PRODUCTS OF DISTRIBUTIONS IN COLOMBEAU ALGEBRA

Marija Miteva

COST Action 15225

11.05.2018

- Introduction to the distribution theory
- Main problems that the theory of distributions is concerned with
- Construction of Colombeau algebra
- Results on Colombeau products of distributions

- Theory of distributions1950s
  - Mathematical meaning of many concepts in science that were defined heuristic previously
  - Dirac δ function and its derivatives
  - The properties were defined heuristically, to be appropriate to the experimental results and to be appropriate for analysis and solving the problems that they characterize
  - The most operations with those concepts remain without mathematical support

$$\delta(x)=\begin{cases}0, & x\neq 0\\ \infty & x=0\end{cases} \qquad \int_{-\infty}^{\infty}\delta(x)\,dx=1 \tag{1}$$

- Need for generalizing the notion of function
- Distribution (generalized function)
- Laurent Schwartz, '*Theory of Distributions*' (1950)

  - Many concepts that can not be described with functions can be described with distributions
  - Concepts that can be described with functions can also be described with distributions

- $\mathcal{D} = \mathcal{D}(\mathbf{R}^n)$ - space of smooth functions with compact support (test functions)
- *Generalized function (distribution) is continuous linear mapping* $f : \mathcal{D} \to \mathbf{C}$

$$f(\varphi) = \langle f, \varphi \rangle \qquad (2)$$

$\mathcal{D}' = \mathcal{D}'(\mathbf{R}^n)$ - the space of distributions with domain $\mathcal{D}$

- f - locally integrative function

$$f(\varphi) = \langle f, \varphi \rangle = \int_{\mathbf{R}^n} f(x)\varphi(x)\,dx \qquad (3)$$

- f - *regular distribution*

- *Singular distributions*
  - Dirac δ distribution:

$$\langle \delta, \varphi \rangle = \varphi(0) \qquad (4)$$

- Differentiation of distributions

$$f \in \mathcal{D}'(\mathbf{R}) \qquad \varphi \in \mathcal{D}(\mathbf{R})$$

- $$\left\langle f^{(n)}(x), \varphi(x) \right\rangle = (-1)^n \left\langle f(x), \varphi^{(n)}(x) \right\rangle \tag{5}$$

$$f \in \mathcal{D}'(\mathbf{R}^n) \qquad \varphi \in \mathcal{D}(\mathbf{R}^n)$$

$$D^k = \prod_{i=1}^{n} \left(\frac{\partial}{\partial x_i} \right)^{k_i} \qquad k_i \in \mathbf{N_0} \qquad k = \sum_{i=1}^{n} k_i$$

- $$\left\langle D^k f, \varphi \right\rangle = (-1)^k \left\langle f, D^k \varphi \right\rangle \tag{6}$$

- **Arbitrary ordered derivative of function will always exist if we consider that function as generalized function (distribution)**

- Two main problems for the distribution theory:

  - **Product of distributions** (not any two distributions can always be multiplied)

  - the product of distributions is not associative operation

  - **Differentiating the product of distributions** (the product of distributions not always satisfy the Leibniz rule)

- The application of distributions in non linear systems needs products of singular distributions
- Attempts for defining product of distributions that will be generalization of existing products
- **Regularization method**

$\varphi_n \to \delta(x)$ - delta sequence; f - distribution

$$f_n(x) = (f * \varphi_n)(x) = \langle f(y), \varphi_n(x-y) \rangle \tag{7}$$

- Sequence of smooth functions (f_n) ; $f_n \to f$
- (f_n) - *regularization of the distribution* f
- $$fg = \lim_{n\to\infty} (f * \varphi_n) \cdot (g * \varphi_n) \tag{8}$$

Introduction – sequential approach

- with regularization method:
- $$\frac{1}{x}\cdot\delta = -\frac{1}{2}\delta' \qquad (9)$$

- but, $\delta \cdot \delta = \delta^2$ is not defined neither with the regularization method

- Overcoming the problem with product of distributions

- Construction of algebra *A* with properties:

1) Contains the space of distributions $\mathcal{D}'(\mathbf{R}^n)$ and $f(x) \equiv 1$ is neutral element in *A*
2) There exist linear differential operators $\partial_i : A \to A$ which satisfy the Leibnitz's rule
3) ∂_i generalizes the notion of derivation on the space of distributions
4) The product in *A* generalizes the product of continuous functions

