4457. *Proposed by Hung Nguyen Viet.*

Prove that for all $-\frac{\pi}{2} < x, y < \frac{\pi}{2}$, $x \neq -y$, we have that

$$\tan^2 x + \tan^2 y + \cot^2(x+y) \geq 1.$$

There were 22 correct solutions; we present four variants here.

Solution 1, due, independently, to Michel Bataille, Ganbat Batmunkh, Martin Lukarevski, Marie-Nicole Gras, C.R. Pranesachar, Ioaanis Sifkas, Kevin Soto Palacios, and the Missouri State University Problem Solving Group.

Let $a = \tan x$ and $b = \tan y$. Then $\cot(x+y) = (1-ab)/(a+b)$ and

$$\tan^2 x + \tan^2 y + \cot^2(x+y) - 1 = a^2 + b^2 + \frac{(1-ab)^2}{(a+b)^2} - 1.$$

Multiplying this quantity by $(a+b)^2$ yields

$$\begin{aligned}(a^4 + b^4 + 2a^3b + 2ab^3 + 3a^2b^2 + 1) - (a^2 + b^2 + 4ab)\\ &= (a^2 + ab + b^2)^2 + 1 - (a^2 + b^2 + 4ab)\\ &= (a^2 + ab + b^2 - 1)^2 + (a-b)^2.\end{aligned}$$

Since this quantity is nonnegative, the result follows.

Equality occurs if and only if $a = b$ and $a^2+ab+b^2 = 1$, if and only if $x = y = \pm\pi/6$.

Comment from the editor. For the difference between the two sides, Devis Alvarado and Walther Janous obtained a fraction with the numerator

$$\left[\left(a+b-\tfrac{2}{\sqrt{3}}\right)^2 + (a-\tfrac{1}{\sqrt{3}})^2 + (b-\tfrac{1}{\sqrt{3}})^2\right]\left[\left(a+b+\tfrac{2}{\sqrt{3}}\right)^2 + (a+\tfrac{1}{\sqrt{3}})^2 + (b+\tfrac{1}{\sqrt{3}})^2\right].$$

Solution 2, by Digby Smith.

If $|x| + |y| > \pi/2$, then at least one of $|x|$ and $|y|$ exceeds $\pi/4$ and the left side exceeds 1. Since $0 \leq |x+y| \leq |x| + |y|$, then

$$\cot^2(x+y) = \cot^2(|x+y|) \geq \cot^2(|x|+|y|).$$

Thus, we may suppose that x and y are both nonnegative. If $x+y = \pi/2$, then the left side exceeds 2. Suppose $x + y \neq \pi/2$. Then, using the arithmetic-geometric means inequality, we have that

$$\begin{aligned}&\tan^2 x + \tan^2 y + \frac{1}{\tan^2(x+y)}\\ &= \frac{1}{2}\left[(\tan^2 x + \tan^2 y) + \left(\tan^2 x + \frac{1}{\tan^2(x+y)}\right) + \left(\tan^2 y + \frac{1}{\tan^2(x+y)}\right)\right]\\ &\geq \tan x \tan y + \frac{\tan x}{\tan(x+y)} + \frac{\tan y}{\tan(x+y)} = \tan x \tan y + \frac{\tan x + \tan y}{\tan(x+y)}\\ &= \tan x \tan y + 1 - \tan x \tan y = 1.\end{aligned}$$