FOR A CORRELATION BETWEEN A CLASS OF SECOND ORDER LINEAR DIFFERENTIAL EQUATIONS AND A CLASS OF SYSTEMS OF FIRST ORDER DIFFERENTIAL EQUATIONS

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Abstract. In this paper, a class of second order linear differential equations and a class of systems of first order differential equations are considered. By a method of transformation, the results for a correlation between them are obtained.

1. INTRODUCTION

In this paper, we consider the second order linear differential equations

$$(t-a)(t-b)x''+(\alpha_1 t+\alpha_2)x'+\beta_1 x=0, a\neq b \tag{A}$$

and the system of first order differential equations

$$\begin{aligned}(t-a)x_1'+Ax_1+Bx_2&=0\\(t-b)x_2'+Cx_1+Dx_2&=0\end{aligned} \tag{B}$$

By replacing $x_1=\frac{1}{C}x_3, x_1'=\frac{1}{C}x_3', C\neq 0$ in the system (2), the system

$$\begin{aligned}(t-a)x_3'+Ax_3+BCx_2&=0\\(t-b)x_2'+x_3+Dx_2&=0\end{aligned} \tag{C}$$

is obtained.

On the 7th Macedonian symposium of the differential equations, Professor Boro Piperevski gave the following dilemma “Can a second order linear differential equation (A) transform in one or in more systems of first order differential equations (B)?” Sure, the second order linear differential equation (A) can transform in systems of differential equations from the type which is different from the systems (B). In mathematical literature until this moment does not have some result which is an answer to this question. In this paper, this dilemma will be solved. The consequence of this result is a sequence of differential equations of type (A) whose integrality depends on only one of them. An interesting case is when the class systems of differential equations from first-order of type (B) are considered as a linear matrix differential equation from first-order. This case is presented in more papers as [6,7,8,9,10,11,12]. Finally, a new dilemma for interpretation of the term reductability is presented.

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Considering a second order homogeneous linear differential equation of type $P_2(x)y''+P_1(x)y'+\lambda_n P_0(x)y=0$ where $P_i(x),(i=0,1,2)$ are polynomials and λ_n is a parameter. According to Brenke [4], this equation will have polynomial solutions of degree n for each $n\in\mathbb{N}$ with an appropriate value of the parameter λ_n, if $P_2(x), P_1(x)$, and $P_0(x)$ are polynomials of the second, first and zero degrees respectively. Also, the general formula for the series of polynomial solutions for the equation as well as certain conditions for their orthogonality with appropriate weight is shown.

Note that in the case when all members of the sequence $(\lambda_n), n=0,1,2,...$, are different, then λ_n are called their own values, and the polynomials y_n own functions.

Special cases of such known orthogonal polynomials are the polynomials of Legendre, Jacobi, Tschebyscheff, Hermite, Laguerre and others which are used in numerical mathematics.

Let us mention the classic results regarding polynomial solutions of the very important hypergeometric differential equation, as an equation with polynomial coefficients. Its solutions are special functions, especially the Jacobi, Legendre, Tschebyscheff polynomials, which belong to the class of classical orthogonal polynomials for which there are corresponding formulas, based on Rodrigues' famous formula.

In fact, this formula was obtained by Rodrigues O. in 1814 for a polynomial solution of a special differential equation of Legendre, but there are the other classical polynomials that are expressed in a similar way.

This special class of differential equations (1) is obtained when Laplace's partial differential equation is transformed into spherical coordinates and afterward, it is required its solution to be a product of functions that depend on only one variable.

In the theory of partial equations, the classical results for the solution of the internal problem of Dirihle for the contour problem for the Laplace's partial differential equation in the sphere are known. By transforming in spheres' coordinates and by using the Fourier method to the separation of the variables, the differential equations are obtained. Their solution is the classes of orthogonal

Legendre polynomials which are the own functions for the appropriate Sturm-Lowville task. Therefore, the solutions of the Laplace's partial differential equations in a type of homogeneous polynomials from the same degree are obtained. Spherical harmonic functions are called.

Remark 1.1. The term reductability of linear homogeneous differential equations has two interpretations. Reduction in a wider interpretation is a reduction of an equation of a system of linear homogeneous differential equations of lower order and that reduction can be more significant, i.e. it can be reduced to multiple classes of linear homogeneous differential equations systems from a lower order.

Definition 1.1. (Frobenius): A linear homogeneous differential equation whose coefficients are unambiguous functions is called more predictable according to Frobenius if there is no common solution with a linear homogeneous differential equation with coefficients unambiguous lower order functions. Otherwise, it is called reductive according to Frobenius. for example [1,2,3,4,5].