Schwartz's impossibility result

- functions $f(x) = x$ и $g(x) = |x|$ are considered

- derivative of their classical product:

$$\partial\left(x|x|\right) = 2|x| \quad (10)$$

$$\partial^2\left(x|x|\right) = 2\partial\left(|x|\right) \quad (11)$$

- Derivative of their product in *A*

$$\partial\left(x\cdot|x|\right) = |x| + x\cdot\partial\left(|x|\right) \quad (12)$$

$$\partial^2\left(x\cdot|x|\right) = 2\partial\left(|x|\right) + x\cdot\partial^2\left(|x|\right) \quad (13)$$

$$\partial^2\left(x\cdot|x|\right) = 2\partial\left(|x|\right) + 2x\cdot\delta \quad (14)$$

- Schwartz's impossibility result
  - From (11) and (14) follows : $x \cdot \delta = 0$ (15)

- *Theorem*: In *A*, if $x \cdot a = 0$ then $a = 0$.

- From (15) $\Rightarrow$ $\delta = 0$.

- New theory of generalized functions, more general then the theory of distributions
- Jean-Francois Colombeau
  - *New generalized function and multiplication of distributions* (1984)
  - *Elementary introduction to new generalized functions* (1985)

- **Colombeau algebra**
  - The product in the algebra generalizes classical product of C^∞ - functions

Colombeau algebra - construction

- $\mathbf{N_0} = \mathbf{N} \cup \{0\}$ - non negative integers
- $\mathcal{D}(\mathbf{R}^n)$ - the space of C^∞ - functions $\varphi : \mathbf{R}^n \to \mathbf{C}$ with compact support

- for $j \in \mathbf{N_0}$ and $q \in \mathbf{N_0}$ next sets are defined:

$$A_0(\mathbf{R}^n) = \left\{ \varphi(x) \in \mathcal{D}(\mathbf{R}^n) \middle| \int_{\mathbf{R}^n} \varphi(x)\,dx = 1 \right\}$$

$$A_q(\mathbf{R}^n) = \left\{ \varphi(x) \in D(\mathbf{R}^n) \middle| \int_{\mathbf{R}^n} \varphi(x)\,dx = 1,\ \int_{\mathbf{R}^n} x^j \varphi(x)\,dx = 0; 1 \le |j| \le q \right\}$$

$$q \ge 1 \qquad x = (x_1, x_2, \dots, x_n) \in \mathbf{R}^n \qquad j = (j_1, j_2, \dots, j_n) \in \mathbf{N}^n$$

$$|j| = j_1 + j_2 + \dots + j_n \qquad x^j = (x_1)^{j_1} (x_2)^{j_2} \cdots (x_n)^{j_n}$$

Colombeau algebra - construction

- $A_0 \supset A_1 \supset A_2 \supset A_3 \dots$

- *Theorem:* The sets A_q are non empty.

  Proof: J.F.Colombeau, *Elementary Introduction to New Generalized Functions* (1985)

- For $\varphi \in A_q\left(\mathbf{R}^n\right)$ and $\varepsilon > 0$ we denote:

$$\varphi_\varepsilon\left(x\right) = \frac{1}{\varepsilon^n}\varphi\left(\frac{x}{\varepsilon}\right) \tag{16}$$

$$\check{\varphi}(x) = \varphi\left(-x\right) \tag{17}$$

Colombeau algebra - construction

- $\mathcal{E}\left(\mathbf{R}^n\right)$ - algebra of functions $f\left(\varphi,x\right):A_0\left(\mathbf{R}^n\right)\times\mathbf{R}^n\to\mathbf{C}$ that are infinitely many times differentiable with respect of the second variable x (with fixed test function φ)

- The space $C^\infty\left(\mathbf{R}^n\right)$ is subalgebra of $\mathcal{E}\left(\mathbf{R}^n\right)$ (those functions that don't depend of φ)

- Embedding of distributions is such that embedding of $C^\infty\left(\mathbf{R}^n\right)$ - functions is identity

Colombeau algebra - construction

- $\mathcal{E}_M\left[\mathbf{R}^n\right]$ - subalgebra of $\mathcal{E}\left(\mathbf{R}^n\right)$ with elements such that for every compact subset K from $\mathbf{R}^n$ and every $p \in \mathbf{N_0}$ there exists $q \in \mathbf{N}$ such that for arbitrary $\varphi \in A_q\left(\mathbf{R}^n\right)$ there exist $c > 0,\ \eta > 0$ and the relation holds:

$$\sup_{x \in K}\left|\partial^p f\left(\varphi_\varepsilon, x\right)\right| \le c\varepsilon^{-q} \qquad (18)$$

for $0 < \varepsilon < \eta$.

