Sum of Powers of the Sides of a Triangle

11984 [2017, 466]. *Proposed by Daniel Sitaru, Drobeta Turnu Severin, Romania.* Let a, b, and c be the lengths of the sides of a triangle with inradius r. Prove $a^6+b^6+c^6 \geq 5184r^6$.

Solution by Leonard Giugiuc, Drobeta Turnu Severin, Romania. We first prove the well-known inequality $a+b+c \geq 6\sqrt{3}r$. Writing $a+b+c=2s$, where s is the semiperimeter and recalling that

$$\frac{s}{r}=\cot\frac{A}{2}\cot\frac{B}{2}\cot\frac{C}{2}=\cot\frac{A}{2}+\cot\frac{B}{2}+\cot\frac{C}{2}, \qquad (*)$$

we see that this follows from Jensen's inequality in the form

$$\cot\frac{A}{2}+\cot\frac{B}{2}+\cot\frac{C}{2}\geq 3\cot\frac{A+B+C}{6}=3\cot\frac{\pi}{6}=3\sqrt{3}.$$

The requested inequality follows from combining this with the power mean inequality $(a^6+b^6+c^6)/3 \geq ((a+b+c)/3)^6$.

Editorial comment. Motivated by the power mean inequality, we examine the inequalities

$$\left(\frac{a^p+b^p+c^p}{3}\right)^{1/p}\geq 2\sqrt{3}r.$$

The requested inequality is the case $p=6$, so to solve the problem it suffices to prove this for any $p \leq 6$. When $p=1$ we get the much stronger inequality (Question 1273 posed by M. E. Fauquembergue in 1878 in *Nouv. Ann. Math.* 37, p. 475, or item 5.11 in Bottema, O. et al. (1969), *Geometric Inequalities*, Groningen: Wolters-Noordhoff) that is proved above and was cited or reproved by many solvers. The strongest version that is in the literature seems to be the case $p=-1$, the inequality $\frac{\sqrt{3}}{2r}\geq\frac{1}{a}+\frac{1}{b}+\frac{1}{c}$. This appears (without proof) on page 342 of Posamentier, A. S., Lehmann, I. (2012), *The Secrets of Triangles*, New York: Prometheus Books. The $p=-2$ case is the strongest member of this family that holds. It asserts

$$\frac{1}{4r^2}\geq\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}.$$

To prove it, we let $s=(a+b+c)/2$ as above and let $x=s-a$, $y=s-b$, and $z=s-c$. Using $(*)$, we see that this inequality becomes

$$\frac{x+y+z}{4xyz}\geq\frac{1}{(y+z)^2}+\frac{1}{(z+x)^2}+\frac{1}{(x+y)^2},$$

which follows by summing three inequalities

$$\frac{1}{4yz}\geq\frac{1}{(y+z)^2}, \quad \frac{1}{4xz}\geq\frac{1}{(x+z)^2}. \quad \text{and} \quad \frac{1}{4xy}\geq\frac{1}{(x+y)^2},$$

each a form of the AM–GM inequality.

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