103.H (Peter Shiu)

Find positive integers a, b, c and A, B, C such that

$$\int_0^1 \frac{a(x(1-x))^4 + b(x(1-x))^{12}}{c(x^2+1)}\,dx = \frac{355}{113} - \pi$$

and

$$\int_0^1 \frac{A(x(1-x))^{12} + B(x(1-x))^{20}}{C(x^2+1)}\,dx = \frac{104348}{33215} - \pi.$$

Answer: The smallest triples of integer values are

$$a = 4127,\ b = 1\,225\,785,\ c = 19\,616\,687$$

and

$$A = 206\,953\,445,\ B = 15\,866\,451,\ C = 7\,373\,066\,576.$$

This problem attracted a large number of correct, tightly argued solutions. Several respondents commented that the integrals in **103.H** are extensions of D. P. Dalzell's famous integral from [1]

$$\int_0^1 \frac{(x(1-x))^4}{x^2+1}\,dx = \frac{22}{7} - \pi$$

to the third and fifth convergents $\frac{355}{113}, \frac{104348}{33215}$ in the continued fraction expansion for π.

Most solutions followed the following pattern.

Let $I_n = \int_0^1 \frac{(x(1-x))^n}{x^2+1}\,dx$ and evaluate $I_4 = \frac{22}{7} - \pi$, $I_{12} = \frac{431302721}{8580495} - 16\pi$. Thus the condition $\frac{a}{c}I_4 + \frac{b}{c}I_{12} = \frac{355}{113} - \pi$ leads to the simultaneous equations

$$\frac{22}{7}\frac{a}{c} + \frac{431302721}{8580495}\frac{b}{c} = \frac{355}{113} \text{ and } \frac{a}{c} + 16.\frac{b}{c} = 1$$

which solve to give $\frac{a}{c} = \frac{4127}{19616687}$ and $\frac{b}{c} = \frac{1225785}{19616687}$ and thus the answer given above for a, b, c.

The calculation for A, B, C follows the same lines using

$$I_{20} = \frac{26856502742629699}{33393321606645} - 256\pi.$$

The variations in the solutions occurred in the evaluation of I_n. Many were content to use computer algebra systems, but others gave schemes that could be implemented by hand. The shortest route, given by Seán Stewart and, in an essentially equivalent form, by Stan Dolan is to use the identity

$$\frac{x^{4n}(1-x)^{4n}}{x^2+1} = \left(x^6 - 4x^5 + 5x^4 - 4x^2 + 4\right)\sum_{k=0}^{n-1}(-4)^{n-1-k}x^{4k}(1-x)^{4k} + \frac{(-4)^n}{x^2+1}.$$

This occurs in [2] and may be established by induction.

Integrating, using $\int_0^1 x^m (1 - x)^n \, dx = \frac{m!\,n!}{(m + n + 1)!}$ in the terms of the summation, gives $I_{4n} =$

$$(-1)^n 4^{n-1}\pi + (-1)^{n+1} 4^{n-1} \sum_{k=0}^{n-1} (-1)^k 2^{4-2k} \frac{(4k)!\,(4k+3)!}{(8k+7)!} \left(820k^3 + 1533k^2 + 902k + 165\right)$$

and substituting $n = 1, 3, 5$ gives I_4, I_{12}, I_{20} needed above.

The proposer, Peter Shiu, proved the more general result below, which makes it clear why *positive* integers occur in **103.H**.

