International Journal of Applied Mathematics

Volume 35 No. 1 2022, 147-159

ISSN: 1311-1728 (printed version); ISSN: 1314-8060 (on-line version)
doi: http://dx.doi.org/10.12732/ijam.v35i1.11

PRODUCT OF THE GENERALIZED FUNCTIONS $x_-{}^{-k}$ AND $\delta^{(p)}(x)$ IN COLOMBEAU ALGEBRA

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Abstract: In this paper the product of the distributions $x_-{}^{-k}$ and $\delta^{(p)}(x)$ is derived. The result is obtained in Colombeau algebra of generalized functions which contains the space of Schwartz distributions as a subspace and has a notion of 'association' that allows us to evaluate the results in terms of distributions.

AMS Subject Classification: 46F10, 46F30
Key Words: distribution; Colombeau algebra; Colombeau generalized function; multiplication of distributions

Received: October 7, 2021

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1. Introduction

The large employment of distributions in different mathematical fields and other natural sciences has imposed the need for solving two main problems that distributional theory comes across: multiplication of distributions (not any two distributions can always be multiplied) and differentiating the product of distributions (the product of distributions not always satisfy the Leibniz rule). Therefore many attempts have been made to define the product of distributions [7, 8, 11], or rather to enlarge the number of existing products. Many attempts have also been made to include the distributions into differential algebras [24].

According to the distribution theory [2, 14], we can distinguish two complementary points of view:

The first one is that distribution can be considered as a continuous linear functional f acting on a smooth function φ with compact support, i.e. we have a linear map $\varphi \to \langle f, \varphi \rangle$ where φ is called *test function*.

The second one is sequential approach: taking a sequence of smooth functions (φ_n) converging to the Dirac δ function, we obtain a family of regularization (f_n) by the convolution product

$$f_n(x) = (f * \varphi_n)(x) = \langle f(y), \varphi_n(x-y) \rangle, \tag{1}$$

which converges weakly to the distribution f. We identify all the sequences that converge weakly to the same limit and consider them as an equivalence class. The elements of each equivalence class are called *representatives* of the appropriate distribution f. This way we obtain sequential representation of distributions. Some authors use an equivalence classes of nets of regularization, i.e. the δ - net $(\varphi_\varepsilon)_{\varepsilon>0}$ defined with $\varphi_\varepsilon = \frac{1}{\varepsilon}\varphi\left(\frac{x}{\varepsilon}\right)$.

By the regularization process, the non-linear structure is lost in a way identifying sequences with their limit. Actually, all the operations then are done on the regularized functions (the sequences of smooth functions) and with the inverse process starting from the result, the function is returned from the regularization. So, we have to get nonlinear theory of generalized functions that will work with regularization.

The optimal solution for overcoming the problems that Schwartz theory of distributions is concerned with, was offered by J.F. Colombeau [15, 16]. He constructed an associative differential algebra of generalized functions $\mathcal{G}(\mathbf{R})$ which contains the space $\mathcal{D}'(\mathbf{R})$ of distributions as subspace and the algebra of C^∞-functions as subalgebra. This theory of generalized functions of Colombeau actually generalizes the theory of Schwartz distributions: these new Colombeau generalized functions can be differentiated in the same way as distributions, but when multiplication and other nonlinear operations are concerned, it is significant that the result of these operations always exists in this algebra as Colombeau generalized function. (How Colombeau algebra $\mathcal{G}$ can be used for treating linear and nonlinear problems including singularities one can see in [9]). These new generalized functions are very much related to the distributions in the sense that their definition may be considered as a natural extension of Schwartz definition of distribution.

The notion 'association' in $\mathcal{G}$ is a faithful generalization of the equality of distributions and enables us to interpret results in terms of distributions again.

Colombeau algebra was constructed in a way that many problems with multiplication of distributions could be avoid. Due to these properties, Colombeau theory has reached large application in different natural sciences and engineering, especially in fields where products of distributions with coinciding singularities are considered. About applications of Colombeau theory of generalized functions one can read papers [17, 1, 18, 29, 27, 28, 12, 26, 19, 25, 13].

