ASYMMETRIC AUCTION MECHANISM VS BILATERAL TRADE INEFFICIENCY THEOREM

Dushko Josheski[1]; Mico Apostolov[2]

Abstract

In this paper asymmetric auctions have been revisited and have been tested in order to proof that Myerson-Satterthwaite theorem does hold when auction is the mechanism of trade. This result is actually an extension of the theorem. In asymmetric auctions bidders are of different types (different CDF's) i.e. the follow different distribution types, convergence achieved is inefficient, we present a case of double auctions also that is inefficient though efficiency there can be "improved" through k-level of thinking.

Key Words: *Myerson-Satterthwaite theorem, asymmetric auctions, BNE*
JEL classification: *D44, D82*

Introduction

Myerson-Satterthwaite theorem was introduced in Myerson, Satterthwaite (1983) .Informally this theorem explained that: there is no efficient way for two parties to trade a good when each party has varying private density valuations for the object that are unknown to other party, without forcing other party to trade with loss. Proofs of this theorem are provided in the auction theory graduate textbooks such as Khrisna (2009) and Milgrom(2004). This theory relates back to most famous adverse selection problem posed as the lemons problem, as Akerlof (1970). As in this example the assumptions of M-S theorem are posed: Individual rationality:$U_b, U_s \geq 0$, weak balanced budget (the auctioneer does not subsidize trade). But the Bayesian-Nash equilibrium is not incentive compatible (trade participants namely seller's cheat), $\forall v_b' : U_b(v_B, v_b') \ngeq U_b(v_B, v_b')$ and it is not ex-post Pareto efficient that the item should be given to then one that vales most but here his value is not equal to the expected quality (there are costs of dishonesty). Market produces gains only for sellers and loss only for the buyers, so this trade is not efficient. This the basic motivation of this paper. A typical feature of auctions is the presence of asymmetric information (see Klemperer (1999),Gibbons (1992)), the appropriate concept therefore is Bayesian Nash equilibrium[3], (Kajii, A., Morris, S. (1997), Harsanyi, John C.,(1967/1968)). A trade with private preferences (known to him) may demand more favorable terms than he is in truth willing to accept, and such behavior will lessen the gains from trade or will make some to even trade with loss, Rustichini, A., Satterthwaite, M. A., Williams, S. R., (1994). Auctions are type of games where players payoff depends on other's types of market participants, e.g. Akerlof (1970), and this market models where participants

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[3]A Bayesian Nash equilibrium is defined as a strategy profile that maximizes the expected payoff for each player given their beliefs and given the strategies played by the other players. That is, a strategy profile θ is a Bayesian Nash equilibrium if and only if for every player i, keeping the strategies of every other player fixed, strategy θ_i maximizes the expected payoff of player i according to his beliefs. Or in general BNE equilibrium is a Nash equilibrium of a Bayesian game :$\mathrm{Eu_i}(\mathrm{s_i}|\mathrm{s_{-i}}, \theta_\mathrm{i}) \geq \mathrm{Eu_i}(\mathrm{s'_i}|\mathrm{s_{-i}}, \theta_\mathrm{i}), \forall \mathrm{s'_i}\,(\theta_\mathrm{i}) \in \mathrm{S_i}$,where $s_i \in \Theta_i \to A_i$,and $\theta_\mathrm{i} \in \Theta_i$ also utilities are $u_i: A_1 \times A_2 \times \dots \dots . . \times A_i \times \theta_1 \times \theta_2 \times \dots \dots . \times \theta_i \to \mathbb{R}$,where $a_i \in A_i$ denotes finite action set.

have information that affects other player's payoffs are called adverse selection models. Most notable advances in the theory of auctions from 1960's and 1970's include: Vickrey (1961, 1962, 1976), Wilson (1967, 1969, 1977, 1979), Cassady, 1967, Griesmer, Levitan and Shubik (1967), Ortega (1968), Rothkopf (1969), Hurwicz (1973), Holmstrom (1977, 1979), Green and Laffont (1977), Milgrom (1979), Myerson (1979), etc. Seminal paper in the literature of asymmetric auctions is written by Maskin, Riley (2000),previously bidders were risk neutral and each bidder has a private valuation different from the others (different cumulative distribution functions and probability density functions), the bidders possess symmetric information, expected payments are functions of their bids, McAfee, McMillan,(1987). Symmetric beliefs are rejected in this paper. Which means that Revenue equivalence theorem (RET) will not apply here. FPA-First price auction and SPA-Second price auction where winners play second best price will not exert same revenues. On this topic (optimal auctions) furthermore Myerson (1981), designed Bayesian-optimal mechanism where it makes use of virtual valuations (virtual values are the derivative of the revenue curve).First in the paper formal statement of the MS theorem is given followed by proofs of inefficiency in Asymmetric N-bidder auctions .Ex-ante Asymmetric information among the bidders is the source of inefficiency, Hafalir,I., Krishna,V.(2009).