  - Functions in $\mathcal{E}\left(\mathbf{R}^n\right)$ whish derivatives are bounded on compact subsets with negative powers of ε

Colombeau algebra - construction

- $f \in C\left(\mathbf{R}^n\right)$
- With the mapping:

$$F(\varphi, x) = \int_{\mathbf{R}^n} f(y)\varphi(y-x)\,dy = \int_{\mathbf{R}^n} f(x+t)\varphi(t)\,dt \qquad (19)$$

the space $C\left(\mathbf{R}^n\right)$ is embedded in $\mathcal{E}_M\left[\mathbf{R}^n\right]$.

- $C^\infty\left(\mathbf{R}^n\right)$ is contained in $\mathcal{E}_M\left[\mathbf{R}^n\right]$ in a way that $f(x)$ are those functions $f(\varphi, x)$ that don't depend of φ .

- $C^\infty\left(\mathbf{R}^n\right) \subset C\left(\mathbf{R}^n\right)$ are embedded in $\mathcal{E}_M\left[\mathbf{R}^n\right]$ with (19).

$$f(x) \neq \int_{R^n} f(x+\varepsilon t)\varphi(t)\,dt \qquad (20)$$

- Ideal such that the difference in (20) will vanish

Colombeau algebra - construction

- $\mathcal{I}\left[\mathbf{R}^n\right]$ is an indeal in $\mathcal{E}_M\left[\mathbf{R}^n\right]$ consisting of functions $f\left(\varphi, x\right)$ such that for every compact subset K of $\mathbf{R}^n$ and each $p \in \mathbf{N_0}$ there exist $q \in \mathbf{N}$ such that for any $r \geq q$ and each $\varphi \in A_r\left(\mathbf{R}^n\right)$ there exist $c > 0,\ \eta > 0$ and it holds:

$$\sup_{x \in K}\left|\partial^p f\left(\varphi_\varepsilon, x\right)\right| \leq c\varepsilon^{r-q} \tag{21}$$

For $0 < \varepsilon < \eta$.

- Te elements of $\mathcal{I}\left[\mathbf{R}^n\right]$ are called null functions

Colombeau algebra - construction

- $$f(x) \neq \int_{\mathbf{R}^n} f(x+\varepsilon t)\varphi(t)\,dt \tag{22}$$
- $$F(\varphi, x) = f(x) - \int_{\mathbf{R}^n} f(x+t)\varphi(t)\,dt \tag{23}$$
- $$F(\varphi_\varepsilon, x) = -\int_{\mathbf{R}^n} \left[f(x+\varepsilon t) - f(x) \right] \varphi(t)\,dt \tag{24}$$
- $$F(\varphi_\varepsilon, x) \in \mathcal{I}\left[\mathbf{R}^n\right]$$
- The difference in (22) vanishes in the factor algebra $\dfrac{\mathcal{E}_M\left[\mathbf{R}^n\right]}{\mathcal{I}\left[\mathbf{R}^n\right]}$

Colombeau algebra - construction

- ***Generalized functions in Colombeau theory are elements of*** *quotient algebra*

$$\mathcal{G} \equiv \mathcal{G}\left(\mathbf{R}^n\right) = \frac{\mathcal{E}_M\left[\mathbf{R}^n\right]}{\mathcal{I}\left[\mathbf{R}^n\right]} \tag{25}$$

- In $\mathcal{E}_M\left[\mathbf{R}^n\right]$ the equivalence relation is defined ‘ $\sim$ ‘ :

$$F_1 \sim F_2 \Leftrightarrow F_1 - F_2 \in \mathcal{I}\left[\mathbf{R}^n\right] \tag{26}$$

- The generalized functions in Colombeau theory are an equivalence classes of smooth functions
- New generalized functions (Colombeau generalized functions)

Embedding of distributions in Colombeau algebra

- $f \in C^{\infty}(\mathbf{R}^n) \qquad f \to f(\varphi, x) \in \mathcal{G}(\mathbf{R}^n)$

$$f(\varphi, x) = f(x) \tag{27}$$

- $f \in C(\mathbf{R}^n) \qquad f \to f(\varphi, x) \in \mathcal{G}(\mathbf{R}^n)$

$$f(\varphi, x) = \int_{\mathbf{R}^n} f(y)\varphi(y-x)\,dy = \int_{\mathbf{R}^n} f(x+y)\varphi(y)\,dy \tag{28}$$

