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Auction theory and a note on game mechanisms

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Abstract
This paper will review important topics on the subject of auction theory and mechanism design, these include: efficiency first and foremost, also revenue comparison between different types of auctions and the issue of incentive compatibility, individual rationality with the general idea and proof that bilateral trade is inefficient. Mechanism design theory tells us that if buyers and sellers both have private information full efficiency is impossible, however Vickrey auction (single unit auction) will be efficient i.e. will put the goods in the hands of the buyers that value them most. However, the conclusion from this paper is that because of overvaluation of bidders the main result is inefficient, i.e. bids are too high. When weak and strong bidders are compared the main conclusion is that strong bidders' expected payoff is higher in second price auction (SPA), while weak bidder prefers first price auction (FPA) bid.

Keywords: asymmetric auctions, first price auction, Green-Laffont, Myerson-Satterthwaite, second price auction.

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Introduction
This paper will provide technical survey of the auction theory and mechanism design theory. A game of auction and mechanism design is a game of private information types (private cumulative distribution functions). In First price auctions bidders may be bid shading and in Second price auctions bidders are bidding their true valuation. In the asymmetric type of auctions Revenue equivalence theorem does not hold and bidders do not know the other bidders' valuation. The result is not socially optimal i.e. is not Pareto efficient. In general, Vickrey-Clarke-Groves (VCG) mechanism (Vickrey, 1961, Clarke, 1971, Groves, 1973) is an auction that performs specific task of dividing items between people where the goal is to maximize the

sum of values of all agents. Previous type of mechanism is opposed by (Myerson, Satterthwaite, 1983) that proposes that all bilateral trade is inefficient.

The paper is organized as follows; Literature review on auction theory and auction game mechanisms is followed by the research methodology. Next a chapter in symmetric auctions follows, with a sequent subchapter(s) on first price auction (FPA), second price auction (SPA), and revenue equivalence theorem (RET) theorem. Next chapter is on asymmetric auctions with subchapters on weak and strong bidders with some examples is completed. And finally, Vickrey-Clarke-Groves (VCG) mechanism is compared with Myerson-Satterthwaite mechanism. Final chapter is conclusion.

Literature review

Auction theory was founded in 1960's although early research had little impact on practice. Most notable advances in the theory of auctions from that era (1960's and 1970's) include: Vickrey (1961, 1962, 1976), Wilson (1967, 1969, 1977, 1979), Cassady (1967), Griesmer, Levitan, Shubik (1967), Ortega (1968), Rothkopf (1969), Hurwicz (1973), Holmstrom (1977, 1979), Green and Laffont (1977), Milgrom (1979), Myerson (1979), etc. In the next decade a benchmark model of auction was defined in (McAfee, McMillan, 1987). In that model all bidders are risk neutral, each bidder has a private valuation different from the others (different cumulative distribution functions and probability density functions), the bidders possess symmetric information, expected payments are functions of their bids. At the beginning of the decade (Laffont, Maskin, 1980), proposed differentiable approach to a dominant strategy mechanism, (Myerson, 1981) designed Bayesian-optimal mechanism that makes use of virtual valuations (virtual values are the derivative of the revenue curve). Riley and Samuelson (1981) provided a more general result on Revenue equivalence theorem. Other notable research from the 1980's on the auctions topic includes: Grossman and Sanford (1981), Milgrom and Weber (1982, a-b), Rassenti, Bulfin and Smith (1982), Engelbrecht-Wiggans, Milgrom and Weber (1983), Gabrielle, Gale and Sotomayor (1986), where the subject of multidimensional auctions has been investigated (Milgrom, 1986, Graham, Marshall, 1987, Holmstrom, Milgrom, 1987, McAfee, McMillan, 1987, Bullow, Roberts, 1989, Hansen, 1988, Klemperer, Meyer, 1989).

Auctions are type of games where players payoff depends on other's types of market participants e.g., Akerlof (1970), and this market models where participants have information that affects other player's payoffs are called adverse selection models. Although the treatment of adverse selection in auction theory has history since 1960's, yet the largest part of the auction theory, puts adverse selection aside to focus on the private values case, in which every type of participants' utility depends on its own type. The 1990's were an era of "putting auction theory at work" (Milgrom, 2004). Since 1994 auction theorist have designed a spectrum of sales for countries everywhere in the world. By 1996 auction theory became so influential that its founder William Vickrey was awarded a Nobel prize in economic sciences. More distinguished research on auctions in 1990's and 2000's (not all, the aim of this part is just the introduction to the essential literature on the subject) includes: Kerry, Zender, 1993, Levin, Smith, 1994, Avery, 1998, Ausubel, Schwartz, 1999, Dasgupta, Maskin, 2000, Faruk, Stacchetti, 2000, Maskin, Riley, 2000, etc.

Although is a theoretical field in the economics, auction theory is also being used as a tool to inform the design of the real-world auctions, and most notably auctions for the privatization of the public sector companies or the sale of licenses for use in the telecommunications or more general in the electromagnetic spectrum. Then we turn our analysis on the auction mechanisms namely VCG mechanism we are opposing to Green-Laffont and Myerson-Satterthwaite.

Research methodology

This paper makes a use of simulation on auctions, and use of mathematics in proofs of the theorems associated with the auction mechanism. Auctions that were simulated were First price auction (FPA), Second price auction (SPA), and asymmetric auctions. Mathematical proofs are included on the Revenue equivalence theorem, proof on the expected payoffs in FPA and SPA that expected payoff is higher for the weak bidder in the FPA auction. Proof on Green-Laffont theorem (Green, Laffont, 1977, 1979) is also included, followed by the proof on Myerson- Satterthwaite theorem.

Symmetric auctions: Introduction to the analysis of the common auction types

Here we take into consideration auctions with Independent private values (IPV). The number of bidders is $n, N \equiv \{1, \dots n\}$, and the set of possible bids is $[0, \infty)$. Player's bid is denoted as b_i where subscript i denotes ith bidder. The bidder's valuation is denoted as v_i . The bidder's distributions are denoted by F.

First price auction sealed bid

In the Sealed bid First Price auction the bidders submit their bids in envelopes. The highest bidder is the wined when the auction has ended, and he pays the price that he bids:

$$b_i = \Pi_i = \begin{cases} v_i - b_i, \text{if } b_i = \max\limits_{i \in N} b_i \\ 0, \text{if } b_i \neq \max\limits_{i \in N} b_i \end{cases} \quad (1)$$

Bidder i wins the auction whenever he submits the highest bid, but only when $\max\limits_{i \neq 1} \beta(v_i) = \beta(\max\limits_{i \neq 1} v_i) < b$, where β here represents the bidder's strategy. Since β is increasing then $\max\limits_{i=1} \beta(v_i) = \beta(Y_i)$, where Y_i is some random variable, and $Y_1 \equiv Y_1^{N-1}$, the highest of $N-1$ values. Bidder 1 wins whenever $\beta(Y_1) < b, \vee \equiv \forall Y_1 < \beta_1(b)$. His expected payoff is therefore:

$$\Pi_i = F\big(\beta^{-1}(b)\big) \times (v - b) \quad (2)$$

When maximizing this with respect to b yields:

$$\frac{f\big(\beta^{-1}(b)\big)}{\beta'(\beta^{-1}(b))}(v - b) - F\big(\beta^{-1}(b)\big) = 0 \quad (3)$$

where $f = F'$ is the density the random variable Y_1. When a symmetric equilibrium exists i.e., when $\beta(v) = b$ which yields following differential equation:

$$F(v)\beta'(v) + f(v)\beta(v) = vf(v) \equiv \frac{d}{dv}\big(F(v)\beta(v)\big) = vf(v) \quad (4)$$

