into two sets A and B so that the sum of the mth powers of elements of the two sets are equal for $0 \le m \le k$. However, Prouhet had stated this more generally for j sets. Within $\{0, \ldots, j^{k+1} - 1\}$, let A_i consist of those numbers whose entries in their j-ary expansions have sum congruent to i modulo j. The result is that the sum of the mth powers of the numbers in A_i is independent of i, for each m with $0 \le m \le k$. In addition to a proof in the cited article, other proofs appear in D. H. Lehmer (1947), The Tarry–Escott problem, *Scripta Math.* 13: 37–41, and E. M. Wright (1949), Equal sums of like powers, *Proc. Edinburgh Math. Soc.* (2)8: 138–142.

The Tarry–Escott Problem appeared again as recently as Problem 10284 [1993, 185; 1995, 843] in this MONTHLY. The problem has a substantial literature, including a short book: A. Gloden (1944), *Mehrgradige Gleichungen*, 2nd ed., Groningen: P. Noordhoff.

For the problem of determining the set $\mathcal{P}_k$ of integers for which such splittings occur, the case $k = 3$ considered here was solved in D. W. Boyd (1997), On a problem of Byrnes concerning polynomials with restricted coefficients, *Math. Comp.* 66: 1697–1703. The set $\mathcal{P}_k$ is now known for k up to 7; see J. Buhler, S. Golan, R. Pratt, and S. Wagon (2019), Symmetric Littlewood polynomials, spectral-null codes, and equipowerful partitions, arxiv.org/abs/1912.03491.

Also solved by K. David & A. van Groningen, S. M. Gagola Jr., K. Gatesman, Y. J. Ionin, M. E. Kidwell & M. D. Meyerson, P. Lalonde (Canada), O. P. Lossers (Netherlands), R. Molinari, M. Reid, N. C. Singer, R. Tauraso (Italy), M. Wildon, GCHQ Problem Solving Group (UK), Missouri State University Problem Solving Group, and the proposers.

Maximizing the Area of an Incenter Triangle

12086 [2019, 82]. *Proposed by Miguel Ochoa Sanchez, Lima, Peru, and Leonard Giugiuc, Drobeta Turnu Severin, Romania.* Let ABC be a triangle with right angle at A, and let H be the foot of the altitude from A. Let M, N, and P be the incenters of triangles ABH, ABC, and ACH, respectively. Prove that the ratio of the area of triangle MNP to the area of triangle ABC is at most $(\sqrt{2} - 1)^3/2$, and determine when equality holds.

Solution by Dmitry Fleischman, Santa Monica, CA. Let the sides of the triangle be denoted a, b, and c, as usual, and let the inradii of $\triangle ABC$, $\triangle ACH$, and $\triangle ABH$ be denoted r_A, r_B, and r_C, respectively. As is well known, the altitude on the hypotenuse of a right triangle divides the triangle into two smaller triangles that are similar to it. All corresponding sides, as well as any other corresponding linear measurements such as altitudes and inradii, are in the proportion $a : b : c$, which are the hypotenuse lengths of the three triangles. In particular, the inradii r_A, r_B, and r_C are in the proportion $a : b : c$.

Let $K(XYZ)$ denote the area of $\triangle XYZ$. We determine $K(MNP)/K(ABC)$ by computing the two ratios $K(MNP)/K(BNC)$ and $K(BNC)/K(ABC)$.

The angle bisector at B contains both M and N, and likewise the angle bisector at C contains both P and N. Hence B, M, and N are collinear, as are C, P, and N. Let the projections of M, N, and P onto BC be denoted M', N', and P', respectively. Since $\triangle MNP$ and $\triangle BNC$ share the angle at N, the ratio of their areas is the product of NM/NB and NP/NC. By similar triangles,

$$\frac{NM}{NB} = 1 - \frac{MB}{NB} = 1 - \frac{MM'}{NN'} = 1 - \frac{r_C}{r_A} = 1 - \frac{c}{a},$$

and a similar calculation shows that $NP/NC = 1 - b/a$. Thus

$$\frac{K(MNP)}{K(BNC)} = \left(1 - \frac{b}{a}\right)\left(1 - \frac{c}{a}\right) = 1 - \frac{b+c}{a} + \frac{bc}{a^2}.$$