Myerson-Satterthwaite theorem

Theorem Myerson-Satterwhaite (notation): $\exists s \sim F_s(\underline{s}, \overline{s}) > 0; \exists b \sim F_b(\underline{b}, \overline{b})$, where F_s and F_b are common knowledge. In the DRG (direct revelation game) traders s-seller and b-bidder report their values and the outcome is selected, an outcome specifies probability of trade p,and the terms of trade x (payoffs).A DRG is a pair of outcome functions where (p, x): $p(s, b)$ is a probability of trade and $x(s, b)$ are thus the expected payments from buyer to seller. Utilities are given as:

equation 1

$$u(s,b) = x(s,b) - s(p,b) \; ; \; v(b,s) = bp(s,b) - x(s,b) \, .$$

Payoffs are defined as:

equation 2

$$X(s) = \int_{\underline{b}}^{\overline{b}} x(s,b) f_b(b) db \; ; X(b) = \int_{\underline{s}}^{\overline{s}} x(s,b) f_s(s) ds \, .$$

Probabilities of trade are given as:

equation 3

$$P(s) = \int_{\underline{b}}^{\overline{b}} p(s,b) f_b(b) db \; ; P(b) = \int_{\underline{s}}^{\overline{s}} p(s,b) f_s(s) ds \, .$$

Interim utilities are given as:

equation 4

$$U(s) = X(s) - P(s); V(b) = bP(b) - X(b) \, .$$

Incentive compatible mechanism (IC) $(p, x)$is given as:

equation 5

$$\text{IC:} \, U(s) \geq X(s') - P(s'); V(b) \geq P(b) - X(b).$$

Incentive rational mechanism (IR) is:

equation 6

$$\forall s \in [\underline{s}, \overline{s}] \vee \forall b \in [\underline{b}, \overline{b}], U(s) \geq 0\,; V(b) \geq 0\,.$$

Lemma IC: The mechanism is IC if and only if $P(s)$ is increasing and $P(b)$ decreasing and:

equation 7

$$\begin{cases} U(s) = U(\underline{s}) + \int_{\underline{s}}^{\overline{s}} P(s)(\theta)d\theta \\ V(b) = V(\underline{b}) + \int_{\underline{b}}^{\overline{b}} P\,(b)(\theta)d\theta\,. \end{cases}$$

Lemma 1 proof: Form previous definition we know that $U(s') \geq X(s') - s'P(s'); U(s) \geq X(s) - sP(s)$

equation 8

$$\begin{cases} U(s) \geq X(s') - sP(s') = U(s') + (s' - s)P(s'), \\ U(s') \geq X(s) - s'P(s) = U(s) + (s - s')P(s). \end{cases}$$

If we subtract these inequalities it will yield:

equation 9

$$(s' - s)P(s) \geq U(s) - U(s') \geq (s' - s)P(s').$$

Now if we take that $s' > s$ implies that $P(s)$ is decreasing, if we divide by $(s' - s)$ and letting $s' \rightarrow s$ yields $\frac{dU(s)}{ds} = -P(s)$ and integrating produces IC(s'). The same is true for the buyer. To prove the IC for the seller it is suffice to show that following applies:

equation 10

$$s[P(s) - P(s')] + [X(s') - X(s)] \leq 0 \;\forall\, s, s' \in [\underline{s}, \overline{s}\,]\,.$$

Now from previous by substituting for $X(s)$ and $X(s')$ and by using IC(s') the following will yield:

equation 11

$$X(s) = sP(s) + U(\overline{s}) + \int_{\underline{s}}^{\overline{s}} P(\theta)d\theta\,.$$

And following to hold:

equation 12

$$\begin{aligned} 0 \geq s[P(s)P(s\,)] + \; s\,P(s\,) + \int_{s'}^{\overline{s}} P(\theta)d\theta \; - sP(s) \\ - \int_{s}^{\overline{s}} P(\theta)d\theta = (s' - s)P(s') + \int_{s'}^{s} P(\theta)d\theta \\ = \int_{s'}^{s} [P(\theta) - P(s')]d\theta\,. \end{aligned}$$

Previous holds only because $P(\cdot)$ is decreasing.

Lemma IR: IC mechanism is individually rational IR if and only if:

equation 13

$$U(\overline{s}) \geq 0 \vee V(\underline{b}) \geq 0\,.$$

Corollary:

equation 14

$$U(\overline{s}) + V(\underline{b}) = \int_{\underline{b}}^{\overline{b}} \int_{\underline{s}}^{\overline{s}} \left[b - \frac{1 - F(b)}{F(b)} - s - \frac{1 - F(s)}{F(s)} \right] \mathrm{p(s,b)f(s)f(b)dsdb} \geq 0 \,.$$

Proof: From the IC we know that following holds:

equation 15

$$X(s) = sP(s) + U(\overline{s}) + \int_{\underline{s}}^{\overline{s}} P(\theta) d\theta \;.$$

Now from the corollary:

equation 16

$$\int_{\underline{b}}^{\overline{b}} \int_{\underline{s}}^{\overline{s}} \mathrm{x(s,b)f(s)f(b)ds} = U(\overline{s}) + \int_{\underline{b}}^{\overline{b}} \int_{\underline{s}}^{\overline{s}} \mathrm{sp(s,b)f(s)f(b)dsdb} + \int_{\underline{b}}^{\overline{b}} \int_{\underline{s}}^{\overline{s}} \mathrm{p(s,b)F(s)f(b)dsdb}.$$

The third term in the right side follows since:

equation 17

$$\int_{\underline{b}}^{\overline{b}} \int_{\underline{s}}^{\overline{s}} \mathrm{p(\theta,b)F(s)f(b)d\theta db} = \int_{\underline{b}}^{\overline{b}} \int_{\underline{s}}^{\theta} \mathrm{p(\theta,b)F(s)f(b)d\theta db} = \int_{\underline{s}}^{\overline{s}} \mathrm{p(s,b)F(s)f(b)dsdb} \,.$$