And then since $\beta(0) = 0$ gives $\beta(v) = \frac{1}{F^{n-1}(v)} \int_0^v F^{n-1}(s)\, ds$. In the previous expression s is signal and $s = v$, or $s \neq v$, or $s = y$. Then:

$$\beta(v) = \frac{1}{F^{n-1}(v)} \int_0^v yf(y)\, dy = E[Y_1 | Y_1 < v] \quad (5)$$

If the valuation is not equal to signal i.e., if bidder cheats, then: $b(v_i) = v - \frac{1}{F^{n-1}(v)} \int_r^v F^{n-1}(s)\, ds$. Variance of the rice in the First Price auction that is equivalent to the Dutch type of auction is given as in (Vickrey, 1961):

$$\sigma_{pd}^2 = \int (p_d - \bar{p}_d)^2 dp(v) = \int_0^1 \left(\frac{n-1}{n} v - \frac{n-1}{n+1}\right)^2 nv^{n-1} dv = \frac{(n-1)(1-n)}{n^2} \quad (6)$$

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In the Dutch model of auction (Dutch type of auction. is a type of auction, that begins with highest bid that decreases until some auctioneer does accept the price, or accepts proposed reserve price), buyers gain is $v - p = v - \left[\frac{n-1}{n}\right] v = \frac{v}{n}$, where v is the highest drawn value, range of possible gains is from 0 to $\frac{1}{n}$.

Sealed bid second price auction

This auction is also known as Vickrey type auction (English auction) (English auctions are such type where it is begun with lowest (reserve prices) and it is going to higher price. English auctions are of open type). In Vickrey type of auction, bidders submit their bids sealed, without knowing other members in the auction bids, and in this type of auction highest bid wins, but the price paid is second highest bid. This result is Pareto optimal. This auction is efficient because the winner is the auctioneer for whom the lot has highest value. In the SPA each bidder submits its bid, and strategy for a bidder is a function:$\beta_i: [0, \omega] \to \mathbb{R}_{++}$,which determines his or her any bid.

$$\Pi_i = \begin{cases} v_i - \max\limits_{i\in N} b_i \ \text{ if } b_i > \max\limits_{j\neq i} b_j \\ \quad 0, \text{if } b_i < \max\limits_{j\neq i} b_j \end{cases} \tag{7}$$

In the SPA auction weakly, dominant strategy is to bid in accordance with $\beta(v) = v$. For the SPA auction variance is given as:

$$\sigma^2_{c_d} = \int_0^1 \left(v - \frac{n-1}{n+1}\right)^2 n(n-1)(v^{n-2} - v^{n-1})dv \tag{8}$$

Revenue equivalence theorem

Mechanisms with the same outcome in BNE (Bayesian Nash Equilibria) have the same expected revenue. This theory confirms that if there are n risk neutral agents, that do independent and personal evaluation of some auction good , and valuation follows cumulative distribution $F(v)$, which is ascending probability distribution of a continuous set of choices $(v, \bar{v})$ .Than every auction mechanism (every institution auction), in which lot will be allocated towards the agent for which it has highest value $\bar{v}$, and every agent with a valuation of good v has utility 0, generates exact same revenue, which lead every bidder to make the same payment. Revenue equivalence theorem $N \equiv \{1, \dots, n\}$, $CDF = F(\cdot)$,winning bid: $b_i = \max\limits_{i\in N} b_i \ b_i > \max\limits_{j\neq i} b_j$, $\omega_l \to \Pi_i = 0$, with all these settings different types of auction generate same expected revenues (profit) from any auction chosen.

Proof of the theorem provided in (Levine, 2014), from the Fundamental theorem of calculus:

$$\exists\, \Pi(v) = \Pi(0) + \int_0^v \Pi'(s)ds = \int_0^v s ds = \frac{1}{2}v^2 \tag{9}$$

Previous tells the expected profit of bidder with valuation v. The average profit of bidder is given as:

$$E_v[\Pi(v)] = \int_0^1 \Pi(v)dv = \int_0^1 \frac{1}{2}v^2 dv = \frac{1}{6} \tag{10}$$

An average bidder 1 can expect profit of 1/6 and a bidder 2 can expect profit of 1/6 and therefore the expected total bidder profit is 1/3. Or, $\sum_{i=1}^{n} \Pi(v) = 2 \cdot E_v[\Pi(v)] = \frac{1}{3}$. The value of surplus (the value that is created by transferring the object to the winner) is $\max\{v_1, v_2\}$ is given as $E(S) = E \max\{v_1, v_2\} = \frac{2}{3}$, this means that the total surplus is $S = 2/3$, and the bidders expect each 1/6 profit, which makes 1/3 left over. This must go to seller as revenue since: $\text{surplus} = \text{total revenue} + \text{bidders profit}$ so:

$$E\left[\sum_{i=1}^{n} \Pi(v)\right] = E[S] - E[\Pi(v)] \tag{11}$$

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First price auctions Nash equilibrium

A Bayes Nash equilibrium for an auction is a bid-function profile such that : $b = (b_1(\cdot), \dots, b_n(\cdot))$ such that for each bidder i and each possible value v_i for bidder i, the bid $b_i(v_i)$ maximizes bidder i's expected payoff given the vector $b_i = (b_1(\cdot), \dots, b_{i-1}(\cdot), b_i + 1(\cdot), \dots, b_n(\cdot))$ of bid functions for the other $n - 1$ bidders.

$$\beta(v) = x - \int_{\omega_l}^{\omega_h} \left(\frac{F(y)}{F(x)}\right)^{n-1} dy \tag{12}$$

where in previous expression $F(y) = 1 - F(v)$, $F(v)$ is a CDF function. And, x are signals drawn from private values so $x = v$. The maximal bid is given with the following expression: $\bar{b} = b(1) = 1 - \int_0^1 F^{n-1}(s)\,ds$. Probability to win is given as:

$$\Pr\left(v_i \leq \beta^{-1}(b)\right) = F\left(\beta^{-1}(b)\right) \wedge \Pr\left(v_{j \neq 1} \leq \beta^{-1}(b)\right)^{n-1} = F\left(\beta^{-1}(b)\right)^{n-1} \tag{13}$$

Earnings of the player are given as $(v_i - b)(F\left(\beta^{-1}(b)\right)^{n-1}$. Each player bid is calculated as first derivative with respect to b:

$$\frac{\partial}{\partial b}(v_i - b)(F\left(\beta^{-1}(b)\right)^{n-1} = v - \int_{\omega_l}^{\omega_h} \left(\frac{F(x)}{F(v)}\right)^{n-1} dx \tag{14}$$

In the case of uniform distribution, where CDF is $F(x) = \frac{x - \omega_L}{\omega_H - \omega_L}$, and PDF is given as: $f(x) = \frac{1}{\omega_H - \omega_L}$. Boundaries are [0,1], and $F(v) = v$,so by proposition:

$$\beta(v) = x - \int_{\omega_l}^{\omega_h} \left(\frac{F(x)}{F(v)}\right)^{n-1} dx = v - \int_0^{\omega_h} \left(\frac{x}{v}\right)^{n-1} dx = v - \frac{1}{n} \cdot v = \frac{(n-1)}{n} \cdot v \tag{15}$$