2. MAINS RESULTS

Let the system is given

$$\begin{aligned}(t-a)x_1' + Ax_1 + Bx_2 &= 0\\ (t-b)x_2' + x_1 + Dx_2 &= 0\end{aligned} \tag{1}$$

The equations which correspond to this system are

$$(t-a)(t-b)x_1'' + [(A+D+1)t-(A+1)b-Da]x_1' + (AD-B)x_1 = 0 \tag{2}$$

$$(t-a)(t-b)x_2'' + [(A+D+1)t-(D+1)a-Ab]x_2' + (AD-B)x_2 = 0 \tag{3}$$

The equation (2) is equivalent to equation

$$(t-a)(t-b)x_1'' + (\alpha_1 t + \alpha_2)x_1' + \beta_1 x_1 = 0,\ a \neq b \tag{4}$$

if the relations

$$a=a,\ b=b,\ A+D+1=\alpha_1,\ -(A+1)b-Da=\alpha_2,\ AD-B=\beta_1 \tag{5}$$

are satisfied. The equation (3) is equivalent to equation

$$(t-a)(t-b)x_2'' + (\alpha_1^* t + \alpha_2^*)x_2' + \beta_1^* x_2 = 0,\ a \neq b \tag{6}$$

if the relations

$$\begin{aligned}&a=a,\ b=b,\ A+D+1=\alpha_1^*,\\ &-(D+1)a-Ab=\alpha_2^*,\ AD-B=\beta_1^*\end{aligned} \tag{7}$$

are satisfied. The equation (3) or the equation (6) can be written in the form

$$(t-a)(t-b)x_2'' + (\alpha_1 t + b - a + \alpha_2)x_2' + \beta_1 x_2 = 0,\ a \neq b \tag{6*}$$

if an equation (4) is given.

Therefore, a system of type (1) corresponds to two differential equations (2) and (3) of type (4) i.e. (6). Therefore, if the coefficients of the system (1) with the relations (5) and (7) are known, then the equations (4) and (6) can be found, with the help of the connections (5) and (7).

Let, us ask the opposite question: how many systems of type (1) (if exist) suit the equation (4)?

Let the coefficients of equation (4) are known. From the relations (5), the equations

$$a=a,\ b=b,\ A=\frac{b-a+a\alpha_1+\alpha_2}{a-b},\ D=\frac{-\alpha_2-b\alpha_1}{a-b},\ B=AD-\beta_1 \tag{8}$$

and the system

$$(t-a)x_1' + Ax_1 + Bx_2 = 0$$
$$(t-b)x_2' + x_1 + Dx_2 = 0 \quad (9)$$

are obtained i.e. the system (1).

From the relations (7), the equations

$$a = a, b = b, D^* = \frac{b-a-b\alpha_1-\alpha_2}{a-b}, A^* = \frac{\alpha_2 + a\alpha_1}{a-b}, B^* = A^*D^* - \beta_1 \quad (10)$$

and the system

$$(t-a)y_1' + A^* y_1 + B^* y_2 = 0$$
$$(t-b)y_2' + y_1 + D^* y_2 = 0 \quad (11)$$

are obtained.

Let us see which two equations correspond to this new system (11).

By using (1), (2) and (3), the equations

$$(t-a)(t-b)y_1'' + [(A^* + D^* + 1)t - (A^* + 1)b - D^* a]y_1' + (A^*D^* - B^*)y_1 = 0 \quad (12)$$

$$(t-a)(t-b)y_2'' + [(A^* + D^* + 1)t - (D^* + 1)a - A^* b]y_2' + (A^*D^* - B^*)y_2 = 0 \quad (13)$$

are obtained or according to (10), the equations

$$(t-a)(t-b)y_1'' + (\alpha_1 t + a - b + \alpha_2)y_1' + \beta_1 y_1 = 0, a \neq b \quad (14)$$

$$(t-a)(t-b)y_2'' + (\alpha_1 t + \alpha_2)y_2' + \beta_1 y_2 = 0, a \neq b \quad (15)$$

are obtained.

So, the equation (4) corresponds to two systems: the system (9), which is the same as the system (1), and the system (11).

The equation (4) is the same as the equation (15) for $y_2 = x_1$. Equation (14) appears as a new equation.

