation. We approach this problem by maximal regularity and then apply an existence theorem for this type of evolution equation. We use as an assumption that one particular result on the solvability of a degenerate oblique derivative problem extends in an appropriate way.

Keywords: Maximal regularity; evolution equations; Stefan problem.

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1. Maximal Regularity and Evolution Equations

The Stefan problem, see [13, 14], is a free boundary problem that models phase transition phenomena of two or more materials. It was the subject of research for many scientists from different areas, see [4, 11, 12]. An evolution equations approach for solving a particular kind of Stefan problem was used in [7], which led in [8] to the study of the associated center manifold.

In this paper, we will use the concept of maximal regularity obtaining solutions of a more general Stefan problem. For more details concerning maximal regularity we refer the interested reader to [6].

Definition 1.1. Let $1 < p < \infty$ and let $J = [0, T]$. The closed and densely defined operator A in Banach space X has *maximal L^p-regularity* if there is $C > 0$ such that for all $f \in L^p(J, X)$ the inhomogeneous Cauchy problem

$$\begin{aligned} u'(t) &= Au(t) + f(t), \quad t \in J \\ u(0) &= u_0 \end{aligned}$$

has a unique solution u, $u \in L^p(J, \mathcal{D}(A))$, $u' \in L^p(J, X)$, that also satisfies the *a priori* estimate

$$\|u'\|_{L^p(J,X)} + \|Au\|_{L^p(J,X)} \leq C\|f\|_{L^p(J,X)}$$

for certain $C > 0$ depending only on A and p.

One necessary condition for maximal L^p-regularity gives the following:

Remark 1.1. Due to a result of G. Dore, when the operator A has maximal L^p-regularity in the Banach space X for some $1 < p < \infty$, then $-A$ is the generator of a bounded analytic semigroup on X.

In the case when X is a Hilbert space, then by a theorem of de Simon from [3] the condition is also sufficient. We are more interested in sufficient conditions for maximal regularity since we want to use it in our applications. Concerning sufficient conditions, we recall the H^∞-calculus for sectorial operators A having spectrum in a sector

$$\Sigma_\phi := \{z \in \mathbb{C} : |\arg(z)| < \phi,\ z \neq 0\}.$$

Definition 1.2 (Bounded H^∞-calculus, McIntosh [10]). Let A be a sectorial operator in the Banach space X, with spectrum in the sector Σ_ϕ. We say that A has bounded $H^\infty(\Sigma_\phi)$-calculus, if there is $\phi \in (0, \pi)$ such that

$$\|h(A)\|_{\mathcal{L}(X)} \leq C\|h\|_\infty, \quad \forall h \in H_0^\infty(\Sigma_\phi)$$

where $h(A)$ is defined through Cauchy's integral formula, see [6].

It is known that operators with H^∞-calculus have maximal regularity.

We will use the next result which states that under certain conditions pseudo-differential operators have bounded H^∞-calculus [1].

Theorem 1.1 (Bilyj, Schrohe, Seiler). *Let M be a compact manifold and $A : C^\infty(M) \to C^\infty(M)$ a pseudo-differential operator of order $m \geq 0$ with symbol $a \in S^m_{\rho,\delta}$ where $0 \leq \delta < \rho \leq 1$. Assume there are constants $c, C > 0$, such that for $x, \xi \in \mathbb{R}^n, |\xi| \geq C$, the spectrum of $a(x,\xi)$ is contained in $\Sigma_\varphi \cap \{|\mu| \geq c\}$ and also for $\mu \in \Sigma_\varphi$ we have*

$$|\partial_\xi^\alpha \partial_x^\beta a(x,\xi)|\ |(a(x,\xi) - \mu)^{-1}| \leq c_{\alpha\beta}\langle\xi\rangle^{-\rho|\alpha|+\delta|\beta|}.$$

Then the operator $A + \tilde{c}$ has bounded H^∞-calculus for sufficiently large $\tilde{c} > 0$.