In the FPA distributions expected revenues are calculated as:

$$E\left(b(v_i)\right) = v_i * \frac{(b_i)^{n-1}}{\left(\frac{n-1}{n-1+a}\right)^{n-1}} \text{ or } EE\left(b(v_i)\right) = v_i * \frac{(b_i)^{n-1}}{\alpha^{n-1}} \tag{16}$$

where a is CRRA coefficient, and $\alpha = \left(\frac{n-1}{n-1+a}\right) = \frac{b_i}{v_i}$ this is because $b_i = \alpha * v_i$. The CRRA utility function is given as (Arrow, 1965):

$$u = \begin{cases} \frac{1}{1-\alpha} c^{1-\alpha} \text{ if } \alpha > 0\,, \alpha \neq 1 \\ \ln c \text{ if } \alpha = 1 \end{cases} \text{ when } \alpha = 1 \Rightarrow \lim_{n \to \infty} \frac{c^{1-\alpha} - 1}{1-\alpha} = \ln(c) \tag{17}$$

Elasticity of substitution is $\sigma = \frac{1}{\alpha}$, and $MRS = \frac{c_2}{c_1} = \left(\frac{p_1}{p_2}\right)^{\sigma}$,when $\alpha \in [0,1]$ than FPA-bid functions are in form:

$$b(v_i) = v - \frac{1}{F^{\frac{n-1}{1-a}}(v)} \int_r^v F^{\frac{n-1}{1-a}}(x)\,dx \text{ or when } r = 0 \Rightarrow b(v_i) = \frac{n-1}{n-1+a}(v_i) \tag{18}$$

In the previous expression r is a reserve price. Reserve price is a hidden minimum price that the seller is willing to accept for an item. In a reserve price auction, the seller is only obligated to sell the item once the bid amount meets or exceeds the reserve price. A seller can lower, but cannot raise, the reserve price. The general bid function when $r > 0$ and $a \neq 0$ is given as:

$$b(v_i) = v - \frac{1}{F^{\frac{n-1}{1-a}}(v)} * \frac{(a-1)}{(a-n)} \left[v^{\frac{a-n}{a-1}} - v^{\frac{a-n}{a-1}}\right] \tag{19}$$

If the coefficient of risk aversion is CARA (Constant absolute risk aversion), i.e., $u(c) = 1 - e^{-ac}$, where $a > 0$, then the bidding function is given as

$$b(v_i) = v + \frac{1}{a} \ln \left(1 - \frac{e^{-av}}{F^{n-1}(v)} \int_{e^{ar}}^{e^{av}} [F(a^{-1} \ln x]^{N-1} dx\right. \tag{20}$$

Or when the reserve price and CARA coefficient are set:

$$b(v_i) = v + \frac{1}{a} \ln \left(1 - \frac{e^{-av}}{v^{n-1}} * \frac{1}{2a} * (e^{2av} - e^{2ar}) * \log(v)^{n-1}\right) \tag{21}$$

Second price auctions Nash equilibrium

For the SPA auctions weakly, dominant strategy is to bid truthfully $\mathbf{b} = \mathbf{v_i}$. SPA with uncertain number of bidders will still have the same strategy and Nash equilibrium since it will not change the uncertainty of players attendance. The probability to win an auction is given as: $\mathbf{P}(\mathbf{v_i}) = \mathbf{v_i}$, the expected payoff if $\mathbf{b_i}$wins is given as: $\frac{\mathbf{v_i^2}}{\mathbf{2}}$. The expected profit in SPA auction is given as:

$$E\big(b(v_i)\big) = v_i\left(v_i - \frac{v_i}{2}\right) = \frac{v_i^2}{2} \tag{22}$$

Expected revenue in CDF of SPA by (Kunimoto, 2008), is given as:

$$\text{Revenue}_{(\text{SPA},r=0,a\in\mathbb{R}^+)} = F(y)^n + nF(y)^{n-1}\big(1 - F(y)\big) = nF(y)^{n-1} - (n-1)F(y)^n \tag{23}$$

Or in the reserve price auction SPA auction CDF of revenue is given as;

$$\text{Revenue}_{(\text{SPA},r\in\mathbb{R},a\in\mathbb{R}^+)} = r * r^{n-1} + nF(y)^{n-1} - (n-1)F(y)^n \tag{24}$$

Expected revenue from first price auctions and Second price auctions

Bidder with lowest bid has expected profit zero $E(b(v_l) = 0$,therefore:

$$E(b(v_i) = E(b(v_l) + \int_0^v b'(x)dx = 0 + \int_0^v xdx = \frac{v^2}{2} \tag{25}$$

The expected profit for each bidder is:

$$E\big(b_1(v)\big) = E\big(b_2(v)\big) = \int_0^1 b(v)dv = \int_0^1 \frac{v^2}{2}dv = \frac{1}{6} \tag{26}$$

Total bidder profit is $\frac{1}{3}$, and expected revenue is equal to expected surplus minus expected total bidder profit $E(R) = \frac{2}{3} - \frac{1}{3} = \frac{1}{3}$.

First price auctions with uncertain number of bidders

Let $p_1(1)$ denotes the probability that player 1 believe that he will be only one present at the auction (the only participant). Since this is symmetric auction with only two bidders we can write this as follows: $p_1(1) = p, p_1(2) = 1 - p$. CDF of the distributed values will be:

$$F(v) = p(v - b) + (1 - p)F\big(\beta^{-1}(b)\big)(v - b) \tag{27}$$

One can maximize by taking derivative with respect to b:

$$\frac{dF(v)}{d(b)} = -p - (1-p)F\big(\beta^{-1}(b)\big) + (1-p)(1-v)f\big(\beta^{-1}(b)\big) \cdot \frac{1}{\beta'(\beta^{-1}(b))} = 0 \tag{28}$$

Since $v = \beta^{-1}(b)$:

$$\frac{dF(v)}{d(b)} = -p - (1-p)F(v) + \cdot \frac{(1-p)\big(v - \beta(v)f(v)\big)}{\beta'} = 0 \tag{29}$$

Or:

$$\frac{dF(v)}{d(b)} = \big((p(1-p)F)\beta\big)' = (1-p)vf(v) \tag{30}$$

The equilibrium bid function is given as:

$$\beta = \frac{(1-p)\int_o^v xf(x)dx}{p(1-p)F(v)} \tag{31}$$

Uncertain number of bidder Nash-bid equilibrium is:

$$b_{i,FPA} = \frac{(1-p)v^2}{2(p(1-p)\cdot v\,)} = \frac{0.5v^2}{2(1+v)} \tag{32}$$

Symmetric equilibrium “Envelope theorem” Approach

Often convenient approach to identify the necessary conditions for symmetric equilibrium is to exploit the envelope theorem (Levin, 2014). Let’s suppose that $b(s)$ is a symmetric equilibrium in increasing differentiable strategies, then bidders i equilibrium payoff signal s_i is given as:

$$[\pi(s_i)] = \big(s_i - b(s_i)\big)F^{n-1}(s_i) \tag{33}$$

Best response in equilibrium because i is playing is given as:

$$[\pi(s_i)] = \max_{b_i}(s_i - b_i)F^{n-1}\left(b_i^{-1}(b_i)\right) \tag{34}$$

Applying the envelope theorem one can obtain following:

$$\frac{d}{ds}\pi(s_i)_{s=s_i} = F^{n-1}\left(b^{-1}b(s_i)\right) = F^{n-1}(s_i) \tag{35}$$

And also, from previous expression we obtain that:

$$[\pi(s_i)] = \pi(\underline{s}) + \int_{\underline{s}}^{\overline{s}} F^{n-1}(\tilde{s})d\,\tilde{s} \tag{36}$$

