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Content

GENERALIZED FRACTIONAL CALCULUS OPERATORS OF RIEMANN-LIOUVILLE AND CAPUTO TYPE AND CAUCHY PROBLEMS

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Both in classical and in Fractional Calculus (FC), the notions of derivatives and integrals (of first, second, etc., or arbitrary non-integer order, respectively) are basic and co-related. In the classical Calculus, the traditional way of introducing these objects is to start with derivatives and differentiability and then to proceed with the notions of integral (primitive) and definite integral as basic construction for the primitive. In contrast, one of the most conventional approaches in FC is first to define the Riemann-Liouville (R-L) integral of fractional order and then to introduce a fractional order derivative, say, in the R-L or in the Caputo sense, as a suitable composition of an integer-order differentiation operator and the R-L integral.

In this talk we survey several types of generalized operators of Fractional Calculus (FC), Kiryakova [1]. They appear to be useful in modeling various phenomena and systems in natural and human sciences, including physics, engineering, chemistry, control theory, etc., by means of fractional order (and multi-order) differential equations, see e.g. in [2]. We start with the Riemann-Liouville and Caputo derivatives and the Erdélyi-Kober operators. Then the multiple Erdélyi-Kober fractional integrals and derivatives of Riemann-Liouville type of multi-order $(\delta_1, ..., \delta_m)$ are introduced as their generalizations. Further, we define and compare the properties of the Caputo-type multiple Erdélyi-Kober derivatives, see in [3]. The generalized operators of FC are in form of integro-differential operators with suitably chosen particular cases of the H- and G-functions in the kernels.

Finally, we illustrate the theory with some examples of the generalized FC operators and of Cauchy problems for fractional order (FO) differential equations involving these operators, and compare the cases with R-L and Caputo type initial values. The role of the special functions (SF) of FC is emphasized, especially of the G- and H-functions and the Mittag-Leffler function and its generalizations, serving as kernel-functions and explicit solutions of FO equations and systems.

References:

[1] V. Kiryakova, *Generalized Fractional Calculus and Applications*, Longman & J. Wiley, 1994;

[2] V. Kiryakova (Ed.-in-Chief), *Fractional Calculus and Applied Analysis* (Int. J. for Theory and Appl.), Vol. 1 (1998)– Vol. 21 (2008), at: https://www.degruyter.com/view/j/fca;

[3] V. Kiryakova and Yu. Luchko, Riemann-Liouville and Caputo type multiple Erdélyi-Kober operators, *Centr. Eur. J. of Phys.* **11**, No 10 (2013), 1314–1336; etc.

FRACTIONAL MODELS OF WAVE PROPAGATION IN VISCOELASTIC MEDIA

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Fractional calculus is a powerful tool for modeling phenomena arising in diverse fields such as mechanics, physics, engineering, economics, finance, medicine, biology, chemistry, etc. It deals with derivatives and integrals of arbitrary real (or even complex) order, thus extending capabilities of the classical calculus, but also introducing novelties in theoretical and applied research. The focus of this talk is on the investigation of waves in viscoelastic media through the constitutive equation containing fractional derivatives of various type.

So far, the classical wave equation has been generalized for the case of viscoelastic materials by the use of fractional derivatives of constant real order (cf. [1,2]). Our most recent study introduces a distributed order fractional model to describe wave propagation in viscoelastic infinite media, and examines existence and uniqueness of fundamental solutions for the corresponding generalized Cauchy problem. Some particular cases of distributed order fractional constitutive stress-strain relations will be presented in more details, as well as numerical experiments, in order to illustrate theoretical results.

This talk is based on joint work with Lj. Oparnica and D. Zorica.

References:

[1] S. Konjik, Lj. Oparnica, D. Zorica, *Waves in fractional Zener type viscoelastic media*, J. Math. Anal. Appl., **365**(1) (2010), 259-268.

[2] S. Konjik, L. Oparnica, D. Zorica, *Waves in viscoelastic media described by a linear fractional model*, Integral Transforms Spec. Funct., **22**(4-5) (2011), 283-291.

CONVOLUTION IN QUASIANALYTIC CLASSES OF GELFAND-SHILOV TYPE

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We construct a special class of ultrapolynomials and use them to construct parametrices in generalised Gelfand-Shilov spaces that have as a special cases the Fourier hyperfunctions and Fourier ultra-hyperfunctions. We apply them in the study of topological and structural properties of several quasianalytic spaces of functions and ultradistributions. As a consequence, we develop a convolution theory for quasianalytic ultradistributions of Gelfand-Shilov type.