In connection with maximal regularity, we will use the following result [2]. For the definition of interpolation spaces that is related to maximal regularity, see [9].

Theorem 1.2 (Clement-Li). *Let E_0 and E_1 be two Banach spaces such that $E_1 \hookrightarrow E_0$ and E_1 is dense in E_0. Let $u_0 \in (E_0, E_1)_{1-\frac{1}{p},p}$ and let $F \in \mathrm{Lip}(U, E_0)$, $A \in \mathrm{Lip}(U, \mathcal{L}(E_1, E_0))$, where U is an open, bounded neighborhood containing u_0,*

$U \subset (E_0, E_1)_{1-\frac{1}{p},p}$. *Assume in addition that* $A(u_0)$ *has maximal regularity. Then there is* $0 < \tau \leq T$, *such that the quasilinear parabolic evolution equation*

$$\dot{u} + A(u)u = F(u), \quad \text{on } J_\tau = (0, T)$$
$$u(0) = u_0$$

has an unique solution

$$u \in L^p(J_\tau, E_1) \cap W^{1,p}(J_\tau, E_0) \cap C(J_\tau, E_{1-\frac{1}{p},p}).$$

2. The Stefan Problem

The classical Stefan problem consists of finding a boundary between two phases and the temperature on the boundary. We propose a model with surface tension and kinetic undercooling which reflects the relaxation dynamics. We denote by Ω_t a smooth time-dependent domain in $\mathbb{R}^n$ with boundary Γ_t.

$$\begin{cases} \Delta u = 0 & \text{in } \Omega_t, \\ V + \partial_\nu u = 0 & \text{on } \Gamma_t, \\ u = aV + \kappa & \text{on } \Gamma_t, \\ \Gamma(0) = \Gamma_0. \end{cases}$$

The boundary condition $u = aV + \kappa$ is given with a non-negative function $a \geq 0$ expresses the temperature as a function of the local normal velocity V and the normal curvature κ of the phase boundary. We assume that the initial geometry Γ_0 is in the Sobolev class $W^{3+s,p}$, $p > n$, $s > 0$. The problem with strictly positive function $a > 0$ is much easier and is solved in [7].

Using the so-called Hanzawa transformation [5], the free boundary problem can be transformed to a problem with fixed domain $D \subset \mathbb{R}^n$ that has boundary $\partial D = \Sigma$.

$$\begin{cases} \mathcal{A}(\rho)v = 0 & \text{in } D, \\ v + \delta\mathcal{B}(\rho)v = H(\rho) & \text{on } \Sigma, \\ \partial_t \rho + L_\rho \mathcal{B}(\rho)v = 0 & \text{on } \Sigma, \\ \rho(0) = \rho_0 & \text{on } \Sigma. \end{cases}$$

The first two equations form a boundary value problem

$$\begin{aligned} \mathcal{A}(\rho)v &= 0 && \text{in } D, \\ v + \delta\mathcal{B}(\rho)v &= H(\rho) && \text{on } \Sigma. \end{aligned}$$

We assume that the initial geometry Γ_0 is in the class $C^{3,\alpha}$. The operator $\mathcal{A}(\rho)$ is a second-order operator

$$A(\rho)v = \sum_{i,j} a_{ij}(\rho)\partial_{ij}^2 v + \sum_i a_i(\rho)\partial_i v,$$

and for its coefficients the following regularity holds:

$$a_{ij}(\rho) \in W^{2+s,p}(D), \quad a_i(\rho) \in W^{1+s,p}(D).$$

3. Extension of Taira's Theorem and Solution to the Stefan Problem

In order to solve the Stefan problem, we conjecture that there is an extension of a theorem of Taira [15]. The assumption is that there is a unique solution $u \in W^{2+s,p}(D)$ to the following boundary value problem:

$$\begin{aligned} &Au = f \quad \text{in } D, \\ &Lu = \mu \frac{\partial u}{\partial \mathbf{n}} + \gamma u = \varphi \quad \text{on } \partial D. \end{aligned}$$