By using (10) and (8) the system (11) can be write as

$$(t-a)y_1' + (A+1)y_1 + (-A + D - 1 + B)y_2 = 0$$
$$(t-b)y_2' + y_1 + (D-1)y_2 = 0 \quad (11^*)$$

Let, the same procedure for the system (11*) is repeated i.e. for the new equation (12), i.e. (14), putting $A^* = A+1, B^* = -A + D - 1 + B, D^* = D - 1$. By the same procedure is obtained the following system

$$(t-a)z_1' + (A^* + 1)z_1 + (-A^* + D^* - 1 + B^*)z_2 = 0$$
$$(t-b)z_2' + z_1 + (D^* - 1)z_2 = 0 \quad (16)$$

The following equations correspond to the system (16),

$$(t-a)(t-b)z_1'' + [(A^* + D^* + 1)t - (A^* + 2)b - (D^* - 1)a]z_1' + (A^*D^* - B^*)z_1 = 0 \quad (17)$$

$$(t-a)(t-b)z_2'' + [(A^* + D^* + 1)t - (A^* + 1)b - D^* a]z_2' + (A^*D^* - B^*)z_2 = 0 \quad (18)$$

Obviously $y_1 = z_2$, but a new equation is an equation (17) for the function z_1.

In accordance with the corresponding shifts, the equations (17) and (18) are

$$(t-a)(t-b)z_1'' + [(A+D+1)t-(A+3)b-(D-2)a]z_1' + (AD-B)z_1 = 0 \quad (17^*)$$

$$(t-a)(t-b)z_2'' + [(A+D+1)t-(A+2)b-(D-1)a]z_2' + (AD-B)z_2 = 0 \quad (18^*)$$

i.e.

$$(t-a)(t-b)z_1'' + (\alpha_1 t + 2a - 2b + \alpha_2)z_1' + \beta_1 z_1 = 0, a \neq b \quad (17^{**})$$

$$(t-a)(t-b)z_2'' + (\alpha_1 t + a - b + \alpha_2)z_2' + \beta_1 z_2 = 0, a \neq b \quad (18^{**})$$

The equations (17) and (18) correspond to the system

$$\begin{aligned} &(t-a)z_1' + (A+2)z_1 + (2D-2A+B-4)z_2 = 0 \\ &(t-b)z_2' + z_1 + (D-2)z_2 = 0 \end{aligned} \quad (16^*)$$

Theorem 1. Let equation (4) be given. Then the equation (4) is reductive to only two systems of first order differential equations of type (1).

Proof. Let the differential equation (4) is given. By the method of transformations of a unique way from the formulas (5) and (7), the two systems (1) and (11) i.e. (11*) are obtained.

It is interesting that the free member of the polynomial coefficient before the first derivative of the equations (4), (6^*), (14), (17**) changes with changing of the systems and that

$$-(a-b)+\alpha_2, \quad \alpha_2, \quad a-b+\alpha_2, \quad 2(a-b)+\alpha_2, \dots \quad n(a-b)+\alpha_2, n \in \mathbb{Z}$$

After a series of transformations, we can obtain an equation of type

$$(t-a)(t-b)x'' + [\alpha_1 t + n(a-b) + \alpha_2]x' + \beta_1 x = 0, a \neq b \quad (19)$$

Theorem 2. Let the differential equation (4) is given. Then there is a sequence of differential equations of type (19).

Proof. Let a differential equation is given that is reductive on the systems (1) and (11*). The system (1) corresponds to the differential equation of type (6*). By the method of transformations from equation (4) with the formulas (7), we obtained the system (11*) to which corresponds the equation (14). By continuing the method of mathematical induction, the sequence of differential equations of type (19) is obtained.

We summarize: Let the equation

$$(t-a)(t-b)x_1'' + (\alpha_1 t + \alpha_2)x_1' + \beta_1 x_1 = 0, a \neq b$$

is given.

Let x_{10} be one of its particular solutions. From the system (9)

$$Bx_{20} = -[(t-a)x_{10}' + Ax_{10}]$$

is obtained, where x_{20} is a particular solution to the equation

$$(t-a)(t-b)x_2'' + (\alpha_1 t + b - a + \alpha_2)x_2' + \beta_1 x_2 = 0, a \neq b$$

and

$$a = a, b = b, A = \frac{b-a+a\alpha_1+\alpha_2}{a-b}, D = \frac{-\alpha_2 - b\alpha_1}{a-b}, B = AD - \beta_1$$

From the system (11*)

$$y_{10} = -[(t-b)x_{10}' + (D-1)x_{10}]$$

is obtained, where y_{10} is a particular solution to the equation

$$(t-a)(t-b)y_1'' + (\alpha_1 t + a - b + \alpha_2)y_1' + \beta_1 y_1 = 0,\ a \neq b$$

From the system (16*)

$$z_{10} = -[(t-b)y_{10}' + (D-2)y_{10}]$$

is obtained, where z_{10} is a particular solution to the equation

$$(t-a)(t-b)z_1'' + (\alpha_1 t + 2(a-b) + \alpha_2)z_1' + \beta_1 z_1 = 0,\ a \neq b$$

By continuing the procedure, the following result is obtained.