Analogously for the buyer follows that:

equation 18

$$\int_{\underline{b}}^{\overline{b}} \int_{\underline{s}}^{\overline{s}} \mathrm{x(s,b)f(s)f(b)dsdb} = -V(\underline{b}) + \int_{\underline{b}}^{\overline{b}} \int_{\underline{s}}^{\overline{s}} \mathrm{bp(s,b)F(s)f(b)dsdb} - \int_{\underline{b}}^{\overline{b}} \int_{\underline{s}}^{\overline{s}} \mathrm{p(s,b)F(s)}\big(1 - F(\mathrm{b})\big)\mathrm{dsdb} \;.$$

Now if we equate the both right hand sides proof is completed:

equation 19

$$\begin{aligned} V(\underline{b}) &= \int_{\underline{b}}^{\overline{b}} \int_{\underline{s}}^{\overline{s}} \mathrm{p(s,b)F(s)}\big(1 - F(\mathrm{b})\big)\mathrm{dsdb} - \int_{\underline{b}}^{\overline{b}} \int_{\underline{s}}^{\overline{s}} \mathrm{bp(s,b)F(s)f(b)dsdb} \\ &= \int_{\underline{s}}^{\overline{s}} \mathrm{p(s,b)F(s)f(b)dsdb} \,. \; \blacksquare \end{aligned}$$

IR mechanism is proved since $V(\underline{b}) \geq 0$.

Theorem Myerson-Satterthwaite (continued): It is not common knowledge that if trade gains exist i.e. the supports of the CDF functions (Cumulative distributions) of traders have non-empty intersections) then no IC (incentive compatibility) and IR (individual rationality) trading mechanism can be ex-post efficient.
Proof: A trading mechanism is ex-post efficient if and only if trade occurs whenever $s \leq b$

equation 20

$$p(s,b) = \begin{cases} 1 \; if \; s \leq \; b \\ 0 \; if \; s > b \end{cases} .$$

In the previous expression $p(s,b)$ is a probability of trade which takes value 1 if trade occurs and zero if it doesn't. To prove that ex-post efficiency cannot be attained, it is enough to show that inequality $(*)$ in the corollary hence:

equation 21

$$\int_{\underline{b}}^{\overline{b}} \int_{\underline{s}}^{\min(b,\overline{s})} \left[b - \frac{1 - F(b)}{f(b)} - s - \frac{F(s)}{f(s)} \right] f(s) f(b) dsdb \,.$$

Previous expression equals to:

equation 22

$$\int_{\underline{b}}^{\overline{b}}\int_{\underline{s}}^{\min(b,\overline{s})}[bf(b)+F(b)-1]f(s)dsdb$$
$$-\int_{\underline{b}}^{\overline{b}}\int_{\underline{s}}^{\min(b,\overline{s})}[sf(s)+F(s)]f(b)dsd=$$
$$-\int_{\underline{b}}^{\overline{s}}[1-F(\theta)]F(\theta)d\theta<0,\underline{b}<\overline{s}\,.\blacksquare$$

Previous result is proof of Myerson-Satterthwaite theorem about trade inefficiency. Some weaker efficiency criterion is Pareto optimality, one may use that criterion if ex-post efficiency does not work.
Definition: BNE equilibrium.

A Bayesian Nash equilibrium is defined as a strategy profile that maximizes the expected payoff for each player given their beliefs and given the strategies played by the other players. That is, a strategy profile θ is a Bayesian Nash equilibrium if and only if for every player i, keeping the strategies of every other player fixed, strategy θ_i maximizes the expected payoff of player i according to his beliefs. Or in general BNE equilibrium is a Nash equilibrium of a Bayesian game
$Eu_i(s_i|s_{-i},\theta_i)\geq Eu_i(s'_i|s_{-i},\theta_i),\forall s'_i\,(\theta_i)\in S_i$,*where* $s_i\in\Theta_i\rightarrow A_i$,*and* $\theta_i\in\Theta_i$ *also utilities are* $u_i\colon A_1\times A_2\times\dots\dots..\times A_i\times\theta_1\times\theta_2\times\dots\dots\times\theta_i\rightarrow\mathbb{R}$,*where* $a_i\in A_i$ *denotes finite action set.*

Definition: Incentive compatibility (Bayesian Incentive compatibility (BIC)
A mechanism $f(p_1,\dots,p_n)$ is called incentive compatible if for every player i $v_i\in V_i,\dots,v_n$ and $\forall v_i\in V_i$ $\forall v'_b\colon U_b(v_B,v'_b)\geq\ U_b(v_B,v'_b)$ where v'_b is the value for the buyer ex-post trade. Valuation function is given as:$v_i\colon A\rightarrow\Re$, and $f(p_1,\dots,p_n)$ is representing the payment functions.

Definition: Direct revelation mechanism

A direct revelation mechanism is a social choice function $f\colon V_1\times\dots..\times V_n\rightarrow A$ and a vector of payment functions $p_1,\dots.,p_n$ where $p_i\colon V_1\times\dots.\times V_n\rightarrow\Re$

Asymmetric auctions

There exists literature in the subject of asymmetric auctions namely: Maskin, Riley (2000), Fibich, Gavious (2003), Fibich, Gavish (2011), Güth,et al. (2005),Gayle, Richard (2008), Hubbard, et al. (2013).