The operator A is a second-order strongly elliptic operator

$$Au(x) = \sum_{i,j=1}^{n} a_{ij}(x)\partial_{ij}^2 u(x) + \sum_{i=1}^{n} a_i(x)\partial_i u(x)$$

with

$$a_{ij} \in W^{2+s,p}(D), \quad a_i \in W^{1+s,p}(D),$$

where $s > 0$, $p > n$ and $sp > n$. Moreover, we assume that the following conditions are satisfied:

(i) $\mu(x) \geq 0$ on ∂D

(ii) $\gg (x) \geq 0$ on $\partial D, \gg (x) > 0$ on $M := \{x \in \partial D : \mu(x) = 0\}$

and $f \in W^{s,p}(D)$, $\varphi \in W_{\mathcal{B}}^{1+s-1/p,p}(\partial D)$.

In Taira's theorem the coefficients are smooth, $a_{ij} \in C^\infty(D), a_i \in C^\infty(D)$, and the functions f and φ have less regularity: $f \in L^p(D)$, $\varphi \in W_{\mathcal{B}}^{1-1/p,p}(\partial D)$. We conjecture that if we impose more regularity to the functions f and φ, then, the solution, if it exists, has more regularity as well.

We now go back to the evolution equation for the Stefan problem in order to find the initial free boundary.

4. The Evolution Equation

The last two equations in the Stefan problem give the evolution equation

$$\begin{cases} \partial_t \rho + L_\rho \mathcal{B}(\rho) v = 0 & \text{on } \Sigma, \\ \rho(0) = \rho_0 & \text{on } \Sigma. \end{cases}$$

Let $\mathcal{S}(\cdot)$ be the conjectured solution operator of the boundary problem. If we put $v = \mathcal{S}(\rho)H(\rho)$ into this evolution equation, we obtain the fully nonlinear evolution equation

$$\begin{cases} \partial_t \rho + L_\rho \mathcal{B}(\rho)\mathcal{S}(\rho)H(\rho) = 0, \\ \rho(0) = \rho_0. \end{cases}$$

This evolution equation can be linearized by using the quasilinear structure of the mean curvature operator $H(\rho)$. It can be written as

$$H(\rho) = P(\rho)\rho + Q(\rho),$$

where $P(\rho)$ is a second-order uniformly elliptic differential operator and $Q(\rho)$ is an analytic function depending on the first- and second-order derivatives of ρ, see [7].

Then the evolution equation becomes quasilinear

$$\begin{aligned} \partial_t \rho + A(\rho)\rho &= F(\rho), \\ \rho(0) &= \rho_0. \end{aligned} \tag{4.1}$$

For arbitrary $\varphi < \pi$, we define the sector $S(\varphi)$ by

$$S(\varphi) = \{re^{it} \in \mathbb{C} : r \geq 0, \varphi \leq t \leq 2\pi - \varphi\}.$$

and show the following theorem.

Theorem 4.1. *The operator $A(\rho_0)$ has $\mathcal{H}^\infty$-calculus with respect to $S(\varphi)$ and consequently it has maximal regularity.*

Proof. The operator $A(\rho_0)$ has the form

$$A(\rho_0) = L_{\rho_0}\Lambda(I + a\Lambda)^{-1}P(\rho_0)$$

with a strictly positive function L_{ρ_0}. We show that the above composition satisfies the assumptions of Theorem 1.1. $A(\rho_0)$ is the composition of four operators:

(i) the positive differential operator $P(\rho_0)$
(ii) the pseudo-differential operator $C(0) = (I + a\Lambda)^{-1}$
(iii) the Dirichlet–Neumann operator Λ
(iv) the multiplication with L_{ρ_0}.