The talk is based on collaborative works with Stevan Pilipović and Jasson Vindas

References:

[1] S. Pilipović, B. Prangoski, J. Vindas, *On quasianalytic classes of Gelfand Shilov type. Parametrix and convolution*, J. Math. Pures Appl., https://doi.org/10.1016/j.matpur.2017.10.008

STOCKWEL TRANSFORM

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The Stockwell transform is a hybrid of the Wavelet transform and the short-time Fourier transform. We define this transform and its synthesis operator on a test function spaces $\mathcal{S}(\mathbb{R})$ and $\mathcal{S}(\mathbb{Y}), \mathbb{Y} = \mathbb{R}^n \times \mathbb{R}^n \backslash \{0\}$, and extend these definitions on the appropriate spaces of distributions via duality approach. We study the quasiasymptotic behaviour of distributions with respect to the asymptotic of their Stockwell transform. Several Abelian and Tauberian type results at the origin and at infinity are proven.

Second part of the talk is dedicated to resolution of wavefront sets with respect to the Stockwell transform. Equivalence between the new definition and the definition in the sense of Hormander is given, for the cases when

the dimension is $n = 1, 2, 4, 8$. At the end, construction of generalization for arbitrary dimension is presented.

WAVE FRONT SETS WITH RESPECT TO BANACH SPACES OF ULTRADISTRIBUTIONS. CHARACTERISATION VIA THE SHORT-TIME FOURIER TRANSFORM

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We define ultradistributional wave front sets with respect to translation-modulation invariant Banach spaces of ultradistributions having solid Fourier image. The main result is their characterisation by the short-time Fourier transform.

TWO CLASSES OF SPECIAL FUNCTIONS OF FRACTIONAL CALCULUS OF MITTAG-LEFFLER TYPE

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Both classes of $2m$- and $3m$-parametric Mittag-Leffler functions, introduced in [1] and [2], are considered. Different integrals and derivatives of an arbitrary order are calculated in the first class, such as Riemann-Liouville and Erdélyi–Kober integrals and derivatives, as well as multiple Erdélyi–Kober fractional integration operators (see e.g. [1], [3]). It turns out the results of these actions are elements of the second class (up to a constant). Various interesting relations are obtained between the functions of the second class and the calculated integrals and derivatives.

The work is under the bilateral agreements of BAS with SANU and MANU.

References:

[1] V. Kiryakova, Multiple (multiindex) Mittag-Leffler functions and relations to generalized fractional calculus. J. Comput. Appl. Math. 118 (2000), 241-259; doi:10.1016/S0377-0427(00)00292-2.

[2] J. Paneva-Konovska, Multi-index (3m-parametric) Mittag-Leffler functions and fractional calculus. Compt. rend. Acad. bulg. Sci. 64, No 8 (2011), 1089-1098; available at: http://www.proceedings.bas.bg/.

[3] J. Paneva-Konovska, Differential and integral relations in the class of multi-index Mittag-Leffler functions, Fract. Calc. Appl. Anal., 21, No1 (2018), 254-265; DOI:10.1515/fca-2018-0016;

GENERALIZED ENTROPIES BASED ON Q-FRACTIONAL CALCULUS

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In the past, there was extensive work on information measures which generalize the Shannon entropy. These measures, called generalized entropies, found an application in a number of different fields, such as economics, physics, chemistry, biology, linguistics, medicine, cognitive sciences, computer sciences and social sciences. The mathematical structure of the generalized entropies is particularly important and a special attention was directed to the establishment of axiomatic systems for their characterization. Thus, in our previous work, we characterized a wide class of generalized entropies with a generalization of celebrated Shannon-Khinchin axiomatic system, which states that the entropy is unique function which is continuous, maximized for uniform distribution, expandable and strongly pseudo-additive.

An alternative approach is based on the property by which the Shannon entropy can be represented using the derivative of information generating function, which opens a possibility for characterization of generalized entropies using generalized derivatives. The first effort in this direction was made by Abe who used Jackson q-derivative, instead of the ordinary one, and characterized the Tsallis entropy. The approach was further developed by Borges, Roditi, Chakrabarti and Jagannathan who used two parameter generalization of Jackson q-derivative, the (q, q')-derivative, to obtain the Sharma-Mittal-Taneja entropy. Moreover, Ubriaco introduced a fractional entropy based on Riemann-Liouville fractional derivative.