In this paper we obtain some results about products of distributions in Colombeau algebra, in terms of associated distributions. Other products of distributions, evaluated in the same way can be found in [3, 5, 6, 20, 10, 21, 22, 23]. The results can be reformulated as regularized products in the classical distribution theory.

2. Colombeau algebra

In this section we will introduce basic notations and definitions from the Colombeau theory.

Let $\mathbf{N_0}$ be the set of non-negative integers, i.e. $\mathbf{N_0} = \mathbf{N} \cup \{0\}$. For $q \in \mathbf{N_0}$ we denote

$$A_q(\mathbf{R}) = \left\{ \varphi(x) \in \mathcal{D}(\mathbf{R}) \,\middle|\, \int_{\mathbf{R}} \varphi(x)\, dx = 1 \right.$$
$$\left. \text{and} \quad \int_{\mathbf{R}} x^j \varphi(x)\, dx = 0,\ j = 1, ..., q \right\},$$

where $\mathcal{D}(\mathbf{R})$ is the space of all C^∞ functions $\varphi : \mathbf{R} \to \mathbf{C}$ with compact support. The elements of the set $A_q(\mathbf{R})$ are called *test functions*. It is obvious that $A_1 \supset A_2 \supset A_3 ...$. Also, $A_k \neq \emptyset$ for all $k \in \mathbf{N}$. For $\varphi \in A_q(\mathbf{R})$ and $\varepsilon > 0$ it is denoted $\varphi_\varepsilon = \frac{1}{\varepsilon}\varphi\left(\frac{x}{\varepsilon}\right)$ and $\overset{\vee}{\varphi}(x) = \varphi(-x)$.

Looking to obtain an algebra containing the space of distributions, which elements could be multiplied and differentiated as well as C^∞ functions, Colombeau started with $\mathcal{E}(\mathbf{R})$, the algebra of functions $f(\varphi, x): A_0(\mathbf{R}) \times \mathbf{R} \to \mathbf{C}$ that are infinitely differentiable with respect to the second variable, x. The embedding of distributions into such an algebra must be done in a way that the embedding of C^∞ functions will be identity. Let f and g be C^∞ functions. Taking the sequence $(f * \varphi_\varepsilon)_{\varepsilon>0}$, which converges to f in $\mathcal{D}'$, as a representative of f, we obtain an embedding of $\mathcal{D}'$ into $\mathcal{E}(\mathbf{R})$. So, if we consider f and g as a distributions, we look at the sequences $(f * \varphi_\varepsilon)_{\varepsilon>0}$ and $(g * \varphi_\varepsilon)_{\varepsilon>0}$. The product of f and g as a distributions not always coincide with their classical product considered as a distribution, i.e.

$$(f * \varphi_\varepsilon)(g * \varphi_\varepsilon) \neq (fg) * \varphi_\varepsilon.$$

The idea therefore is to find an ideal $\mathcal{I}[\mathbf{R}]$ such that this difference will vanish in the resulting quotient. In order to determine $\mathcal{I}[\mathbf{R}]$ it is obviously enough to find an ideal containing the differences $((f * \varphi_\varepsilon) - f)_{\varepsilon>0}$.

Expanding the last term in a Taylor series and having in mind properties of $\varphi(x)$ as an element of $A_q(\mathbf{R})$ we can see that it will vanish faster then any power of ε, uniformly on compact sets, in all derivatives. The set of these differences will not be an ideal in $\mathcal{E}(\mathbf{R})$ but in a set of a sequences which derivatives are bounded uniformly on compact sets by negative power of ε. These sequences are called 'moderate' sequences and the set containing them is denoted with $\mathcal{E}_M[\mathbf{R}]$.