And here just to make a remark that: as b_s is increasing, a bidder with signal $\underline{s}$ will never win the auction and henceforth $\pi(\underline{s}) = 0$. If we combine previous two expressions we will get the equilibrium bidding strategy (again also by dropping subscript (i) :

$$b(s) = s - \frac{\int_{\underline{s}}^{\overline{s}} F^{n-1}(\tilde{s})d\,\tilde{s}}{F^{n-1}(s)} \tag{37}$$

Again, we have showed necessary conditions for an equilibrium i.e., any increasing differentiable symmetric equilibrium must involve the strategy $b(s)$. To check sufficiency (that $b(s)$ actually is an equilibrium), we can exploit the fact that $b(s)$ is increasing (Since the CDF is an increasing function, with a fact that F^(n-1) ($\tilde{s}$) is probability that bidder's true value is s or less), and satisfies the envelope formula to show that it must be a selection from i's best response given the other bidder's use the strategy b(s). Now as a proposition here we may put that both the First order conditions (FOC's). Now the revenue from the First price auction (FPA) it is the expected winning bid of the bidder with the highest signal $\mathbb{E}[s^{1:n}]$, now $F^{n-1}(s)$ is the probability that if one takes $n-1$ draws from F, all will be below s i.e.,:

$$b(s) = s - \frac{\int_{\underline{s}}^{\overline{s}} F^{n-1}(\tilde{s})d\,\tilde{s}}{F^{n-1}(s)} = \frac{1}{F^{n-1}(s)}\int_{\underline{s}}^{\overline{s}} \tilde{s}dF^{n-1}(\tilde{s}) = \mathbb{E}[s^{1:n-1}|s^{1:n-1} \le s] \tag{38}$$

That solves if a bidder has a signal s, he sets his bid to equal the highest expectation of the highest of other $n-1$ values, conditional on the notion that all those values are being less than his own, so now the expected revenue is:

$$\mathbb{E}[b(s^{1:n})] = \mathbb{E}[s^{1:n-1}|s^{1:n-1} \le s^{1:n}] = \mathbb{E}[s^{2:n}] \tag{39}$$

Asymmetric auctions

Basic setup

There exist set: $\Theta = \{1,2,\dots,N\}$,of types of bidders. And $\forall\theta \in \{1,2,\dots,N\}$ and $\exists n(\theta) \ge 1$, which are bidders of type θ . Bidders of type θ draw an IPV for the object from CDF $F: [\omega_H, \omega_L] \to R$ .It is assumed that $F \in C^2\left((\omega_H, \omega_L)\right)$ and $f \equiv F' > 0$, on ω_H. The inverse of equilibrium bidding strategy (Maskin, Riley, 2000, Fibich, Gavish, 2011)) is given as:

$$v'_i(b) = \frac{F_i\left(\beta^{-1}(b)\right)}{f_i(\beta^{-1}(b))} = \left[\left(\frac{1}{n-1}\sum_{j=1}^{n}\frac{1}{v_j(b)-b}\right) - \frac{1}{v_i(b)-b}\right], i = 1,\dots,n \tag{40}$$

Inverse bid functions are solutions that gives profit maximization problem:

$$\frac{\partial U_i(b;v_i)}{\partial b} = (v_i - b)\sum_{j=1,j\ne1}^{n}\left(\prod_{k=1,k\ne1}^{n}F_k(v_k(b))\right) f_j\left(v_j(b)\right)v_j'(b) - \prod_{j=1,j\ne1}^{n}F_j\left(v_j(b)\right) = 0 \tag{41}$$

Maximization problem here is given as in:

$$\max_b U_i(b;v_i) = (v_i\text{-}b)\prod_{j=1,j\ne1}^{n}F_j\left(v_j(b)\right), i=1,\dots n \tag{42}$$

where one solution is:

$$\sum_{j=1,j\ne1}^{n}\frac{f_j\left(v_j(b)\right)v_j'(b)}{F_j(v_j(b))} - \frac{1}{v_i(b)-b}, i = 1,\dots n \tag{43}$$

Or bidder chooses to maximize his expected surplus $S = \pi_i$ as in (McAfee, McMillan, 1987):

$$\pi_i = (v_i - b_i)F(v)^{n-1} \quad \partial\pi_i/\partial b_i = 0, \frac{dy}{dx} = \frac{\partial\pi_i}{\partial v_i} = F\left(\beta^{-1}(b_1)\right)^{n-1} \tag{44}$$

Bidders expected revenue in FPA asymmetric auction is given as:

$$E_i\,(p,b_i,v_i) = k_i \int_r^{b(\omega_h)} \left[F_i^{-1}(\ell_i(v)) - v\right] \cdot \frac{\ell_i'(v)}{\ell_i(v)} \prod_{j=1}^{n} \left[\ell_j(v)\right]^{k_j} dv \quad (45)$$

where in previous expression $k := \frac{(2-\lambda+\mu)}{1-\lambda}$, and bidder maximizes:

$$\beta\left(\beta^{-1}(b_1)\right) = \arg \max_{u\in(0,\omega_h)} (v-u)\cdot\left[F_i(\lambda_i(u))\right]^{k_i-1} \prod_{j\neq 1}\left[F_j\left(\lambda_j(u)\right)\right]^{k_j} \quad (46)$$

$\exists u = \sum_{i=1}^{n} u_i$, where u_i denotes the player of type i. Where in previous expressions $\ell_i(v) = F_i(\lambda_i(v))$, and probabilities of winning the reserve price auction are given as:

$$p_i(r) = k_i \int_r^{\omega_h} \frac{\ell_i'(v)}{\ell_i(v)} \prod_{j=1}^{n} \left[\ell_j(v)\right]^{k_j} dv \quad (47)$$

Auctioneer expected revenue is given with the following expression:

$$E(p,b_i,v_i) = \omega_h - r\prod_{j=1}^{n}\left[F_j(r)\right]^{k_j} - \int_r^{b(\omega_h)} \frac{\ell_i'(v)}{\ell_i(v)} \prod_{j=1}^{n} \left[\ell_j(v)\right]^{k_j} dv \quad (48)$$

Here $U(p_i, E_i, r) = p_i \cdot (r - E_i)$, by the envelope theorem optimal values are denoted by asterisk $U^{*\prime}(r) = p^*(r)$ and one can integrate to obtain the previous result.

$$U^*(x) = \int_0^r p^*(v)dv \quad (49)$$

Following is sort of prove of RET, that in a way expected revenue depends on the optimal auction price and that revenue does not depend on the auction mechanism.

Equilibrium for strong and weak bidders

Now, let b_S be an equilibrium bid of a strong bidder and b_w is an equilibrium bid of a weak bidder. We have following problem to maximize:

$$\max_b F_w\left(b_w^{-1}(b)\right)(v_S - b) \quad (50)$$

FOC for previous expression is given as:

$$\frac{f_w\left(b_w^{-1}(b)\right)}{F_w\left(b_w^{-1}(b)\right)} \cdot (b_w^{-1})'(b) - \frac{1}{b_S^{-1}(b)-b} \quad (51)$$

In the previous expression $b_S^{-1}(b) = v_S$ or the weak bidder valuation. Now, the weak bidder's problem is given as in the following expression:

$$\max_b F_S\left(b_S^{-1}(b)\right)(v_w - b) \quad (52)$$

Weak bidder's FOC is given as:

$$\frac{f_S\left(b_S^{-1}(b)\right)}{F_S\left(b_S^{-1}(b)\right)} \cdot (b_S^{-1})'(b) - \frac{1}{b_w^{-1}(b)-b} \quad (53)$$

In the previous expression if we set that $v_w = b_w^{-1}(b)$ and that the last expression equals zero than we get:

$$\frac{f_S\left(b_S^{-1}(b)\right)}{F_S\left(b_S^{-1}(b)\right)} \cdot (b_S^{-1})'(b) = \frac{1}{v_w-b} \quad (54)$$

Theorem: Suppose that $F_S(v) \leq F_w(v)$, meaning that F_S conditionally first-order stochastically dominates F_w. Then when one compares FPA and SPA, both uniformly distributed following applies: $\forall b_S^{-1}(b) = v_S, \because E\left(b_{FPA}(v)\right) < E\left(b_{SPA}(v)\right)$ for $b_S^{-1}(b) = v_S$ and $\forall b_w^{-1}(b) \neq v_w, \because E\left(b_{FPA}(v)\right) > E\left(b_{SPA}(v)\right)$ for $b_w^{-1}(b) \neq v_w$.