The composition is therefore a pseudo-differential operator with local symbols $p(x, \xi)$ in $S^3_{1,1/2}$. We want to show:

There are constants $c > 0$, $R \geq 0$ with the following property: for all multi-indices α, β, there are constants $C_{\alpha,\beta}$ such that for all x and ξ with $|\xi| \geq R$ and all $\mu \in S(\varphi)$ we have

(i) $|p(x, \xi) - \mu| \geq c, \quad \mu \in S(\varphi),$
(ii) $\dfrac{|\partial_\xi^\alpha \partial_x^\beta p(x, \xi)|}{|p(x, \xi) - \mu|} \leq C_{\alpha,\beta}\langle \xi \rangle^{-|\alpha|+|\beta|/2}.$

We know that the local symbol λ of Λ has the principal symbol $|\xi|_x$. For simplicity we denote it in all estimates by $|\xi|$. We also denote by $d = d(x, \xi)$ the local symbol of $P(\rho_0)$ and by d_0 the principal symbol. Then d_0 is strictly positive and it holds $d_0(x, \xi) \geq c|\xi|^2$ for sufficiently large $|\xi|$ with a constant $c > 0$.

The function L is irrelevant to (i) and (ii) and therefore can be ignored. Now, let

$$p_0(x, \xi) = |\xi|(1 + a(x)|\xi|)^{-1} d_0(x, \xi).$$

Then p_0 is strictly positive for $\xi \neq 0$, since $d_0 > 0$ and $p(x, \xi)p_0^{-1}(x, \xi) \to 1$ for $|\xi| \to \infty$, uniformly in x. Hence, we have that for $|\xi| \geq 1$

$$|\mu - p(x, \xi)| \geq c/2|p_0(x, \xi)|.$$

Since $|p_0(x, \xi)| \geq 1$ for $|\xi| \geq 1$, we obtain (i).

In order to show (ii), we first remark that except for regularizing terms, $p = \lambda \# c(0) \# d$ is the Leibniz product of three symbols. Since for the composition we have an asymptotic expansion, it is sufficient to show (ii) for the terms of the expansion. We have

$$\begin{aligned}\partial_\xi^\alpha \partial_x^\beta p(x, \xi) \sim \sum \frac{1}{\sigma!\tau!} &(\partial_\xi^{\alpha_1+\sigma} \partial_x^{\beta_1} \lambda(x, \xi))(\partial_\xi^{\alpha_2+\tau} \partial_x^{\beta_2+\sigma} c_j(x, \xi; 0)) \\ &\times (\partial_\xi^{\alpha_3} \partial_x^{\beta_3+\tau} d(x, \xi)),\end{aligned}$$

where the sum is taken over all multi-indices σ, τ, $j = 0, 1, \ldots$, all $\alpha_1, \alpha_2, \alpha_3$ which add up to α, and all $\beta_1, \beta_2, \beta_3$ which add up to β.

Now, let $\sigma, \tau, j, \alpha_j, \beta_k$ be fixed. Using the symbol estimates for λ and δ, the terms on the right-hand side can be estimated by

$$\begin{aligned} &O(\langle\xi\rangle^{1-|\alpha_1|-|\sigma|}|c_0(x, \xi, 0)|\langle\xi\rangle^{-j/2-|\alpha_2|-|\tau|+|\beta_2|/2+|\sigma|/2}\langle\xi\rangle^{2-|\alpha_3|})) \\ &\quad = O(\langle\xi\rangle^{3-j/2-|\alpha|-|\tau|-|\sigma|/2+|\beta|/2}|c_0(x, \xi; 0)|). \end{aligned} \tag{4.2}$$

The principal symbol of $C(0) = (1 + a\Lambda)^{-1}$ is $c_0(x, \xi; 0) = (1 + a(x)|\xi|)^{-1}$. Since $|\delta(x, \xi)| \geq c\langle\xi\rangle^2$ for suitable $c > 0$ and large $|\xi|$, we have $p_0(x, \xi) \geq c'\langle\xi\rangle^3|c_0(x, \xi; 0)|$ and therefore (ii) follows. □

5. Solution of the Stefan Problem

Stefan problem is reduced to a single quasilinear evolution equation (4.1). Its solution follows from our previous results and allows us to obtain solution for the Stefan problem.