Finally, the generalized functions of Colombeau are elements of the quotient algebra

$$\mathcal{G} \equiv \mathcal{G}(\mathbf{R}) = \frac{\mathcal{E}_M[\mathbf{R}]}{\mathcal{I}[\mathbf{R}]},$$

where, as explained before, $\mathcal{E}_M[\mathbf{R}]$ is the subalgebra of 'moderate' functions such that for each compact subset K of $\mathbf{R}$ and any $p \in \mathbf{N_0}$ there is a $q \in \mathbf{N}$ such that, for each $\varphi \in A_q(\mathbf{R})$ there are $c > 0$, $\eta > 0$ and it holds:

$$\sup_{x \in K} |\partial^p f(\varphi_\varepsilon, x)| \leq c\varepsilon^{-q}$$

for $0 < \varepsilon < \eta$ and $\mathcal{I}[\mathbf{R}]$ is an ideal of $\mathcal{E}_M[\mathbf{R}]$ consisting of all functions $f(\varphi, x)$ such that for each compact subset K of $\mathbf{R}$ and any $p \in \mathbf{N_0}$ there is a $q \in \mathbf{N}$ such that for every $r \geq q$ and each $\varphi \in A_r(\mathbf{R})$ there are $c > 0$, $\eta > 0$ and it holds:

$$\sup_{x \in K} |\partial^p f(\varphi_\varepsilon, x)| \leq c\varepsilon^{r-q}$$

for $0 < \varepsilon < \eta$. Elements of $\mathcal{I}\left[\mathbf{R}\right]$ are also known as 'null' functions or 'negligible' functions.

The Colombeau algebra $\mathcal{G}\left(\mathbf{R}\right)$ contains the distributions on $\mathbf{R}$ canonically embedded as a $\mathbf{C}$-vector subspace by the map:

$$i : \mathcal{D}'\left(\mathbf{R}\right) \to \mathcal{G}\left(\mathbf{R}\right) : \, u \to \widetilde{u} = \left\{ \widetilde{u}\left(\varphi, x\right) = \left(u * \overset{\vee}{\varphi} \right)\left(x\right) : \varphi \in A_q\left(\mathbf{R}\right) \right\},$$

where $*$ denotes the convolution product of two distributions and is given by: $\left(f * g\right)\left(x\right) = \int\limits_{\mathbf{R}} f\left(y\right) g\left(x - y\right) dy.$

According to the above, we can also write:

$$\widetilde{u}\left(\varphi, x\right) = \left\langle u\left(y\right), \varphi\left(y - x\right)\right\rangle,$$

where $\left\langle u, \varphi \right\rangle$ denotes the integral $\int\limits_{\mathbf{R}} u\left(x\right)\varphi\left(x\right) dx.$

An element $f \in \mathcal{G}$ (a generalized function of Colombeau) is actually an equivalence class $[f] = [f_\varepsilon + \mathcal{I}]$ of an element $f_\varepsilon \in \mathcal{E}_M$ which is called *representative* of f. Multiplication and differentiation of generalized functions are performed on arbitrary representatives of the respective generalized functions.

With the next two definitions, we introduce the notion of 'association'.

Definition 1. Generalized functions $f, g \in \mathcal{G}\left(\mathbf{R}\right)$ are said to be *associated*, denoted $f \approx g$, if for some representatives $f\left(\varphi_\varepsilon, x\right)$ and $g\left(\varphi_\varepsilon, x\right)$ and arbitrary $\psi\left(x\right) \in \mathcal{D}\left(\mathbf{R}\right)$ there is a $q \in \mathbf{N_0}$ such that for any $\varphi\left(x\right) \in A_q\left(\mathbf{R}\right)$:

$$\lim_{\varepsilon \to 0_+} \int\limits_{\mathbf{R}} \left| f\left(\varphi_\varepsilon, x\right) - g\left(\varphi_\varepsilon, x\right) \right| \psi\left(x\right) dx = 0.$$

Definition 2. A generalized function $f \in \mathcal{G}$ is said to admit some $u \in \mathcal{D}'\left(\mathbf{R}\right)$ as 'associated distribution', denoted $f \approx u$, if for some representative $f\left(\varphi_\varepsilon, x\right)$ of f and any $\psi\left(x\right) \in \mathcal{D}\left(\mathbf{R}\right)$ there is a $q \in \mathbf{N_0}$ such that for any $\varphi\left(x\right) \in A_q\left(\mathbf{R}\right)$

$$\lim_{\varepsilon \to 0_+} \int\limits_{\mathbf{R}} f\left(\varphi_\varepsilon, x\right) \psi\left(x\right) dx = \left\langle u, \psi \right\rangle.$$

These definitions are independent of the representatives chosen and the distribution associated, if it exists, is unique. The association is a faithful generalization of the equality of distributions.