Basic setup
There exist set: $\Theta=\{1,2,\dots,N\}$,of types of bidders. And $\forall\theta\in\ \{1,2,\dots,N\}$ and $\exists n(\theta)\geq 1$, which are bidders of type θ . Bidders of type θ draw an IPV for the object from CDF $F\colon[\omega_H,\omega_L]\rightarrow R$ .It is assumed that $F\in C^2\big((\omega_H,\omega_L)\big)$ and $f\equiv F'>0$, on ω_H . The inverse of equilibrium bidding strategy, Maskin and Riley (2000) and Fibich and Gavish (2011) is given as:

equation 23

$$v'_i(b) = \frac{F_i(\beta^{-1}(b))}{f_i(\beta^{-1}(b))} = \left[\left(\frac{1}{n-1}\sum_{j=1}^{n}\frac{1}{v_j(b)-b}\right) - \frac{1}{v_i(b)-b}\right], i = 1, \dots, n \, .$$

Inverse bid functions are solutions that gives profit maximization problem:

equation 24

$$\frac{\partial U_i(b;v_i)}{\partial b} = (v_i - b)\sum_{j=1,j\neq 1}^{n}\left(\prod_{k=1,k\neq 1}^{n} F_k(v_k(b))\right) f_j\left(v_j(b)\right) v_j'(b) - \prod_{j=1,j\neq 1}^{n} F_j\left(v_j(b)\right) = 0 \, .$$

Maximization problem here is given as in:

$$\max_{b} U_i(b;v_i) = (v_i - b)\prod_{j=1,j\neq 1}^{n} F_j\left(v_j(b)\right), i=1,\dots n \, .$$

Where one solution is:

equation 25

$$\sum_{j=1,j\neq 1}^{n}\frac{f_j\left(v_j(b)\right) v_j'(b)}{F_j(v_j(b))} - \frac{1}{v_i(b)-b}, i = 1, \dots n \, .$$

Or bidder chooses to maximize his expected surplus $S = \pi_i$ as in McAfee and McMillan (1987):

equation 26

$$\pi_i = (v_i - b_i)F(v)^{n-1} \;\; \partial\pi_i/\partial b_i = 0, \frac{\mathrm{dy}}{\mathrm{dx}} = \frac{\partial \pi_i}{\partial v_i} = F\left(\beta^{-1}(b_1)\right)^{n-1} \, .$$

Bidders expected revenue in FPA asymmetric auction is given as:

equation 27

$$E_i\,(p, b_i, v_i) = k_i \int_r^{b(\omega_h)} \left[F_i^{-1}(\ell_i(v)) - v\right] \cdot \frac{\ell_i'(v)}{\ell_i(v)} \prod_{j=1}^{n} \left[\ell_j(v)\right]^{k_j} dv \, .$$

Where in previous expression:$= \frac{(2-\lambda+\mu)}{1-\lambda}$, and bidder maximizes:

equation 28

$$\beta\left(\beta^{-1}(b_1)\right) = arg \max_{u\in(0,\omega_h)} (v-u) \cdot \left[F_i(\lambda_i(u))\right]^{k_i-1} \prod_{j\neq 1} \left[F_j\left(\lambda_j(u)\right)\right]^{k_j} .$$

$\exists u = \sum_{i=1}^{n} u_i$, where u_i denotes the player of type i. Where in previous expressions $\ell_i(v) = F_i(\lambda_i(v))$, and probabilities of winning the reserve price auction are given as:

equation 29

$$p_i(r) = k_i \int_r^{\omega_h} \frac{\ell_i'(v)}{\ell_i(v)} \prod_{j=1}^{n} \left[\ell_j(v)\right]^{k_j} dv \, .$$

Auctioneer expected revenue is given with the following expression:

equation 30

$$E(p, b_i, v_i) = \omega_h - r\prod_{j=1}^{n}\left[F_j(r)\right]^{k_j} - \int_r^{b(\omega_h)} \frac{\ell_i'(v)}{\ell_i(v)} \prod_{j=1}^{n} \left[\ell_j(v)\right]^{k_j} dv \, .$$

Here $U(p_i, E_i, r) = p_i \cdot (r - E_i)$, by the envelope theorem optimal values are denoted by asterisk $U^{*\prime}(r) = p^*(r)$,as in Milgrom (1989), and one can integrate to obtain the previous result.

equation 31

$$U^*(x) = \int_0^r p^*(v)dv \, .$$

Following is sort of prove of RET, that in a way expected revenue depends on the optimal auction price and that revenue does not depend on the auction mechanism. In asymmetric auctions there are stochastically dominant and stochastically weak bidders.

Theorem: Suppose that $F_S(v) \leq F_w(v)$, meaning that F_S conditionally first-order stochastically dominates F_w.Than when one compares FPA and SPA, both uniformly distributed following applies:

1. $\forall b_S^{-1}(b) = v_S, \because E\left(b_{FPA}(v)\right) < E\left(b_{SPA}(v)\right) \, for \, b_S^{-1}(b) = v_S$
2. $\forall b_w^{-1}(b) \neq v_w, \because E\left(b_{FPA}(v)\right) > E\left(b_{SPA}(v)\right) \, for \, b_w^{-1}(b) \neq v_w$