A particular solution to an equation of type

$$(t-a)(t-b)x'' + (\alpha_1 t + n(a-b) + \alpha_2)x' + \beta_1 x = 0,\ a \neq b$$

will be obtained by the formula

$$x_0 = -[(t-b)x_{n-1,0}' + (D-n)x_{n-1,0}]$$

where $x_{n-1,0}$ is a particular solution to the equation

$$(t-a)(t-b)x_{n-1}'' + (\alpha_1 t + (n-1)(a-b) + \alpha_2)x_{n-1}' + \beta_1 x_{n-1} = 0, a \neq b, n \in \mathbb{Z}.$$

Lemma. Let the sequence of differential equations of type (19) is given and let a particular solution $x_{k-1,0}$ of the equation

$$(t-a)(t-b)x_{k-1}'' + (\alpha_1 t + (k-1)(a-b) + \alpha_2)x_{k-1}' + \beta_1 x_{k-1} = 0, a \neq b, k \in \mathbb{Z}$$

is known. Then by formula

$$x_{k,0} = -[(t-b)x_{k-1,0}' + (D-k)x_{k-1,0}],\ k \in \mathbb{Z}$$

the particular solution of the equation

$$(t-a)(t-b)x'' + (\alpha_1 t + k(a-b) + \alpha_2)x' + \beta_1 x = 0,\ a \neq b, k \in \mathbb{Z}$$

is given.

Proof. By the principle of mathematical induction, the result is obtained.

3. EXAMPLE

The result obtained for correlation between a class of second order linear differential equations and a class of systems of first order differential equations, we will shows via example.

Let the second order differential equation

$$(t-1)(t-2)x'' + (-3t+1)x' + 3x = 0 \tag{1'}$$

and the system of first order differential equations

$$\begin{aligned} (t-1)x_1' + x_1 + 2x_2 &= 0 \\ (t-2)x_2' - 4x_1 - 5x_2 &= 0 \end{aligned} \tag{2'}$$

i.e.

$$\begin{aligned} (t-1)x_1' + x_1 - 8x_2 &= 0 \\ (t-2)x_2' + x_1 - 5x_2 &= 0 \end{aligned} \tag{2''}$$

are given. By corresponding transformations of (2'), the equations are obtained

$$(t-1)(t-2)x_1'' + (-3t+1)x_1' + 3x_1 = 0 \quad (3')$$

$$(t-1)(t-2)x_2'' + (-3t+2)x_2' + 3x_2 = 0 \quad (4')$$

where $x_1 = x$. The solution is the functions $x_1 = 12t - 4, x_2 = 3t - 2.$ The new system from (3') is

$$\begin{aligned}(t-1)y_1' + 2y_1 - 15y_2 &= 0\\ (t-2)y_2' + y_1 - 6y_2 &= 0\end{aligned} \quad (5')$$

The equations which suit the system (5') are

$$(t-1)(t-2)y_1'' - 3t\,y_1' + 3y_1 = 0 \quad (6')$$

$$(t-1)(t-2)y_2'' + (-3t+1)\,y_2' + 3y_2 = 0 \quad (7')$$

where $y_2 = x_1$. The solution is the functions

$$y_2 = 12t - 4, y_1 = 60t \text{ or } y_2 = 3t - 1, y_1 = 15t.$$

By the same procedure, a new system from the system (5') is obtained

$$\begin{aligned}(t-1)z_1' + 3z_1 - 24z_2 &= 0\\ (t-2)z_2' + z_1 - 7z_2 &= 0\end{aligned} \quad (8')$$

The equations which suit the system (8') are

$$(t-1)(t-2)z_1'' - (3t+1)z_1' + 3z_1 = 0$$

$$(t-1)(t-2)z_2'' - 3t\,z_2' + 3z_2 = 0$$

where $z_2 = y_1$. The solution is the functions $z_1 = 360t + 120, z_2 = 60t$ or $z_1 = 6t + 2, z_2 = t$.

4. CONCLUSIONS

We concluded that exists a correlation between the class of second order linear differential equations (A) and the class of systems of first order differential equations (C). That means that the second order linear differential equation (4) is reductive with two systems of first order differential equations of type (1). But also for the second order linear differential equation (4) exists a sequence of differential equations of type (19).

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