- $f \in \mathcal{D}'(\mathbf{R}^n) \qquad f \to f(\varphi, x) \in \mathcal{G}(\mathbf{R}^n)$

$$f(\varphi, x) = \left(f * \check{\varphi}\right)(x) = \langle f(y), \varphi(y-x) \rangle = \int_{\mathbf{R}^n} f(y)\varphi(y-x)\,dy \tag{29}$$

Embedding of distributions in Colombeau algebra

- each distribution f is associated with equivalence class $f \in \mathcal{G}$ (*new generalized function*) of an element $f_\varepsilon \in \mathcal{E}_M$
- f_ε - *representative of the distribution* f

- Operations product and differentiation of generalized functions are performed on an arbitrary representative from the classes of those generalized function which is C^∞ - функција

  - Results are independent of the representatives choosen

- The space of smooth functions $C^{\infty}(\mathbf{R}^{n})$ is subalgebra of Colombeau algebra $\mathcal{G}(\mathbf{R}^{n})$.

- Space of continuous functions $C(\mathbf{R}^{n})$ and the space of distributions $\mathcal{D}'\left(\mathbf{R}^{n}\right)$ are not subalgebras of Colombeau algebra $\mathcal{G}(\mathbf{R}^{n})$.
  - If f, g are two continuous functions (or distributions which classical product exists), the embedding of their classical product fg and the product of their embeddings $f \cdot g$ in $\mathcal{G}$ may not coincide.
  - This difference of the products has been overcome introducing the concept of „association“ in $\mathcal{G}$

Association in Colombeau algebra

- Generalized functions $F, G \in \mathcal{G}\left(\mathbf{R}^n\right)$ are said to be ***associated*** ($F \approx G$) if for each representatives $f\left(\varphi_\varepsilon, x\right)$ and $g\left(\varphi_\varepsilon, x\right)$ and each $\psi(x) \in \mathcal{D}\left(\mathbf{R}^n\right)$, there exists $q \in \mathbf{N_0}$ such that for any $\varphi(x) \in A_q\left(\mathbf{R}^n\right)$ holds:

$$\lim_{\varepsilon \to 0_+} \int_{\mathbf{R}^n} \left| f\left(\varphi_\varepsilon, x\right) - g\left(\varphi_\varepsilon, x\right) \right| \psi(x)\, dx = 0 \qquad (30)$$

- Generalized function $F \in \mathcal{G}$ is associated with the distribution $u \in \mathcal{D}'$ ($F \approx u$) if for each representative of that generalized function and arbitrary $f\left(\varphi_\varepsilon, x\right)$ and each $\psi(x) \in \mathcal{D}\left(\mathbf{R}^n\right)$, there exist $q \in \mathbf{N_0}$ such that for any $\varphi(x) \in A_q\left(\mathbf{R}^n\right)$ holds:

$$\lim_{\varepsilon \to 0_+} \int_{\mathbf{R}^n} f\left(\varphi_\varepsilon, x\right) \psi(x)\, dx = \langle u, \psi \rangle \qquad (31)$$

Association in Colombeau algebra

- Above definitions are independent of the representatives chosen

- The distribution associated, if it exists, is unique

- Elements of Colombeau algebra with this process of association is associated to element in $\mathcal{D}'$ which allows us to consider obtained results in the sense of distribution.
  - Not any element in Colombeau algebra has associated distribution!

Association in Colombeau algebra

- *Theorem:* If $f, g \in C(\mathbf{R}^n)$ are two continuous functions, their product $f \cdot g$ in $\mathcal{G}(\mathbf{R})$ is associated with their classical product fg in $C(\mathbf{R}^n)$.

- *Theorem:* If $f \in C^\infty\left(\mathbf{R}^n\right)$ and $T \in \mathcal{D}'\left(\mathbf{R}^n\right)$, the product $f \cdot T$ in $\mathcal{G}\left(\mathbf{R}^n\right)$ is associated with the classical product $f\,T$ in $\mathcal{D}'\left(\mathbf{R}^n\right)$

- *Theorem*: If S and T are two distributions in $\mathcal{D}'\left(\mathbf{R}^n\right)$ and their classical product ST in $\mathcal{D}'\left(\mathbf{R}^n\right)$ exists, then the product of these two distributions $S \cdot T$ in $\mathcal{G}\left(\mathbf{R}^n\right)$ is associated with their classical product ST .