Since $K(BNC) = ar_A/2$ and $K(ABC) = (a+b+c)r_A/2$,

$$\frac{K(BNC)}{K(ABC)} = \frac{a}{a+b+c}.$$

We conclude

$$\frac{K(MNP)}{K(ABC)} = \left(1 - \frac{b+c}{a} + \frac{bc}{a^2}\right)\frac{a}{a+b+c} = \frac{1-(b+c)/a + bc/a^2}{1+(b+c)/a}.$$

Let $t = (b+c)/a$, so that $(t^2-1)/2 = bc/a^2$. We express $K(MNP)/K(ABC)$ as $(t-1)^2/(2t+2)$, which we denote by $f(t)$. Since

$$t = \sin C + \cos C = \sqrt{2}\cos(C - \pi/4)$$

and $0 < C < \pi/2$, we have $1 < t \le \sqrt{2}$. Since

$$f'(t) = \frac{(t-1)(t+3)}{2(t+1)^2},$$

we see that f is increasing on $[1, \sqrt{2}]$. Hence f achieves its maximum on $[1, \sqrt{2}]$ at $t = \sqrt{2}$, and that maximum value is

$$f\left(\sqrt{2}\right) = \frac{(\sqrt{2}-1)^2}{2(\sqrt{2}+1)} = \frac{(\sqrt{2}-1)^3}{2},$$

which was to be shown. Note that $t = \sqrt{2}$ when $\cos(C - \pi/4) = 1$, or when $C = \pi/4$, i.e., when the original triangle is isosceles.

Also solved by S. Amghibech (Canada), M. Bataille (France), H. Chen, P. P. Dályay (Hungary), P. De (India), R. Downes, A. Fanchini (Italy), G. Fera (Italy), K. Gatesman, O. Geupel (Germany), M. Goldenberg & M. Kaplan, W. Janous (Austria), B. Karaivanov (USA) & T. S. Vassilev (Canada), K. T. L. Koo (China), O. Kouba (Syria), J. H. Lindsey II, O. P. Lossers (Netherlands), M. Lukarevski (Macedonia), D. Ş. Marinescu & M. Monea (Romania), C. Mindrila, R. Nandan, C. Pranesachar (India), A. Stadler (Switzerland), R. Stong, K. Sullivan, R. Tauraso (Italy), M. Vowe (Switzerland), T. Wiandt, L. Zhou, GCHQ Problem Solving Group (UK), and the proposer.

Nontrivial Solutions To a Matrix Equation

12087 [2019, 82]. *Proposed by M. L. J. Hautus, Heeze, Netherlands.* Let K be a field, and let A be a linear map from K^n into itself. The equation $X^2 = AX$ has the trivial solutions $X = 0$ and $X = A$. Show that it has a nontrivial solution if and only if the characteristic polynomial $\det(\lambda I - A)$ is reducible, with the following sole exception: If K has two elements, $n = 2$, and A is nilpotent and nonzero, then the characteristic polynomial is reducible, yet $X^2 = AX$ has no nontrivial solutions.

Solution by Koopa Tak Lun Koo, Chinese STEAM Academy, Hong Kong, China. The problem is trivial if $n = 1$ or $A = 0$. Suppose $n \ge 2$ and $A \ne 0$.

Suppose that $X^2 = AX$ has a solution X outside $\{0, A\}$, and consider the characteristic polynomial $\det(\lambda I - A)$. Let $V = \{Xv : v \in K^n\}$. Note that $V \ne \{0\}$. If $V = K^n$, then X is surjective and hence invertible, so $X = A$. Thus V is a proper subspace of K^n. Since $A(Xv) = (AX)v = X^2 v = X(Xv) \in V$ for all $v \in K^n$, we see that A maps V into V. Choose a basis of V and extend it to a basis of K^n. With respect to this basis, A has a matrix representation of the form $\begin{bmatrix} B & C \\ 0 & D \end{bmatrix}$, where both B and D are square with order less than n. Thus $\det(\lambda I - A) = \det(\lambda I - B)\det(\lambda I - D)$, so $\det(\lambda I - A)$ is reducible.

For the converse, suppose $\det(\lambda I - A) = p(\lambda)q(\lambda)$, where $\deg(p) = m$ with $1 \le m < n$. By the Cayley–Hamilton theorem, $p(A)q(A) = q(A)p(A) = 0$.