Proof: For purposes of the proof $b_S(v), b_w(v)$ have the same range so a matching function is defined as: $m(v) \equiv b_w^{-1}\left(b_S(v)\right)$ or as a weak bidder that bids equal to strong bidder in FPA. Since from previous we know that $b_S(v) < b_w(v)$ in FPA, now we know that $m(v) = v$.The strong bidder expected payoff is given as:

equation 32

$$E[\pi(v_i)] = \Pr(b_w(v_w) < b)(v - b) \, .$$

If we take derivative with respect to v we get: $E_v[\pi(v_i)] = \Pr(b_w(v_w) < b)$
And by replacing $b = b_S(v)$, which gives us the following identity:

equation 33

$$E_v[\pi(v_i)] = \Pr\left(b_w(v_w) < b_S(v)\right) = P_r\left(v_w < m(v)\right) \\ = F_w\left(m(v)\right) .$$

Because $\Pr(v < a) = F(a)$ when distribution of values is uniform. By the envelope theorem Milgrom and Ilya 2002 value function for FPA is given as:

equation 34

$$V_S^{FPA}(v) = \int_{\omega_l}^{\omega_h} F_w\left(m(w)\right)dv \, .$$

And for the SPA, where bidder's bid their true valuation (there is no bid shading):

equation 35

$$V_S^{SPA}(v) = \int_{\omega_l}^{\omega_h} F_w(v)dv \, .$$

Since $m(v) < v$ and that F_w is strictly increasing, the strong bidder prefers SPA. For the weak bidders expected payoff for the FPA is given as:

equation 36

$$V_w^{FPA}(v) = \int_{\omega_l}^{\omega_h} F_S\left(\frac{v}{m}\right) ds \, .$$

And for the SPA we have got:

equation 37

$$V_S^{SPA}(v) = \int_{\omega_l}^{\omega_h} F_S(v)dv.$$

Since $m^{-1}(v) > v$, expected payoff is higher for the weak bidder in the FPA.

Revenue equivalence theorem failure

Proposition : the weak bidder values is: $b_w \sim \left[0, \frac{1}{1+x}\right]$, and that strong bidder valuation is distributed as $b_s \sim \left[0, \frac{1}{1-x}\right]$.In equilibrium the weak and strong bidder bid functions are given as:

equation 38

$$b_w^{-1}(b) = \frac{1}{1+(2b)^2} \text{ and } b_s^{-1}(b) = \frac{1}{1-(2b)^2}.$$

FPA and SPA CDFs are given as:

equation 39

$$F_{FPA}(b) = \frac{(1-x^2)(2b)^2}{1-x^2(2b)^4};\ F_{SPA}(b) = 2b - (1-x)b^2.$$

If $x = 0$ both auctions yield revenue. When $x > 0$ the expected revenue is strictly greater for the first price auction than for the English auction (SPA auction). So the exert same revenue only in the case of uniform distribution.

Distributions from Plum class and Cheng class

When bidders are asymmetric solutions to the equilibrium bidding strategies of FPA auctions are difficult to obtain and there are three known classes of distributions for which equilibria in FPA auctions are known. Plum (1992) derives FPA bidding strategies when the distributions belong to class $\mathcal{P}$ consisting of F_1 and F_2 such that:

equation 40

$$F_1(x) = \left(\frac{x}{\omega_1}\right)^{\alpha};\ F_2(x) = \left(\frac{x}{\omega_2}\right)^{\alpha}$$

where $x \in (0,1)$,$\omega_1 = \frac{3}{2}; \omega_2 = 1; a = 3$. In Cheng (2006) FPA bidding strategies belong the class $\mathcal{C}_1$ where : $F_1(x) = \left(\frac{x}{\omega_1}\right)^{\alpha_1};\ F_2(x) = \left(\frac{x}{\omega_2}\right)^{\alpha_1}$, where :$a_1 > a_2 > 0; \omega_2 = \left(\frac{a_2}{a_2+1} * \frac{a_1+1}{a_1}\right); \omega_1 = \frac{3}{2}; \omega_2 = 1; a = 3$. And in Cheng (2007) , FPA bidding strategies belong to $\mathcal{C}_2$ class where :$F_1(x) = \left(\frac{x-1}{a}\right)^{a} \in [1, a+1]$ and $F_2(x) = \exp\left(\frac{a}{a+1}x - a\right) \in [0, a+1]$,where $a > 0$. Hafalir,I., Krishna,V.(2009), prove that RET fails in FOA and FPA with resale , and that FPA with resale is stochastically dominant in terms of revenue.

Double actions

Double auction (Chatterjee, Samuelson 1983)

Two players $N = \{b, s\}$, p_s asking price and p_b where $p_s \geq 0; p_b \geq 0$ values attached to the good by the seller and the buyer are : $\{v_s, v_b\}$ and $0 \leq v_s \leq 1; 0 \leq v_b \leq 1$.Buyer and seller beliefs are : $\mu_b = 1; \mu_s = 1$. The average price if $p_b \geq p_s$ is $p_{avg} = \left(\frac{p_s+p_b}{2}\right)$; and if $p_b < p_s$ then no trade occurs. The payoff function of the buyer and the seller are given as:

equation 41

$$u_s(p_s, p_b; v_s, v_b) = \begin{cases} \frac{p_s+p_b}{2}, p_b \geq p_s \\ v_s; p_b \leq p_s \end{cases};\ u_b(p_s, p_b; v_s, v_b) = \begin{cases} v_b - \frac{p_s+p_b}{2}, p_b \geq p_s \\ 0; p_b \leq p_s \end{cases}$$