Multiplying two distributions in $\mathcal{G}$ as a result it is in general obtained a generalized function which may not always be associated to the third distribution. By *Colombeau product of distributions* is meant the product of their embedding in $\mathcal{G}$ whenever the result admits an associated distribution. If the regularized model product of two distributions exists, then their Colombeau product also exists and it is same with the first one. The relation $f \approx u$ is asymmetric, the distribution u stands on the r.h.s.; the relation $f \approx \widetilde{u}$ is an equivalent relation in $\mathcal{G}$ so it is symmetric in $\mathcal{G}$ and it can also be written as $f - \widetilde{u} \approx 0$. As a final conclusion of this introduction we have a fact that we operate with the elements of $\mathcal{G}$ exactly the same as with the C^∞-functions, because we actually operate with their representatives which are C^∞- functions.

3. Results on some products of distributions

In [4] it was proved that for any $p \in \mathbf{N}$ the product of the generalized functions $\widetilde{x^{-p}}$ and $\widetilde{\delta^{(p-1)}(x)}$ in $\mathcal{G}(\mathbf{R})$ exists and it holds:

$$\widetilde{x^{-p}} \cdot \widetilde{\delta^{(p-1)}(x)} \approx \frac{(-1)^{|p|}(p-1)!}{2(2p-1)!}\delta^{(2p-1)}(x)\,.$$

Also in [20], the product of the generalized functions $\widetilde{\ln|x|}$ and $\widetilde{\delta^{(p-1)}(x)}$ in $\mathcal{G}(\mathbf{R})$ was considered and it was proved that

$$\widetilde{\ln|x|} \cdot \widetilde{\delta^{(p-1)}(x)} \approx \frac{(-1)^p}{p}\delta^{(p-1)}(x)\,.$$

In [21] the product of the generalized functions $\widetilde{x_+{}^{-k}}$ and $\widetilde{\delta^{(p)}(x)}$ is derived in Colombeau algebra and there is proved the relation

$$\widetilde{x_+{}^{-k}} \cdot \widetilde{\delta^{(p)}(x)} \approx \frac{(-1)^k k \cdot p!}{(k+p+1)!}\delta^{(k+p)}(x)\,.$$

Products of the form $x^{-k}\delta^{(p)}(x)$ are useful in quantum renormalization theory in Physics. We will prove the next theorem considering similar products.

Theorem 3. *The product of the generalized functions* $\widetilde{x_-^{-k}}$ *and* $\widetilde{\delta^{(p)}(x)}$ *for* $k = 1, 2, \ldots$ *and* $p = 0, 1, 2, \ldots$ *in* $\mathcal{G}(\mathbf{R})$ *admits an associated distribution and it holds:*

$$\widetilde{x_-^{-k}} \cdot \widetilde{\delta^{(p)}(x)} \approx \frac{k \cdot p!}{(p+k+1)!}\delta^{(k+p)}(x). \tag{2}$$

Proof. By definition we have $x_-^{-k} = \dfrac{-1}{(k-1)!}\dfrac{d^k}{dx^k}(\ln x_-)$. We will embed this distribution into Colombeau algebra. For $\varphi \in A_0(\mathbf{R})$ given test function, using the embedding rule, we have:

$$\widetilde{x_-{}^{-k}}\,(\varphi_\varepsilon, x) = \int\limits_{\mathbf{R}} y_-{}^{-k} \cdot \varphi_\varepsilon\,(y-x)\,dy$$
$$= -\frac{1}{\varepsilon\,(k-1)!}\int\limits_{-\infty}^{0} \frac{\partial^k}{\partial x^k}\,(\ln y_-)\,\varphi\left(\frac{y-x}{\varepsilon}\right)dy$$
$$= \frac{(-1)^{k+1}}{\varepsilon^{k+1}\,(k-1)!}\int\limits_{-\infty}^{0} \ln\left(-y\right)\varphi^{(k)}\left(\frac{y-x}{\varepsilon}\right)dy.$$