Theorem 5.1. *For any initial geometry $\rho(0, .) = \Gamma_0$, with $\rho(0, .) \in C^\infty(\Sigma)$, the Stefan Problem has a unique local solution (v, ρ) on a sufficiently small time interval*

$J_\tau = [0, \tau)$, *such that*

$$v \in C(J_\tau, W^{2+s,p}(D))$$

and

$$\rho \in L^p(J_\tau, W_{\mathcal{B}}^{4+s-\frac{1}{p},p}(\Sigma)) \cap W^{1,p}(J_\tau, W^{2+s-\frac{1}{p},p}(\Sigma)).$$

Proof. The result follows from the fact that the evolution equation for the Stefan problem satisfies the conditions of Theorem 1.2 applied to the Banach couple $E_0 = W^{2+s-1/p,p}(\Sigma)$, $E_1 = W_{\mathcal{B}}^{4+s-1/p,p}(\Sigma)$, and also the conditions of Theorem 4.1. □

References

1. O. Bilyj, E. Schrohe and J. Seiler, H_∞-calculus for hypoelliptic pseudodifferential operators, *Proc. Amer. Math. Soc.* **139** (2010) 1645–1656.
2. P. Clement and S. Li, Abstract parabolic quasilinear equation and application to a groundwater flow problem, *Adv. Math. Sci. Appl.* **3** (1993/1994) 17–32.
3. L. de Simon, Un'applicazione della teoria degli integrali singolari allo studio dell equazioni differenziali lineari astratte del primo ordine, *Rend. Sem. Univ. Padova* **34** (1964) 547–558.
4. A. Friedman and F. Reitich, The Stefan-Problem with small surface tension, *Trans. Amer. Math. Soc.* **328**(2) (1991) 465–515.
5. E. I. Hanzawa, Classical solutions of the Stefan-Problem, *Tohoku Math. J.* **33** (1981) 297–335.
6. P. C. Kunstmann and L. Weis, Maximal L_p-regularity for parabolic equations, Fourier multiplier theorems and H^∞-functional calculus, in *Functional Analytic Methods for Evolution Equations*, Lecture Notes in Mathematics, Vol. 1855 (Springer, 2004).
7. M. Lukarevski, Evolution equations for the Stefan Problem, *Serdica Math. J.* **41** (2015) 333–342.
8. M. Lukarevski, Center manifolds for evolution equations associated with the Stefan problem, *Serdica Math. J.* **43** (2017) 9–20.
9. A. Lunardi, *Analytic Semigroups and Optimal Regularity in Parabolic Problems* (Birkhäuser, Basel, 1995).
10. A. McIntosh, Operators which have an H_∞ functional calculus, *Miniconference on Operator Theory and Partial Differential Equations* (North Ryde, 1986); *Proc. Centre for Mathematics and its Applications*, Vol. 14 (Australian National University, Canberra, 1986), pp. 210–231.
11. A. M. Meirmanov, *The Stefan-Problem* (Walter de Gruyter, Berlin, 1992).
12. W. W. Mullins and R. F. Sekerka, Morphological stability of a particle growing by diffusion or heat flow, *J. Appl. Phys.* **34** (1963) 323.
13. J. Stefan, Über die Theorie der Eisbildung, insbesondere ber die Eisbildung im Polarmeere, *Sitzungsber. Acad. Wiss. Wien Math. — Natur. Cl* **98** (1889) 269–286.
14. J. Stefan, Über die theorie der eisbildung, *Monatsh. Math.* **1** (1890) 1–6.
15. K. Taira, Boundary value problems for elliptic integro-differential operators, *Math. Z.* **222**(2) (1996) 305–327.