In this work we combine the approaches by Abe and Ubriaco and we propose new entropy based on the q-fractional derivative, which was defined and analyzed in our previous research. Thus, we obtain a q-fractional entropy which contains both the fractional and the Tsallis entropies as special cases. In addition, we suggest a new definition for the fractional derivative which generalizes the q-fractional derivative and the (q, q')-derivative, so that the resulting (q, q')-fractional entropy generalizes Sharma-Mittal-Taneja one. We also consider possible applications of presented results in aforementioned research fields.

ON A TIME DISCRETIZATION SCHEME FOR THE FRACTIONAL STOCHASTIC HEAT EQUATION

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In this work, we dealt with the fractional stochastic heat equation perturbed by a multiplicative space-time white noise and driven by the fractional power of the Laplacian. We fulfilled the temporal approximation for such type of equations via the implicit Euler scheme . In particular, we estimated the rate of the strong convergence and we showed its depenence on the fractional power of the Laplacian.

COMPOSITIONS INVOLVING DIRAC-DELTA AND HEAVISIDE FUNCTIONS

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The composition of the distributions x^λ and x_+^μ is evaluated for $\lambda = -1, -2, \dots$ $\mu > 0$ and $\lambda\mu \in \mathbb{Z}^-$. Further results are deduced.

INTERIOR ESTIMATE FOR ELLIPTIC PDE, QUASICONFORMAL AND HQC MAPPINGS BETWEEN LYAPUNOV JORDAN DOMAINS

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We prove that if h is a quasiconformal (shortly qc) mapping of the unit disk $\mathbb{U}$ on Lyapynov domain, then it maps subdomains of Lyapynov type of $\mathbb{U}$, which touch the boundary of $\mathbb{U}$, onto domains of similar type. We can regard this result as "good local approximation of qc mapping h by its restriction to a special Lyapunov domain so that codomain is "locally convex". In particular if h is a hqc (harmonic qc) mapping of $\mathbb{U}$ onto Lyapynov domain, using it, we prove that h is co-Lip on $\mathbb{U}$. It settles an open intriguing problem in the subject and can be regarded as a version of Kellogg- Warschawski theorem for hqc.

We also study the growth of gradient of mappings which satisfy certain PDE equations (or inequalities) using Green-Laplacian formula for functions and its derivatives.

If in addition the considered mappings are quasiconformal (qc) between C^2 domains, we show that they are Lipschitz

MEAN-FIELD RICCATI ODE'S ON COMPACT LIE GROUPS: SYNCHRONIZATION AND SWARMING

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Synchronization of large ensembles of coupled oscillators is a universal phenomena with a great variety of manifestations in Physics and Life Sciences. Paradigmatic model of this kind was introduced by Kuramoto in 1975. An ensemble of Kuramoto oscillators can be described by the system of complex-valued Riccati ODE's on the unit circle S^1:

$$\dot{z}_j = i(fz_j^2 + \omega z_j + \bar{f}), \quad j = 1, \ldots, N. \tag{1}$$

Here, $z_j \in S^1$ represents the state of the oscillator j, $\omega \in \mathbb{R}$ is a natural frequency of the oscillator j and $f = f(t, z_1, \ldots, z_N)$ is a complex-valued global coupling function.

I discuss some relations of the model (1) with Hyperbolic Geometry and Potential Theory in the complex plane.

It turns out that the model (1) is pretty universal and appears in different settings in various theories in Physics.

Furthermore, (1) has fine algebraic and geometric properties and can be extended to some compact Lie groups of higher dimension. For instance, collective motion of coupled particles on the 3-sphere can be described by the system of quaternion-valued Riccati ODE's:

$$\dot{q}_j = q_j f q_j + w_j q_j + q_j u_j - \bar{f}, \quad j = 1, \ldots, N, \tag{2}$$

where q_j is a unit quaternion, w_j and u_j are "pure" quaternions and $f = f(t, q_1, \ldots, q_N)$ is a quaternionic coupling function.

I briefly point out some physical interpretations and technological applications of models (1) and (2).

DYNAMICAL PROPERTIES OF THE SOLUTIONS OF A CERTAIN FRACTIONAL DIFFERENTIAL EQUATIONS

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The question how often the orbit of the semigroup intersects every open non-empty subset in E, leads us to the theory of frequently hypercyclic semigroups and furthermore to the theory of frequently hypercyclic C-distribution semigroups. Here we investigate the degree of frequency in meeting the orbits on certain class of strongly continuous semigroups. Furthermore, the irregular orbits of frequently hypercyclic semigroups will be considered as a continuous generalization of considerations of S. Grivaux. We analyze frequently hypercyclic, q-frequently hypercyclic and (t_k)-hypercyclic strongly continuous semigroups and corresponding analogues of Furstenberg theorem are given. Also the recurrent groups of bounded operators are introduced. Moreover, we introduce and analyze the notions of asymptotically Bloch-periodic functions in Banach spaces and Fréchet spaces. We give a sufficient conditions for the existence and uniqueness of asymptotically Bloch-periodic solutions for a nonlinear differential equation with piecewise constant argument in a Banach space.