The strategies for this game are given as: $p_s(v_s); p_b(v_b)$. The maximization problem is given as:

equation 42

$$\max_{p_s} E_{v_b}\{u_s(p_s,p_b;v_s,v_b)|v_s,p_b(v_b)\}$$

Now if the seller substitutes $p_b(v_b)$, then we have $u_s(p_s,p_b;v_s,v_b)=\begin{cases}\frac{p_s+p_b(v_b)}{2}, p_b(v_b)\geq p_s\\ v_s; p_b(v_b)\leq p_s\end{cases}$ or : $u_s(p_s,p_b;v_s,v_b)=\begin{cases}\frac{p_s+p_b(v_b)}{2}, v_b\geq p_b^{-1}(p_s)\\ v_s; v_b\leq p_b^{-1}(p_s)\end{cases}$. Or the maximization problem now is given as:

equation 43

$$\max_{p_s}\int_{v_b=0}^{p_b^{-1}(s)} v_s dv_b+\int_{v_b=p_b^{-1}(p_s)}^{1}\frac{p_s+p_b(v_b)}{2}dv_b$$

This problem FOC is given as:

equation 44

$$v_s\frac{dp_b^{-1}(p_s)}{dp_s}-\frac{1}{2}\left(p_s+p_b\left(p_b^{-1}(s)\right)\right)\frac{dp_b^{-1}(s)}{dp_s}+\int_{p_b^{-1}(s)}^{1}\frac{1}{2}dv_b=0$$

And because $p_s=p_b\left(p_b^{-1}(p_s)\right)$:

$(v_s-p_s)\frac{dp_b^{-1}(p_s)}{dp_s}+\frac{1}{2}[v_b]_{p_b^{-1}(p_s)}^{1}=0$ Or: $(v_s-p_s)\frac{dp_b^{-1}(p_s)}{dp_s}+[1-p_b^{-1}(p_s)]=0$

And since $p_s^{-1}(\cdot)=q_s(\cdot)$ and $p_b^{-1}(\cdot)=q_b(\cdot)$. The best replies are defined as :

equation 45

$[q_s(p_s)-p_s]q_b'(p_s)+\frac{1}{2}[1-q_b(p_s)]=0$;$[q_b(p_b)-p_s]q_s'(p_b)+\frac{1}{2}q_s(p_b)=0$

$$q_b'(p_b)=\frac{1}{2}\left[3-\frac{q_s(p_b)q_s''(p_b)}{[q_s'(p_b)]^2}\right]$$

Then the SODE is given as:

equation 46

$$[q_s(p_s)-p_s]\left[3-\frac{q_s(p_s)q_s''(p_s)}{[q_s'(p_s)]^2}\right]+\left[1-p_s-\frac{q_s(p_s)}{2q_s'(p_s)}\right]=0$$

This second order differential equation has two parameter solution: $q_s(p_s)=\alpha p_s+\beta$ where $\alpha=\frac{3}{2}$ and $\beta=-\frac{3}{8}$, so $q_s p_s=v_s=\frac{3}{2}p_s-\frac{3}{8}$; $q_b(p_b)=v_b=\frac{3}{2}p_b-\frac{1}{8}\Rightarrow p_s=\frac{2}{3}v_s+\frac{1}{4}; p_b=\frac{2}{3}v_b+\frac{1}{12}$. This is the BNE equilibrium of this game. Double auction game occurs whenever $p_b\geq p_s$ or $\frac{2}{3}v_b+\frac{1}{12}\geq\frac{2}{3}v_s+\frac{1}{4}$ or $v_b\geq v_s+\frac{1}{4}$. Next, we still show the properties of L1 buyers, $b(v)$ is maximized over $b\in[0,1]$:

equation 47

$$\int_0^b\left(v-\left(\frac{s+b}{2}\right)\right)f\left(s_*^{-1}(s)\right)ds+\int_b^1 0ds$$

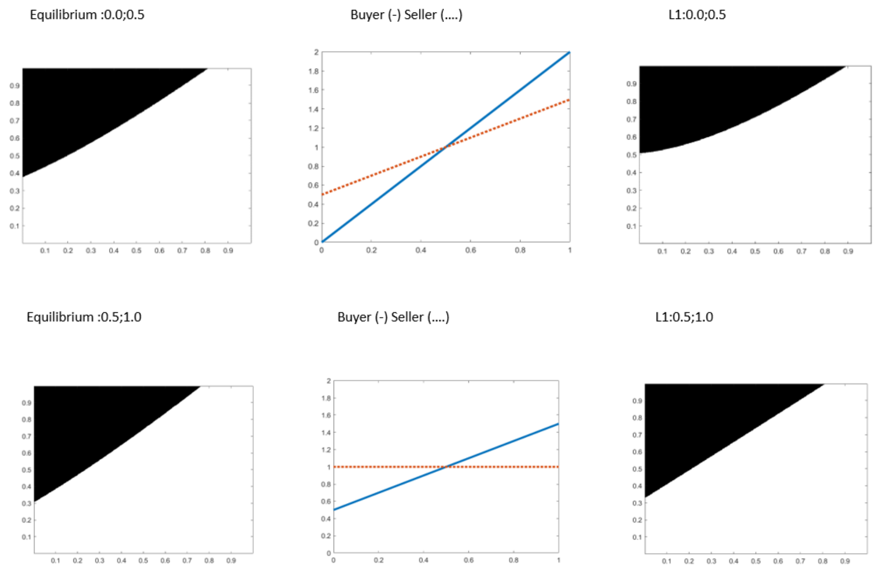
$f\left(s_*^{-1}(s)\right)$ is the density of the equilibrium strategy, equilibrium buyers bid is $b^*(v)$ and $b \in (0,1)$, b is bid and s denotes ask (seller) and $s \in (0,1)$, and $\frac{s+b}{2}$ is the price by which buyer receives the item of auction. Similarly, L1 sellers must maximize over $b \in [0,1]$:

equation 48

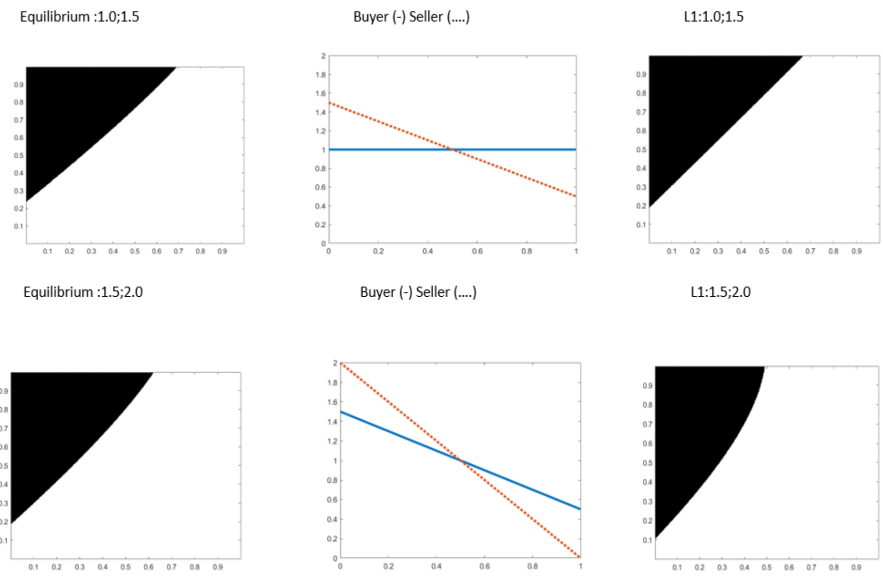
$$\int_s^1 \frac{s+b}{2} db + \int_0^s C db$$

Where in previous expression C denotes the sellers' costs. In Level K thinking agents trade not based on their beliefs about the fundamental value of the item of sale but based on other people beliefs about the value. Level zero player is non-strategic and will chose actions without regard to the actions of other players, Stahl (1993).Next graphically is depicted trade in double auctions within level K framework .Black regions are those where trade may take place and the equilibrium is also depicted ,and L1 player response to L0 players strategy[1]. This figure was produced in MATLAB and the used code was written by Crawford (2015). This is non-equilibrium model of Level K thinking that predicts initial responses to games and is focusing on direct mechanisms. In the Level K game all players think that are most sophisticated. In case of private information predictive power of level K thinking is significantly weakened, Shapiro, Shi, Zillante (2011).L0 player distribution is assumed to be uniform so his value will be mean of the distribution. Reserve price in presence of L0 player is reducing the gap for trade. Most of the computations in the literature show that double auction without reserve price is in fact optimal. the K-level thinking just proves the statement: that no incentive compatible, interim individually rational mechanism can assure ex-post efficiency, is fully compatible to K-level of thinking. This is show on the following graph on the next page.

Figure 1: *Trading regions are black, for the equilibrium incentive mechanisms and the mechanisms that are incentive in the set of L1 IC mechanisms*



[1] Example of Level-K thinking is so called Keynesian beauty contest, introduced by the John Maynard Keynes in chapter 12 of The General Theory of Employment, Interest and Money (1936), where he used to explain price fluctuations in the equity markets.



Conclusion

Bilateral trade is inefficient, but also N-bidder asymmetric auctions are ex-post inefficient. Asymmetries are the source of the inefficiency. This result can be proved numerically also. For instance, Backward shooting method calculates Bayesian Nash equilibrium, though solution is much above the means values that different bidders place on the item of sale. Bayesian Nash equilibrium is allocative inefficient result. In the case of asymmetric auctions, we have proved that Revenue equivalence theorem does not apply for FPA and SPA auctions. Anyway, for the truthful bidders SPA is weakly dominant strategy, and for a weak bidder expected payoff is higher in FPA auction. This means that one party is also forced to trade at loss due to incomplete information. Also, equilibrium is maybe feasible but in no case is efficient. This equilibrium is IR or individually rational for every trading buyer $p \leq b$ and for every seller $p \geq s$,bot not IC or truthful, it is efficient since highest bid wins the auction. Since information about the value of the object sale is not common knowledge and in BNE equilibrium payoffs depend not only on one agent type but also on other types, this game results in winners curse result, where winning agent tends to overpay due to emotional reasons or incomplete information. Or in a case of double auction if $s > b$ no trade takes place (seller wants more than the buyer pays),and if $b > s$ than $p = \frac{s+b}{2}$, the utility of buyer and the seller is zero in case when $s > b$ or for the buyer $u_b = b - p$ if $s \leq b$, and for the seller $u_s = p - s$ if $s \leq b$ this means that there will be trade which will result in the same utilities for the buyer and the seller like : $U_s = p - s = \frac{s+b}{2} - s = \frac{s+b-2s}{2} = \frac{b-s}{2}$; $u_b = b - p = b - \frac{s+b}{2} = \frac{b-s}{2}$ only if the buyers bid is higher than the true valuation of the seller which means that buyers will overpay and trade at loss.