Proof: For purposes of the proof $b_S(v), b_w(v)$ have the same range so a matching function is defined as: $m(v) \equiv b_w^{-1}\left(b_S(v)\right)$ or as a weak bidder that bids equal to strong bidder in FPA. Since from previous we know that $b_S(v) < b_w(v)$ in FPA, now we know that $m(v) = v$. The strong bidder expected payoff is given as:

$$E[\pi(v_i)] = \Pr(b_w(v_w) < b)(v - b) \quad (55)$$

If we take derivative with respect to v we get: $E_v[\pi(v_i)] = \Pr(b_w(v_w) < b)$ and by replacing $b = b_S(v)$, which gives us the following identity:

$$E_v[\pi(v_i)] = \Pr\left(b_w(v_w) < b_S(v)\right) = P_r\left(v_w < m(v)\right) = F_w\left(m(v)\right) \quad (56)$$

Because $\Pr(v < a) = F(a)$ when distribution of values is uniform. By the envelope theorem value function for FPA is given as:

$$V_S^{FPA}(v) = \int_{\omega_l}^{\omega_h} F_w\big(m(w)\big)dv \quad (57)$$

And for the SPA, where bidder's bid their true valuation (there is no bid shading):

$$V_S^{SPA}(v) = \int_{\omega_l}^{\omega_h} F_w(v)dv \quad (58)$$

Since $m(v) < v$ and that F_w is strictly increasing, the strong bidder prefers SPA. For the weak bidders expected payoff for the FPA is given as:

$$V_w^{FPA}(v) = \int_{\omega_l}^{\omega_h} F_S\left(\frac{v}{m}\right)ds \quad (59)$$

And for the SPA we have got:

$$V_S^{SPA}(v) = \int_{\omega_l}^{\omega_h} F_s(v)dv \quad (60)$$

Since $m^{-1}(v) > v$, expected payoff is higher for the weak bidder in the FPA.

Revenue Equivalence Theorem in Asymmetric Auctions

Here we set proposition that with asymmetric bidders, the expected revenue in FPA may exceed that of SPA (English auction). Now Let's suppose that weak bidder and strong bidder follow uniform distribution and that interval for the weak bidder values is: $b_w \sim \left[0, \frac{1}{1+x}\right]$, and that strong bidder valuation is distributed as $b_s \sim \left[0, \frac{1}{1-x}\right]$. So that the strong bidder has wider interval than the weak bidder. If $x = 0$ both $b_w, b_S \sim \mathrm{Uni}[0,1]$ and that inverse equilibrium bid function in the First price auction is given as $b^{-1}(b) = 2b$. A buyer with valuation $2b$ has a probability to win: $\Pr(\mathrm{win}|v_i = 2b) = 2b$ and therefore, expected payment of: $\Pr(\mathrm{win}|v_i = 2b) = 2b(v_i - b_i) = 2b(2b - b) = 2b^2$. When x becomes positive, in English auction, the weak buyer with valuation b wins with probability $2b(1 - x)$, and the expected payment is $2b^2(1 - x)$. In a high-bid auction, bidders do not use $b^{-1}(b) = 2b$, if they did the strong bidder would outbid the weak one by $\left[\frac{1}{2(1-x)}, \frac{1}{2(1+x)}\right]$, and he can reduce his bid n win with probability 1.

For an equilibrium the strong bidder must reduce his bid as a function of his valuation. Then this reduction will make weak bidder to bid more aggressively than $b^{-1}(2b) = 2b$, since the strong buyers bids are distributed more densely than before.

In equilibrium the weak and the strong bid functions are given as:

$$b_w^{-1}(b) = \frac{1}{1+(2b)^2} \text{ and } b_s^{-1}(b) = \frac{1}{1-(2b)^2} \quad (61)$$

The CDF for the winning bid in FPA is:

$$F_{FPA}(b) = F_s(b_s^{-1}(b)F_w\big(b_w^{-1}(b)\big) = (1 - x)b_s^{-1}(b)(1 + x)b_w^{-1}(b) = \frac{(1-x^2)(2b)^2}{1=x^2(2b)^4} \quad (62)$$

For the SPA or English auction (EA) the second valuation is less than b if it's not the case that both valuations are higher. Then we have:

$$F_{SPA}(b) = 1 - \big(1 - F_s(b)\big)\big(1 - F_w(b)\big) = (1 - x)b - (1 - x^2)b^2 = 2b - (1 - x)b^2 \quad (63)$$

The CDF in an open auction is ever increasing in x. If $x = 0$ both auctions yield revenue. When $x > 0$ the expected revenue is strictly greater for the first price auction than for the English auction.

Examples of asymmetric auctions: Different length of interval values in uniform distribution

In the following case we compare same auction type (uniform distribution) with different length of interval values. In this case there are two players $p_1 \sim \mathrm{Uni}(0,1)$ and $p_2 \sim \mathrm{Uni}(0,1)$ (figure 1).

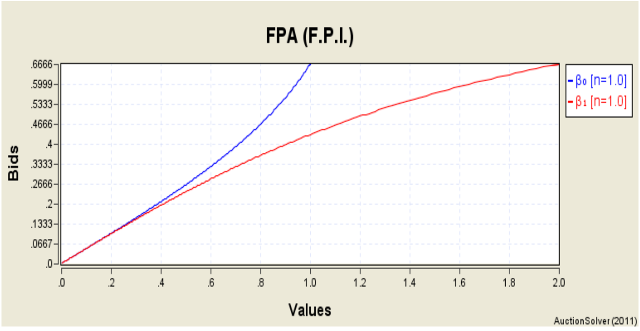

Figure 1 Nash equilibrium for $\mathbf{p_1}$ and $\mathbf{p_2}$ for any given value at $[\mathbf{0,2}]$
Source: Authors own calculations.

So, in the previous example BNE or Bayesian Nash equilibrium is $\bar{b} = 0.666596687674 \approx \frac{2}{3}$ for both p_1 and p_2 if p_1 value is 1 and p_2 value is 2.

Example 1: Same length of interval of values, but different distributions: Normal and Uniform distributions

In this case we are comparing normal and uniform distribution both spanned in the interval $(0,1)$ where CDF of a normal distribution function is given as: $F(x) = \frac{c}{\sqrt{2\pi}} e^{-\frac{(x-0.11)}{0.67}}$ and for the uniform distribution CDF of a function is given as: $F(x) = \frac{x-\omega_l}{\omega_h-\omega_l}$ (figure 2).

