A NEW DEMONSTRATION OF GARFUNKEL-BANKOFF INEQUALITY

by Marian Dincă and Martin Lukarevski

The following inequality

$$tg^2\frac{A}{2}+tg^2\frac{B}{2}+tg^2\frac{C}{2}\geq 2-8\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}\qquad(1)$$

proposed by J.Garfunkel and solution by L.Bankoff in journal Crux Mathematicorum shows that it is equivalent to O.Kooi's inequality:

$$s^2\leq\frac{R(4R+r)^2}{2(2R-r)}\qquad(2)$$

In this article I give a new demonstration

let: $\frac{A}{2}=\frac{\pi}{2}-\alpha,\frac{B}{2}=\frac{\pi}{2}-\beta,\frac{C}{2}=\frac{\pi}{2}-\gamma, where\ \alpha,\beta,\gamma\ the\ angles\ an\ acute\ triangle$

inequality (1) will be equivalent to:

$$ctg^2\alpha+ctg^2\beta+ctg^2\gamma\geq 2-8\cos\alpha\cos\beta\cos\gamma$$

$$a^2=b^2+c^2-2bc\cos\alpha=b^2+c^2-4F\cdot ctg\alpha, where\ a,b,c,F\ the\ sides\ and\ area$$

$$rezult: ctg\alpha=\frac{b^2+c^2-a^2}{4F}, similarly: ctg\beta=\frac{c^2+a^2-b^2}{4F}, ctg\gamma=\frac{b^2+a^2-c^2}{4F}$$

$$16F^2=\sum_{ciclic}a^2\left(b^2+c^2-a^2\right)=\sum_{ciclic}\left(b^2+c^2-a^2\right)\left(c^2+a^2-b^2\right)$$

$$let: b^2+c^2-a^2=x_1\geq 0, c^2+a^2-b^2=y_1\geq 0, b^2+a^2-c^2=z_1\geq 0$$

$$rezult: 16F^2=x_1y_1+y_1z_1+z_1x_1$$

$$8\cos\alpha\cos\beta\cos\gamma=\frac{\left(b^2+c^2-a^2\right)\left(c^2+a^2-b^2\right)\left(b^2+a^2-c^2\right)}{a^2b^2c^2}=\frac{8x_1y_1z_1}{(x_1+y_1)(y_1+z_1)(z_1+x_1)}$$

*we shall obtain*:

$$\frac{x_1^2+y_1^2+z_1^2}{x_1y_1+y_1z_1+z_1x_1}+\frac{8x_1y_1z_1}{(x_1+y_1)(y_1+z_1)(z_1+x_1)}\geq 2, or$$

$$\frac{(x_1+y_1+z_1)^2-2(x_1y_1+y_1z_1+z_1x_1)}{x_1y_1+y_1z_1+z_1x_1}+\frac{8x_1y_1z_1}{(x_1+y_1+z_1)(x_1y_1+y_1z_1+z_1x_1)-x_1y_1z_1}\geq 2$$

$let: x_1 + y_1 + z_1 = p,\ x_1y_1 + y_1z_1 + z_1x_1 = q,\ x_1y_1z_1 = r, rezult:$

$\frac{p^2}{q} - 2 + \frac{8r}{pq - r} \geq 2, equivalent\ to:$

$p^2(pq - r) + 8qr \geq 4(pq - r)q,$

$r(12q - p^2) + p^3q - 4pq^2 \geq 0$

$f(r) = r(12q - p^2) + p^3q - 4pq^2,$

is a first degree equation

and represents the equations of a straight line in the plane

we consider p and q parameters

$because: pq \geq 9r, we\ obtain: r \leq \frac{pq}{9}\ and\ r \geq 0$

it is necessary and sufficient: if and only if : $f\left(\frac{pq}{9}\right) \geq 0\ and\ f(0) \geq 0$

$$f\left(\frac{pq}{9}\right) = \frac{pq}{9}(12q - p^2) + p^3q - 4pq^2 = \frac{12pq^2 - p^3q + 9p^3q - 36pq^2}{9} =$$

$$= \frac{8p^3q - 24pq^2}{9} = \frac{8pq}{9}(p^2 - 3q) \geq 0, because: p^2 - 3q \geq 0, well-known\ inequality$$

$if\ r = 0, rezult\ x_1y_1z_1 = 0,$ let $z_1 = 0, rezult; p = x_1 + y_1, q = x_1y_1$

$and\ f(0) = p^3q - 4pq^2 = pq(p^2 - 4q) \geq 0,$

$because: p^2 - 4q = (x_1 + y_1)^2 - 4x_1y_1 = (x_1 - y_1)^2 \geq 0$

$equality\ for\ p^2 - 3q = 0, rezult:$

$$(x_1 + y_1 + z_1)^2 - 3(x_1y_1 + y_1z_1 + z_1x_1) = 0 = \frac{1}{2}\left[(x_1 - y_1)^2 + (y_1 - z_1)^2 + (z_1 - x_1)^2\right]$$

$rezult: x_1 = y_1 = z_1\ imply; a = b = c, A = B = C$

$and: p^2 - 4q = (x_1 + y_1)^2 - 4x_1y_1 = (x_1 - y_1)^2 = 0$

$rezult: x_1 = y_1\ and\ z_1 = 0, imply: a = b\ and\ b^2 + a^2 - c^2 = 0$

the triangle is rectangular and isosceles

$C = \frac{\pi}{2}\ and\ A = B = \frac{\pi}{4},$

REFERENCE

[1] Problem 825 (proposed by J. Garfunkel, solution by L. Bankoff), Crux Math. 9 (1983), 79 and 10 (1984), 168.

[2] Martin Lukarevski and Dan Stefan Marinescu:A REFINEMENT OF THE KOOI S INEQUALITY, MITTENPUNKT AND APPLICATIONS,Journal Mathematical Inequalities,Volume 13,Number 3(2019),827 –

[3] O. KOOI, Inequalities for the triangle, Simon Stevin, 32 (1958), 97 –101