We will first use the substitution $t=-y$ and obtain:

$$\widetilde{x_-{}^{-k}}\,(\varphi_\varepsilon, x) = \frac{(-1)^{k+1}}{\varepsilon^{k+1}\,(k-1)!}\int\limits_{0}^{\infty} \ln t\cdot\varphi^{(k)}\left(-\frac{t+x}{\varepsilon}\right)dt.$$

We suppose that supp$\varphi(x) \subseteq [-l, l]$, without lost of generality. Then using the substitution $s = -\frac{t+x}{\varepsilon}$ we obtain:

$$\widetilde{x_-{}^{-k}}\,(\varphi_\varepsilon, x) = \frac{(-1)^{k+1}}{\varepsilon^{k+1}\,(k-1)!}\int\limits_{-x}^{-x-\varepsilon l} \ln t\cdot\varphi^{(k)}\left(-\frac{t+x}{\varepsilon}\right)dt$$
$$= \frac{(-1)^{k}}{\varepsilon^{k}\,(k-1)!}\int\limits_{0}^{l} \ln\left(-x-\varepsilon s\right)\cdot\varphi^{(k)}\left(s\right)ds,$$

which gives us the representatives of the distribution x_-^{-k} in Colombeau algebra.

Differentiating in $\mathcal{D}'(\mathbf{R})$, we obtain the representatives of $\widetilde{\delta^{(p)}(x)}$:

$$\widetilde{\delta^{(p)}\,(\varphi_\varepsilon, x)} = \frac{(-1)^p}{\varepsilon^{p+1}}\varphi^{(p)}\left(-\frac{x}{\varepsilon}\right).$$

Let us calculate the product of these two generalized functions embedded in Colombeau algebra. For any $\psi(x) \in \mathcal{D}(\mathbf{R})$ we have:

$$\left\langle \widetilde{x_-{}^{-k}}\,(\varphi_\varepsilon, x)\cdot\widetilde{\delta^{(p)}}\,(\varphi_\varepsilon, x)\,, \psi\,(x)\right\rangle = \int\limits_{-\infty}^{0} \widetilde{x_-{}^{-k}}\,(\varphi_\varepsilon, x)\cdot\widetilde{\delta^{(p)}}\,(\varphi_\varepsilon, x)\,\psi\,(x)\,dx$$

$$=\frac{(-1)^{k+p}}{\varepsilon^{k+p+1}(k-1)!}\int_{-\infty}^{0}\left(\int_{0}^{l}\ln(-x-\varepsilon s)\cdot\varphi^{(k)}(s)\,ds\right)\varphi^{(p)}\left(-\frac{x}{\varepsilon}\right)\psi(x)\,dx.$$

We use that $\operatorname{supp}\varphi(x)\subseteq[-l,\,l]$ and obtain

$$\left\langle\widetilde{x_-^{-k}}(\varphi_\varepsilon,x)\cdot\widetilde{\delta^{(p)}}(\varphi_\varepsilon,x),\psi(x)\right\rangle=$$

$$\frac{(-1)^{k+p+1}}{\varepsilon^{k+p+1}(k-1)!}\int_{0}^{\varepsilon l}\left(\int_{0}^{l}\ln(-x-\varepsilon s)\,\varphi^{(k)}(s)\,ds\right)\varphi^{(p)}\left(-\frac{x}{\varepsilon}\right)\psi(x)\,dx.$$