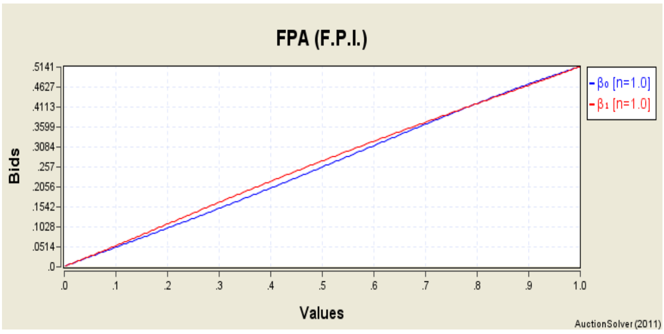

Figure 2 Nash equilibrium for $\mathbf{p_1} \sim \mathbf{Normal}\ (\mathbf{0,1})$ and $\mathbf{p_2} \sim \mathbf{Uni}(\mathbf{0,1})$ for any given value at $[\mathbf{0,1}]$
Source: Authors own calculations.

Example 2: Same length of interval of values, but different distributions: Exponential and Uniform distributions

CDF of the bidder that follows exponential distribution is given as: $F(x) = \frac{1-\exp\left(-\lambda(x-\omega_L\right)}{1-\exp\left(-\lambda(\omega_H-\omega_L\right)}$ (figure 3).

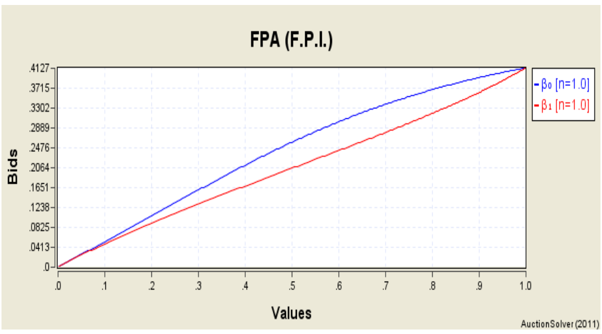

Figure 3 Nash equilibrium for $\mathbf{p_1} \sim \mathbf{exponential}\ (\mathbf{0,1})$ and $\mathbf{p_2} \sim \mathbf{Uni}(\mathbf{0,1})$ for any given value at $[\mathbf{0,1}]$
Source: Authors own calculations.

BNE or Bayesian Nash equilibrium is $\bar{b} = 0.41274184946056 14 \approx \frac{2}{5}$, for both p_1 and p_2.

Example 3: Three bidders' case: Exponential, Normal and Uniform

BNE is $\bar{b} = 0.617585303682194 \approx \frac{3}{5}$, for both p_1 and p_2 and p_3 (figure 4).

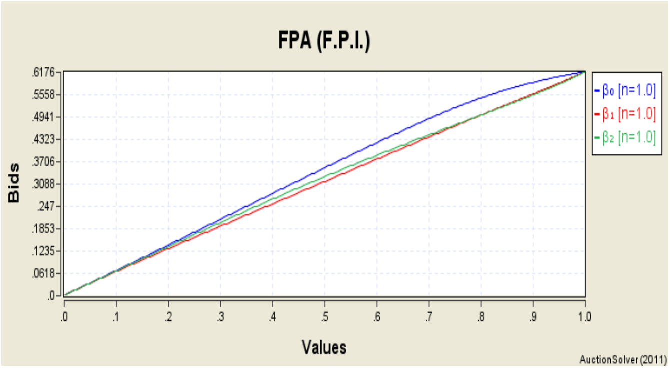

Figure 4 Nash equilibrium for $\mathbf{p_1} \sim \mathbf{exponential}\ (\mathbf{0,1})$ and $\mathbf{p_2} \sim \mathbf{Normal(0,1)}$ and $\mathbf{Uni} \sim (\mathbf{0,1})$ for any given value at $[\mathbf{0,1}]$

Source: Authors own calculations.

Example 4: Two players uniform distributed over $(0,1)$, and one player uniformly distributed over $(0,2)$

On figure 5 two bidders follow $p_1 = p_2 \sim \text{Uni}\ (0{,}1)$, and $p_3 \sim \text{Uni}\ (0{,}2)$, at any given value $(0{,}2)$ In this auction BNE equilibrium is equal to: $\bar{b} = 0.8 \sim \frac{4}{5}$.

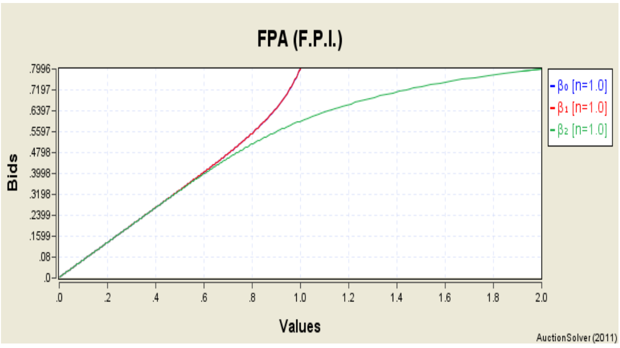

Figure 5 Nash equilibrium for $\mathbf{p_1} \sim \mathbf{Uni}\ (\mathbf{0,1})$ and $\mathbf{p_2} \sim \mathbf{Uni(0,1)}$ and $\mathbf{p_3} \sim \mathbf{Uni(0,2)}$ for any given value at $[\mathbf{0,2}]$

Source: Authors own calculations.

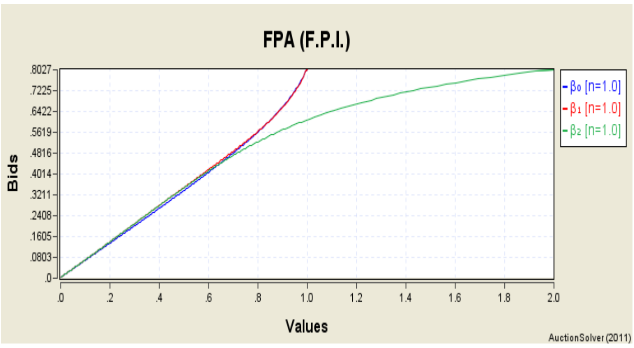

Figure 6 Nash equilibrium for $\mathbf{p_1} \sim \mathbf{Uni}\ (\mathbf{0,1})$ and $\mathbf{p_2} \sim \mathbf{Uni(0,2)}$ and $\mathbf{p_3} \sim \mathbf{Normal(0,1)}$ for any given value at $[\mathbf{0,2}]$

Source: Authors own calculations.

On figure 6 two bidders follow $p_1 = p_2 \sim \mathrm{Uni}\ (0,1)$, and $p_3 \sim \mathrm{Normal}\ (0,1)$, at any given value $(0,2)$ In this auction BNE equilibrium is equal to: $\bar{b} = 0.8$.