NEUTRIX CALCULUS AND Q-CALCULUS

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In this paper, I will present a q-analog (also known as q-calculuc) of the classical derivative, which we can consider as a special case of fractional calculus. Some results from q-calculus involving neutrix calculus will be presented.

FRACTIONAL DERIVATIVE APPLICATION IN GEOMETRIC FUNCTION THEORY

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The study of geometric function theory and univalent functions dates back to the early years of the twentieth century and is one of the most popular areas of research in complex analysis. Initiated by the work of Bieberbach and his contemporaries, the famous conjecture of 1916 became one of the most celebrated problems in mathematics.

Essentially, this theory studies functions f that are analytic in the open unit disk and are normalized such that $f(0) = f'(0) - 1 = 0$, i.e., are of the form $f(z) = z + a_2z^2 + a_3z^3 + \cdots$ and delivers results where the first or the second derivative of f appear in the condition or in the conclusion of the result.

The idea is to replace this derivatives (first and second) with fractional derivative of certain order and find physical interpretation of the obtained results.

COLOMBEAU PRODUCT OF DISTRIBUTIONS

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In this paper some products of distributions are derived. The results are obtained in Colombeau algebra of generalized functions, which is the most relevant algebraic construction for dealing with Schwartz distributions. Colombeau algebra $\mathcal{G}(\mathbf{R})$ contains the space $\mathcal{D}'(\mathbf{R})$ of Schwartz distributions as a subspace and has a notion of 'association' that allows us to evaluate the results in terms of distributions.

APPLICATION OF WAVELETS TO DENSITY ESTIMATION AND NONPARAMETRIC REGRESSION

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Density estimation and regression function estimation have been performed on a specific data set which describes population in the largest city registered in year 2000 for 150 countries, downloaded from World Bank. Gaussian and Epanechnikov kernel for different bandwidth, Fourier series for different number of terms included, Haar based histogram for different bandwidth, Daubechies and Meyer wavelets for different dilatation factor, are the techniques used to get density estimation and regression function estimation. Results have been achieved using programming language Python and its modules: numpy, matplotlib, scipy, sklearn and openpyxl as the most important among the others. Developed scripts can be reused for other data sets with trivial modifications.

PRODUCTS OF DISTRIBUTIONS AND FRACTIONAL CALCULUS

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In this paper results on Colombeau product of distributions $x_+^{-r-1/2}$ and $x_-^{-r-1/2}$ are derived. They are obtained in Colombeau differential algebra $\mathcal{G}(\mathbf{R})$ of generalized functions contains the space $\mathcal{D}'(\mathbf{R})$ of Schwartz distributions as a subspace, and has a notion of "association" that is a faithful generalization of the weak equality in $\mathcal{D}'(\mathbf{R})$. Also some connections between colombeau calculus and fractional calculus are considered.

HISTORY OF FRACTIONAL CALCULUS

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Fractional calculus is a generalization of ordinary differentiation and integration to arbitrary (non-integer) order. The subject is as old as the differential calculus, and goes back to times when Leibnitz and Newton invented differential calculus. The most common notations for -th order derivative of a function $y(t)$ defined in (a, b) are $y^{(\beta)}(t)$. Negative values of correspond to fractional integrals.

In a letter to L'Hospital in 1695 Leibniz raised the following question: "Can the meaning of derivatives with integer order be generalized to derivatives with non-integer orders?" L'Hospital was somewhat curious about that question and replied by another question to Leibniz: "What if the order will be 1/2?" Leibnitz in a letter dated September 30, 1695 the exact birthday of the fractional calculus! replied: "It will lead to a paradox, from which one day useful consequences will be drawn."

The question raised by Leibnitz for a fractional derivative was an ongoing topic for more than 300 years. Many known mathematicians contributed to this theory over the years, among them Liouville, Riemann, Weyl, Fourier, Abel, Lacroix, Leibniz, Grunwald and Letnikov.

In this talk, several definitions of fractional derivatives will be introduced. Examples from different engineering fields will be presented to demonstrate that fractional derivatives arise naturally in many applications.