Then the substitution $u=-\frac{x}{\varepsilon}$ gives us

$$\left\langle\widetilde{x_-^{-k}}(\varphi_\varepsilon,x)\cdot\widetilde{\delta^{(p)}}(\varphi_\varepsilon,x),\psi(x)\right\rangle$$

$$=\frac{(-1)^{k+p+1}}{\varepsilon^{k+p}(k-1)!}\int_{-l}^{0}\left(\int_{0}^{l}\ln(\varepsilon u-\varepsilon s)\,\varphi^{(k)}(s)\,ds\right)\varphi^{(p)}(u)\,\psi(-\varepsilon u)\,du$$

$$=\frac{(-1)^{k+p+1}}{\varepsilon^{k+p}(k-1)!}\int_{-l}^{0}\varphi^{(p)}(u)\,\psi(-\varepsilon u)\int_{0}^{l}\ln(\varepsilon u-\varepsilon s)\,\varphi^{(k)}(s)\,dsdu. \tag{3}$$

Applying the Taylor theorem for the function ψ we have:

$$\psi(-\varepsilon u)=\sum_{i=0}^{p+k}\frac{\psi^{(i)}(0)}{i!}(-\varepsilon u)^i+\frac{\psi^{(p+k+1)}(\eta u)}{(p+k+1)!}(-\varepsilon\eta)^{p+k+1} \tag{4}$$

for $\eta\in(0,1)$.

Using relation (4) in (3) and changing the order of integration we have:

$$\left\langle\widetilde{x_-^{-k}}(\varphi_\varepsilon,x)\cdot\widetilde{\delta^{(p)}}(\varphi_\varepsilon,x),\psi(x)\right\rangle$$

$$=\sum_{i=0}^{p+k}\frac{(-1)^{k+p+i+1}\psi^{(i)}(0)}{\varepsilon^{k+p-i}\cdot i!\,(k-1)!}\int_{-l}^{0}\varphi^{(p)}(u)\,u^i\int_{0}^{l}\ln(\varepsilon u-\varepsilon s)\,\varphi^{(k)}(s)\,dsdu$$

$$=\sum_{i=0}^{p+k}\frac{(-1)^{k+p+i+1}\psi^{(i)}(0)}{\varepsilon^{k+p-i}\cdot i!\,(k-1)!}\cdot J_i+O(\varepsilon),$$

where

$$J_i=\int_{0}^{l}\varphi^{(k)}(s)ds\int_{-l}^{0}\ln(\varepsilon u-\varepsilon s)\cdot u^i\varphi^{(p)}(u)du$$

and $i = 0, 1, \ldots, p+k$.

Next, using integration by parts we have:

$$J_i = \int_0^l \varphi^{(k)}(s)ds \int_{-l}^0 \ln(\varepsilon u - \varepsilon s) \cdot u^i \varphi^{(p)}(u)du$$

$$= \frac{1}{i+1} \int_0^l \varphi^{(k)}(s)ds \int_{-l}^0 \ln(\varepsilon u - \varepsilon s) \cdot u^i \varphi^{(p)}(u)d\left(u^{i+1} - s^{i+1}\right)$$

$$= -\frac{1}{i+1} \int_0^l \varphi^{(k)}(s)ds \int_{-l}^0 \left(u^{i+1} - s^{i+1}\right) \ln(\varepsilon u - \varepsilon s) \varphi^{(p+1)}(u)\, du$$

$$-\frac{1}{i+1} \int_0^l \varphi^{(k)}(s)ds \int_{-l}^0 \frac{u^{i+1} - s^{i+1}}{u-s} \varphi^{(p)}(u)\, du.$$

The first integral in the last term is zero due to the properties of the Colombeau algebra, so we have that:

$$-(i+1)\, J_i = \int_0^l \varphi^{(k)}(s)ds \int_{-l}^0 \frac{u^{i+1} - s^{i+1}}{u-s} \varphi^{(p)}(u)\, du$$

$$= \sum_{m=0}^{i} \int_0^l \varphi^{(k)}(s)ds \int_{-l}^0 u^m s^{i-m} \varphi^{(p)}(u)\, du$$

$$= \sum_{m=0}^{i} \int_0^l s^{i-m} \varphi^{(k)}(s)ds \int_{-l}^0 u^m \varphi^{(p)}(u)\, du.$$