Mechanism design - VCG auction mechanism vs Green-Laffont and Myerson-Satterthwaite theorem

Vickrey-Clarke-Groves, auction is named after (Vickrey, 1961, Clarke, 1971, Groves, 1973) for their papers that generalized the idea. VCG mechanism is a direct quasi-linear mechanism:

$$(\chi, \tilde{v}) = \arg\max_x \sum_i \bar{v}_i(x)$$

$$p_i(\hat{v}) = \textstyle\sum_{j\neq i} \hat{v}_j\left(\chi(\hat{v}_{-i})\right) - \sum_{j\neq i} \hat{v}_j\left(\chi(\hat{v})\right) \quad (64)$$

where $\bar{v} = (\bar{v}_{-i}, \hat{v}_i)$ denotes some best bidder strategy. Here bidder wants to maximize

$$\max_{\hat{v}_i}\left(v_i(\chi(\hat{v})) - p(\hat{v})\right) \text{ or } \max_{\hat{v}_i}\ \left(v_i(\chi(\hat{v})) + \textstyle\sum_{j\neq i} \hat{v}_j(\chi(\hat{v}))\right) \quad (65)$$

If $x \in X$ than we have:

$$\max_x\ \left(v_i(x) + \textstyle\sum_{j\neq i} \hat{v}_j(x)\right) \quad (66)$$

And under the Groves mechanism we have:

$$\chi(\hat{v}) = \arg\max_x(\textstyle\sum_i \hat{v}_i(x)) = \arg\max_x\left(\hat{v}_i(x) + \sum_{j\neq i} \hat{v}_j(x)\right) \quad (67)$$

In the Groves mechanism price constraint is given as following:

$$p_i(\hat{v}) = h_i(\hat{v}_{i-1}) - \textstyle\sum_{j\neq i} \hat{v}_j\left(\chi(\hat{v})\right) \quad (68)$$

Theorem (Green, Laffont, 1977, 1979): An efficient social choice function $C: \mathbb{R}^{X_n} \to X \times \mathbb{R}^n$ can be implemented in dominant strategies for agents with quasilinear utilities only if $p_i(v) = h_i(v_{-1}) - \sum_{j\neq i} v\left(\chi(v)\right)$ where $h_i(\cdot)$ is an arbitrary function of v_{-1}, $v: X \to \mathbb{R}$, a social choice function is :$f(\cdot) = \left(X^* F(\cdot)\right)$ with an efficient valuation is implemented in dominant strategies only if $F_i(\cdot)$ is given by Groves payments. Let's suppose that $\forall i, \exists\{v_i(\cdot, \theta_i) | \theta_i \in \Theta_i\} = Ɉ$, where the last is the set of all valuation functions.

Proof: $F_i(\theta_i, \theta_{i-1}) = \sum_{j\neq i} v_j(x^*(\theta_i, \theta_{i-1}), \theta_j) + h_i(\theta_i, \theta_{i-1}), \exists \theta_i, \hat{\theta}_i, \theta_{-1} \ni h_i(\theta_i, \theta_{-i}) \neq h_i(\hat{\theta}_i, \theta_{-i})$, now we consider the following case: $x^*(\theta_i, \theta_{i-1}) = x^*(\hat{\theta}_i, \theta_{i-1})$, by the IC dominant strategies follows:

$$v_i(x^*(\theta_i, \theta_{i-1}), \theta_i) + F_i(\theta_i, \theta_{-i}) \geq v_i(x^*(\hat{\theta}_i, \theta_{i-1}), \theta_i) + F\_i(\hat{\theta}_i, \theta_{-i})$$

$$v_i(x^*(\theta_i, \theta_{i-1}), \theta_i) + F\_i(\theta_i, \theta_{-i}) \geq v_i(x^*(\hat{\theta}_i, \theta_{i-1}), \theta_i) + F\_i(\hat{\theta}_i, \theta_{-i}) \quad (69)$$

Previous implies that $F_i(\theta_i, \theta_{i-1}) = F_i(\hat{\theta}_i, \theta_{i-1})$, and therefore $h_i(\theta_i, \theta_{-i}) = h_i(\hat{\theta}_i, \theta_{-i})$, that is a contradiction. In the previous theorem statement social choice function has a following meaning: The social choice function f is a function: $f: \Theta_1 \times \ldots\ldots \times \Theta_i \to Y$ that for each profile of agents 'types assign a collective choice $f(\theta_1, \ldots, \theta_i) \in Y$ and a mechanism is $\Gamma = (\theta_1, \ldots, \theta_i)$,$g(\cdot)$ is a collection of sets $(\theta_1, \ldots, \theta_i)$ and an outcome of the function $f: \theta_1 \times \ldots\ldots \times \theta_i \to Y$. Myerson and Satterthwaite (1983) theorem states that there is no efficient way for two parties to trade when they each do not know other party probabilistic varying valuation for it without the forcing of one party to trade at loss. The Myerson-Satterthwaite theorem is a negativity in economics and impossibility that states that no mechanism can be Bayes-Nash incentive compatible (a direct mechanism is Bayesian-Incentive compatible if honest reporting forms a Bayesian-Nash equilibrium). Myerson and Satterthwaite (1983) theorem belong to the class of negativity results in economics (Gibbard, 1973, Satterthwaite, 1975, Arrow, 1970, Green, Laffont, 1977). Theorem states that in many bilateral trade situations with asymmetric information, ex post efficiency is inconsistent with incentive compatibility and individual rationality.

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Theorem Myerson-Satterwhaite (notation): $\exists s \sim F_s(\underline{s}, \overline{s}) > 0; \exists b \sim F_b(\underline{b}, \overline{b})$, where F_s and F_b are common knowledge. In the DRG (direct revelation game) traders s-seller and b-bidder report their values and the outcome is selected, an outcome specifies probability of trade p ,and the terms of trade x (payoffs). A DRG is a pair of outcome functions where (p, x) :$p(s, b)$ is a probability of trade and $x(s, b)$ are thus the expected payments from buyer to seller. Utilities are given as:

$$\begin{aligned} u(s, b) &= x(s, b) - s(p, b) \\ v(b, s) &= bp(s, b) - x(s, b) \end{aligned} \tag{70}$$

Payoffs are defined as:

$$X(s) = \int_{\underline{b}}^{\overline{b}} x(s, b) f_b(b) db \,; X(b) = \int_{\underline{s}}^{\overline{s}} x(s, b) f_s(s) ds \tag{71}$$

Probabilities of trade are given as:

$$P(s) = \int_{\underline{b}}^{\overline{b}} p(s, b) f_b(b) db \,; P(b) = \int_{\underline{s}}^{\overline{s}} p(s, b) f_s(s) ds \tag{72}$$

Interim utilities are given as:

$$U(s) = X(s) - P(s); V(b) = bP(b) - X(b) \tag{73}$$

Incentive compatible mechanism (IC) $(p, x)$is given as:

$$IC: U(s) \geq X(s') - P(s'); V(b) \geq P(b) - X(b) \tag{74}$$

Incentive rational mechanism (IR) is:

$$\forall s \in [\underline{s}, \overline{s}] \vee \forall b \in [\underline{b}, \overline{b}], U(s) \geq 0 \,; V(b) \geq 0 \tag{75}$$

Lemma IC: The mechanism is IC if and only if $P(s)$ is increasing and $P(b)$ decreasing:

$$\begin{cases} U(s) = U(\underline{s}) + \int_{\underline{s}}^{\overline{s}} P(s)(\theta) d\theta \\ V(b) = V(\underline{b}) + \int_{\underline{b}}^{\overline{b}} P\,(b)(\theta) d\theta \end{cases} \tag{76}$$

Lemma 1 proof: From previous definition we know that $U(s') \geq X(s') - s'P(s'); U(s) \geq X(s) - sP(s)$

$$\begin{cases} U(s) \geq X(s') - sP(s') = U(s') + (s' - s)P(s'), \\ U(s') \geq X(s) - s'P(s) = U(s) + (s - s')P(s) \end{cases} \tag{77}$$

If we subtract these inequalities it will yield:

$$(s' - s)P(s) \geq U(s) - U(s') \geq (s' - s)P(s') \tag{78}$$

Now if we take that $s' > s$ implies that $P(s)$ is decreasing, if we divide by $(s' - s)$ and letting $s' \to s$ yields $\frac{dU(s)}{ds} = -P(s)$ and integrating produces IC(s'). The same is true for the buyer. To prove the IC for the seller it is suffice to show that following applies:

$$s[P(s) - P(s')] + [X(s') - X(s)] \leq 0 \; \forall \, s, s' \in [\underline{s}, \overline{s}] \tag{79}$$

From previous by substituting for $X(s)$ and $X(s')$ and by using IC(s') the following will yield:

$$X(s) = sP(s) + U(\overline{s}) + \int_{\underline{s}}^{\overline{s}} P(\theta) d\theta \tag{80}$$

And following to hold:

$$\begin{gathered} 0 \geq s[P(s)P(s)] + s\,P(s) + \int_{s'}^{\overline{s}} P(\theta) d\theta - sP(s) \\ - \int_{s}^{\overline{s}} P(\theta) d\theta = (s' - s)P(s') + \int_{s'}^{s} P(\theta) d\theta = \int_{s'}^{s} [P(\theta) - P(s')] d\theta \end{gathered} \tag{81}$$

Previous holds only because $P(\cdot)$ is decreasing.