References

1. Akerlof, G. A. (1970). The Market for 'Lemons': Quality Uncertainty and the *Market Mechanism. Quarterly Journal of Economics. The MIT Press. 84*(3): 488–500.
2. Cassady, R. (1967). *Auctions and auctioneering*. University of California Press.
3. Cheng, H., (2006). Ranking sealed high-bid and open asymmetric auctions. *Journal of Mathematical Economics 42*, 471–498.
4. Cheng, H., 2007. *Private communication*. in Hafalir, I., Krishna,V. (2009).

5. Crawford,V. P. (2015). Efficient Mechanisms for Level-*k* Bilateral Trading,Department of Economics, University of Oxford Department of Economics, and University of California, San Diego.
6. Gibbons,R. (1992). *Game theory for applied economists*. Princeton University Press.
7. Green, J., Laffont, J, J. (1977). Characterization of satisfactory mechanisms for the revelation of preferences for public goods. *Econometrica, 45*(2), 427-438.
8. Griesmer, J., Levi tan,R., Shubik, M. (1967). Towards a Study of Bidding Processes, Part IV-Games with Unknown Costs, *Naval Research Logistics Quarterly, 14*(4), 415–433.
9. Hafalir, I., Krishna,V. (2009). Revenue and efficiency effects of resale in first-price auctions. *Journal of Mathematical Economics, 45*(9–10).
10. Harsanyi, John C., (1967/1968). Games with Incomplete Information Played by Bayesian Players, I-III. Management Science 14 (3): 159-183 (Part I), 14 (5): 320-334 (Part II), 14 (7): 486-502 (Part III).
11. Holmstrom, B. (1977). *On Incentives and Control in Organizations*: Doctoral thesis, Stanford University.
12. Holmstrom, B. (1979). Groves Schemes on Restricted Domains.*Econometrica 47*(5), 1137-1144.
13. Hurwicz, L.(1973). The Design of Mechanisms for Resource Allocation. *American Economic Review 63*(2) 1–30.
14. Kajii, A., Morris, S. (1997). The Robustness of Equilibria to Incomplete Information. *Econometrica. 65*(6), 1283–1309.
15. Klemperer, P. (1999). Auction Theory: A Guide to the Literature. *Journal of Economic Surveys, 13*(3).
16. Krishna, Vijay. (2009). Auction Theory. Second ed. Academic Press.
17. Maskin, E., Riley, J. G. (2000). Equilibrium in Sealed High Bid Auctions. *Review of Economics Studies, 67*(3), 439–454.
18. McAfee, R. P., McMillan, J. (1987). Auctions with Entry. *Economics Letters, 23*(4), 343-347.
19. Milgrom, Paul, R. (2004). *Putting Auction Theory to Work*. Cambridge, UK: Cambridge UP.
20. Milgrom, Paul, R. (1979). A Convergence Theorem for Competitive Bidding with Differential Information. *Econometrica 47*(3), 679-688
21. Myerson, R. B. (1979). Incentive compatibility and the bargaining problem. *Econometrica, 47*(1), 61-73.
22. Myerson, R. B., Satterthwaite, M. A. (1983). Efficient Mechanisms for Bilateral Trading. *Journal of Economic Theory, 29*(2), 265–281
23. Myerson, R. B. (1981). Optimal Auction Design. Mathematics of *Operations Research, 6*(1) 58-73.
24. Ortega R. A., (1968). Models for Competitive Bidding Under Uncertainty. PhD thesis, Stanford University (and Technical Report No. 8, Department of Operations Research, Stanford University)
25. Plum, M., (1992). Characterization and computation of nash equilibria for auctions with incomplete information. *International Journal of Game Theory 20*, 393–418.
26. Rothkopf, M. H. (1969). A Model of Rational Competitive Bidding. *Management Science, 15*(7), 362-373.
27. Rustichini, A., Satterthwaite, M. A., Williams, S. R., (1994). Convergence to Efficiency in a Simple Market with Incomplete Information. *Econometrica*, *62*(5), 1041–1063.
28. Shapiro, D., Shi, X., Zillante, A. (2011). Level-k Reasoning in a Generalized Beauty Contest: Belk College of Business, University of North Carolina and Department of Economics, University of Toronto.

29. Stahl, D. O. (1993). Evolution of Smart$_n$ Players. *Games and Economic Behavior, 5*(4), 604-617.
30. Vickrey, W. (1961). Counterspeculation, Auctions, and Competitive Sealed Tenders. *The Journal of Finance 16*(1), 8–37.
31. Vickrey, W. (1962). Auction and Bidding Games. In *Recent Advances in Game Theory*. Princeton, NJ: The Princeton University Conference, pp. 15–27.
32. Vickrey, W. (1976). *Auctions Markets and Optimum Allocations*. In Amihud, Y. (ed.), Bidding and Auctioning for Procurement and Allocation: Studies in Game Theory and Mathematical Economics. New York: New York University Press, pp. 13–20.
33. Wilson, R. (1967).Competitive Bidding with Asymmetric Information. *Management Science, 13*(11), 816–820.
34. Wilson, R. (1969). Competitive Bidding with Disparate Information. *Management Science, 15*(7), 446–448.
35. Wilson, R. (1977). A Bidding Model of Perfect Competition. *Review of Economic Studies, 44*(3), 511–518.
36. Wilson, R. (1979). Auctions of Shares. *Quarterly Journal of Economics, 93(4),* 675–689.
37. Wilson, R. (1987). Bidding. The New Palgrave: A Dictionary of Economics. J. Eatwell, M. Milgate, and P. Newman. London: MacMillan Press. Vol, 238–242.