Let $0 \leqslant x \leqslant 1$ and, for $h = 0, 1, 2, \ldots$, write

$$F_h(x) = (x(1 - x))^{4h} \text{ and } J_h = \int_0^1 \frac{F_h(x)}{x^2 + 1} \, dx, \text{ so that } \pi = 4J_0.$$

The equations in the problem are particular cases taken from the following:

Proposition: Let P, Q be coprime positive integers satisfying $\pi < \frac{P}{Q} < \frac{22}{7}$. Then there exists an odd h, and coprime positive integers r, s such that

$$\frac{rJ_h + sJ_{h+2}}{4^{h-1}(r + 16s)} = \frac{P}{Q} - \pi.$$

Proof: From $F_h(x) \equiv (-4)^h \pmod{x^2 + 1}$ we deduce that $\frac{F_h(x) - (-4)^h}{x^2 + 1}$ is a polynomial in x with integer coefficients, and with degree $8h - 2$. On integration we see that, if h is odd, then there are positive integers u_h, v_h, with v_h being a divisor of $\text{LCM}(1, 2, \ldots, 8h - 1)$, such that

$$J_h = \frac{u_h}{v_h} - 4^{h-1}\pi, \Rightarrow \mu_h = \frac{u_h}{4^{h-1}v_h}, \qquad h = 1, 3, 5, \ldots,$$

are rational approximations to π, because $J_h < F_h\left(\frac{1}{2}\right) = 1/4^{4h}$. The fractions μ_h are decreasing with respect to odd h, and the intervals $I_h = \left\{\mu_{h+2} \leqslant \mu < \mu_h\right\}$ $(h = 1, 3, 5, \ldots)$ form a partition of the interval $\pi < \mu < 22/7 = \mu_1$, so that each such $\mu \in I_h$ for a unique h.

For $h = 1, 3, 5, \ldots$, let $(J, J') = (J_h, J_{h+2})$, $(u, v) = (u_h, v_h)$, $(u', v') = (u_{h+2}, v_{h+2})$. Then, for a real $\lambda \geqslant 0$, we now have

$$\lambda J + J' = \frac{\lambda u}{v} + \frac{u'}{v'} - 4^{h-1}(\lambda + 16)\pi.$$

As λ increases in $0 \leqslant \lambda < \infty$, the bilinear expression (in λ)

$$f(\lambda) = f_h(\lambda) = \frac{\frac{\lambda u}{v} + \frac{u'}{v'}}{4^{h-1}(\lambda + 16)}$$

increases strictly in I_h, and $\mu = f(\lambda)$ is rational if, and only if, λ is rational. The inverse $g : I_h \to [0, \infty)$ of f is given by the bilinear expression (in μ)

$$\lambda = g(\mu) = \frac{4^{h+1}\mu - \frac{u'}{v'}}{4^{h-1}\mu - \frac{u}{v}}. \qquad (*)$$

Now, let $\pi < \frac{P}{Q} < \frac{22}{7}$. Then $\mu = \frac{P}{Q} \in I_h$ for a unique h, and u, v, u', v' can be computed accordingly. The reduced fraction $\lambda = g(\mu) = \frac{r}{s}$ can then be evaluated from $(*)$, together with the computation of GCD(r, s). The proposition is proved.

James Mundie and Seán Stewart raised the question of whether there are analogous results for the 2nd and 4th convergents, $\frac{333}{106}$ and $\frac{103993}{33102}$, which approximate π from below.

References

1. D. P. Dalzell, On $\frac{22}{7}$, *Journal London Maths. Soc* **19** (1944) pp. 133-134.
2. H. A. Medina, A sequence of polynomials for approximating inverse tangent, *Amer. Math. Monthly* **113** (February 2006) pp. 156-161.

Correct solutions were received from: S. Dolan, R. Gordon, A. P. Harrison, G. Howlett, M. Lukarevski, A. Mahajan, J. A. Mundie, S. M. Stewart, E. Swylan, L. Wimmer and the proposer Peter Shiu.

Readers will be saddened by the news of the death of Brian Trustrum in December 2019. He was one of the most loyal respondents to Problem Corner over the years. His solutions were characterised by a meticulous attention to detail and a ready willingness to tackle problems from all areas of pure and applied mathematics. Brian and Kathleen Trustrum were regular attendees at the annual MA Conferences and Brian's acuity, thoughtfulness and enthusiastic participation will be greatly missed.

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