Lemma IR: IC mechanism is individually rational IR if and only if:

$$U(\overline{s}) \geq 0 \vee V(\underline{b}) \geq 0 \tag{82}$$

Corollary:

$$U(\overline{s}) + V(\underline{b}) = \int_{\underline{b}}^{\overline{b}} \int_{\underline{s}}^{\overline{s}} \left[b - \frac{1-F(b)}{F(b)} - s - \frac{1-F(s)}{F(s)} \right] p(s, b) f(s) f(b) ds db \geq 0 \tag{83}$$

Proof: From the IC we know that following holds:

$$X(s) = sP(s) + U(\overline{s}) + \int_{\underline{s}}^{\overline{s}} P(\theta)d\theta \tag{84}$$

Now from the corollary:

$$\int_{\underline{b}}^{\overline{b}}\int_{\underline{s}}^{\overline{s}} x(s,b)f(s)f(b)dsdb = U(\overline{s}) + \int_{\underline{b}}^{\overline{b}}\int_{\underline{s}}^{\overline{s}} sp(s,b)f(s)f(b)dsdb + \int_{\underline{b}}^{\overline{b}}\int_{\underline{s}}^{\overline{s}} p(s,b)F(s)f(b)dsdb \tag{85}$$

The third term in the right side follows since:

$$\int_{\underline{b}}^{\overline{b}}\int_{\underline{s}}^{\overline{s}} p(\theta,b)F(s)f(b)d\theta db = \int_{\underline{b}}^{\overline{b}}\int_{\underline{s}}^{\theta} p(\theta,b)F(s)f(b)d\theta db = \int_{\underline{s}}^{\overline{s}} p(s,b)F(s)f(b)dsdb \tag{86}$$

Analogously for the buyer follows that:

$$\int_{\underline{b}}^{\overline{b}}\int_{\underline{s}}^{\overline{s}} x(s,b)f(s)f(b)dsdb = -V(\underline{b}) + \int_{\underline{b}}^{\overline{b}}\int_{\underline{s}}^{\overline{s}} bp(s,b)F(s)f(b)dsdb - \int_{\underline{b}}^{\overline{b}}\int_{\underline{s}}^{\overline{s}} p(s,b)F(s)\big(1-F(b)\big)dsdb \tag{87}$$

Now if we equate the both right hand sides proof is completed:

$$V(\underline{b}) = \int_{\underline{b}}^{\overline{b}}\int_{\underline{s}}^{\overline{s}} p(s,b)F(s)\big(1-F(b)\big)dsdb - \int_{\underline{b}}^{\overline{b}}\int_{\underline{s}}^{\overline{s}} bp(s,b)F(s)f(b)dsdb = \int_{\underline{s}}^{\overline{s}} p(s,b)F(s)f(b)dsdb \tag{88}$$

IR mechanism is proved since $V(\underline{b}) \geq 0$.

Theorem Myerson-Satterthwaite (continued): It is not common knowledge that if trade gains exist i.e., the supports of the CDF functions of traders have non-empty intersections) then no IC and IR trading mechanism can be ex-post efficient.

Proof: A trading mechanism is ex-post efficient if and only if trade occurs whenever $s \leq b$

$$p(s,b) = \begin{cases} 1 \text{ if } s \leq b \\ 0 \text{ if } s > b \end{cases} \tag{89}$$

In the previous expression $p(s,b)$ is a probability of trade which takes value 1 if trade occurs and zero if it doesn't. To prove that ex-post efficiency cannot be attained, it is enough to show that inequality $(*)$ in the corollary hence:

$$\int_{\underline{b}}^{\overline{b}}\int_{\underline{s}}^{\min(b,\overline{s})} \left[b - \frac{1-F(b)}{f(b)} - s - \frac{F(s)}{f(s)}\right] f(s)f(b)dsdb \tag{90}$$

Previous expression equals to:

$$\begin{aligned} &\int_{\underline{b}}^{\overline{b}}\int_{\underline{s}}^{\min(b,\overline{s})} [bf(b) + F(b) - 1]f(s)dsdb - \int_{\underline{b}}^{\overline{b}}\int_{\underline{s}}^{\min(b,\overline{s})} [sf(s) + F(s)]f(b)dsd = \\ &\quad -\int_{\underline{b}}^{\overline{s}} [1 - F(\theta)]F(\theta)d\theta < 0, \underline{b} < \overline{s} \end{aligned} \tag{91}$$

Previous result is proof of Myerson-Satterthwaite theorem about trade inefficiency. Some weaker efficiency criterion is Pareto optimality, one may use that criterion if ex-post efficiency does not work.

Conclusion

Vickrey-Clarke-Groves mechanism provides insight into what mechanism design can achieve. VCG mechanism provides every participant with a dominant strategy and that is to bid truthfully. Incentive compatible direct mechanisms are the center of the attention here because for any Bayesian equilibrium in any bargaining game, there is an equivalent incentive compatible direct mechanism that always yields the same outcomes. This result is called revelation principle. It is difficult to solve for Bayesian equilibria because that includes solving for agents' best response strategies.

On the other hand, Myerson and Satterthwaite (1983), show that there is no efficient way for two parties to trade when each party does not know other party, when the two trade parties have secret and varying valuations for the subject, without forcing one party too trade at loss. This theorem is negativity in economics a negative mirror of the fundamental welfare theorem. In asymmetric auction bidders with asymmetrically distributed private values, second price auctions allocate the item efficiently, whereas the first price auction does not. The revenue is approximately the same, and is independent of bidding rules, as long as at equilibrium wins bidder with highest reservation price and the bidder with lowest reservation price has zero surplus i.e. $v_i - b = p - c = 0$. More striking is that for i.i.d., regular single-item environments the second price auction with reserve price is revenue optimal. Mechanisms differ from auctions in a way that: they are not universal (they can differ from time to time because both the allocation and payment rules depend on a comparison of virtual valuations, that in turn depend on buyers' value distributions), nor they are anonymous (in mechanisms buyers identities matter). Variations in the rules in auctions can affect the participation. Mechanisms are a set of rules created essentially to govern the interactions of the parties.

Also, one important result that was tested and present in this paper (in the Asymmetric auctions section) is the equilibrium of the asymmetric First price auction, this result was most famously reported in (Maskin, Riley 2000). Basic result here is that one's bidder value function stochastically dominates other players' value function. Proposition from the comparisons of mechanism designs is that there in a balanced trade problem there is no mechanism that is efficient, incentive compatible, individually rational, and at a same time balances budget.

Auctions can be approached from different angles from game theory perspective, auctions are Bayesian games of incomplete information, and in mechanism design theory, auctions are allocation mechanisms, furthermore applications of auctions include: procurement, public finance etc. all of which should be evident in the game theoretic and mechanism design approach that we utilized in this paper.

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