The integrals of the form $J_{a,b} = \displaystyle\int_0^l v^a \varphi^{(b)}(v)dv$ are non zero only if $a = b$, and its value is $J_{a,a} = (-1)^a a!$. Thus the only non zero term in the sum above is obtained when $m = p$ and $k = i - m$, i.e. $i = k + p$ and then we have:

$$-(p+k+1)\, J_{k+p} = (-1)^{k+p} k!p!.$$

Finally,

$$J_{p+k} = \frac{(-1)^{k+p+1} k!p!}{p+k+1}.$$

For the Colombeau product of the distributions we consider, we obtain:

$$
\begin{aligned}
&\left\langle \widetilde{x_-^{\,-k}} \left(\varphi_\varepsilon, x\right) \cdot \widetilde{\delta^{(p)}} \left(\varphi_\varepsilon, x\right), \psi\left(x\right) \right\rangle \\
&= \frac{(-1)\cdot\psi^{(k+p)}(0)}{(k+p)!\,(k-1)!} \cdot \frac{(-1)^{k+p+1} k!p!}{(p+k+1)} + O(\varepsilon) \\
&= \frac{(-1)^{k+p} k!p!\psi^{(k+p)}(0)}{(k+p+1)!\,(k-1)!} + O(\varepsilon) \\
&= \frac{k\cdot p!}{(k+p+1)!}(-1)^{k+p}\psi^{(k+p)}(0) + O(\varepsilon) \\
&= \frac{k\cdot p!}{(k+p+1)!}\left\langle \delta^{(k+p)}(x), \psi(x)\right\rangle + O(\varepsilon).
\end{aligned}
$$

Therefore passing to the limit, as $\varepsilon \to 0$, we obtain equation (2) which proves the theorem. □

We will extend the result in the previous theorem in $\mathcal{G}(\mathbf{R}^m)$. For the proof, we need the following lemma proved in [3].

Lemma 4. *Let u and v be distributions in $\mathcal{D}(\mathbf{R}^m)$ such that $u(x) = \prod_{i=1}^m u^i(x_i)$ and $v(x) = \prod_{i=1}^m v^i(x_i)$ with each u^i and v^i in $\mathcal{D}(\mathbf{R})$ and suppose that their embedding in $\mathcal{G}(\mathbf{R})$ satisfy $\widetilde{u^i} \cdot \widetilde{v^i} \approx \omega^i$, for $i = 1, 2, \ldots, m$. Then $\widetilde{u} \cdot \widetilde{v} \approx \omega$, where $\omega(x) = \prod_{i=1}^m \omega^i(x_i)$ is in $\mathcal{D}(\mathbf{R}^m)$.*

Theorem 5. *The product of the generalized functions $\widetilde{x_-^{-k}}$ and $\widetilde{\delta^{(p)}(x)}$ for $k = 1, 2, \ldots$ and $p = 0, 1, 2, \ldots$ in $\mathcal{G}(\mathbf{R}^m)$ admits an associated distribution and it holds:*

$$\widetilde{x_-^{-k}} \cdot \widetilde{\delta^{(p)}(x)} \approx \frac{k\cdot p!}{(p+k+1)!}\delta^{(k+p)}(x). \tag{5}$$

Proof. In this case, due to the tensor structure of the product of the distributions we consider, we can apply the previous lemma and obtain:

$$
\begin{aligned}
\widetilde{x_-^{-k}} \cdot \widetilde{\delta^{(p)}(x)} = \prod_{i=1}^m \widetilde{x_{i-}^{-k_i}} \cdot \widetilde{\delta^{(p_i)}(x_i)} &\approx \prod_{i=1}^m \frac{k_i\cdot p_i!}{(p_i+k_i+1)!}\delta^{(k_i+p_i)}(x_i) \\
&= \frac{k\cdot p!}{(p+k+1)!}\delta^{(k+p)}(x),
\end{aligned}
$$

which completes the proof of the theorem. □

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