Association in Colombeau algebra

- Two distributions embedded in Colombeau algebra are new (Colombeau) generalized functions
- Product of two distributions in $\mathcal{G}$ is in general new (Colombeau) generalized function (for which there may not exist associated distribution)
- *Is for the product of two distributions in $\mathcal{G}$ there exists associated distribution, we say that there exists* ***Colombeau product of those two distributions***
- If the classical product of two distributions exists, then their Colombeau product also exists and is the same with the first one

- ✓ New (Colombeau) generalized functions – generalization of the Schwartz's definition of distributions
- ✓ Solving problems with product of distributions, problems with discontinuous functions, problems with differentiation (all the operations are performed on arbitrary representative which is smooth function)
- ✓ Many products of distributions that are not defined in the classical theory of distributions, exist as Colombeau products
- ✓ Association in Colombeau algebra allows us to consider obtained results in terms of distributions

- New theory of generalized functions, optimal propertis for dealing with distributions – increasing number of scientists who work with Colombeau algebras

- B. Damyanov (1997) in $\mathcal{G}(\mathbf{R})$:

$$x_{+}^{-r-1/2} \cdot x_{-}^{r-1/2} \approx \frac{(-1)^{r} \pi}{2} \delta(x) \tag{32}$$

$r = 0, \pm 1, \pm 2, \ldots$

- B. Damyanov (1999) in $\mathcal{G}\left(\mathbf{R}^{m}\right)$, for $p \in \mathbf{N}^{m}$:

$$x^{-p} \cdot \delta^{(p-1)}(x) \approx \frac{(-1)^{|p|}(p-1)!}{2^{m}(2p-1)!} \delta^{(2p-1)}(x) \tag{33}$$

Results on products of distributions and application of Colombeau algebra

- B. T. Jolevska, T. A. Pacemska (2013) in $\mathcal{G}(\mathbf{R})$:

$$x_+^{-r-1/2} \cdot x_-^{-r-1/2} \approx \frac{\pi}{2(2r)!} \delta^{(2r)}(x) \qquad (34)$$

- **B. Jolevska-Tuneska**, A. Takaci and E. Ozcag: *On differential equations with non-standard coefficients*, Applicable Analysis and Discrete Mathematics, 1 (2007),1-8.

- Application of Colombeau algebra in science and engineering

- Marija Miteva and Biljana Jolevska-Tuneska. *Some Results on Colombeau Product of Distributions*. **Advances in Mathematics: Scientific Journal.** Vol 1, No. 2, 121-126 (2012).

- <u>Theorem1:</u> The product of generalized functions $\ln|x|$ and $\delta^{(s-1)}(x)$, for $s = 0, 1, 2, \ldots$ in $\mathcal{G}(\mathbf{R})$ admits an associated distribution and it holds:

$$\ln|x| \cdot \delta^{(s-1)}(x) \approx \frac{(-1)^s}{s} \delta^{(s-1)}(x) \qquad (35)$$

- Marija Miteva, Biljana Jolevska-Tuneska and Tatjana Atanasova Pacemska. *On Products of Distributions in Colombeau Algebra*. **Mathematical Problems in Engineering**. Vol. 2014, Article ID 910510. \\ doi:10.1155/2014/910510

IF (2013) = 1.383

  - обопштување на (33)

- Theorem 2: The product of generalized functions x_+^{-k} and $\delta^{(p)}(x)$ for $k=1,2,\ldots$ and $p=0,1,2,\ldots$, in $\mathcal{G}(\mathbf{R})$ admits an associated distribution and it holds:

$$x_+^{-k}\cdot\delta^{(p)}(x)\approx\frac{(-1)^k k\cdot p!}{(p+k+1)!}\delta^{(k+p)}(x) \qquad (36)$$

- Marija Miteva, Biljana Jolevska-Tuneska and Tatjana Atanasova Pacemska. *On Products of Distributions in Colombeau Algebra*. **Mathematical Problems in Engineering**. Vol. 2014, Article ID 910510. \\ doi:10.1155/2014/910510

IF (2013) = 1.383

  - обопштување на (33)

- Theorem3: The product of generalized functions x_+^{-k} and $\delta^{(p)}(x)$, for $k=1,2,\dots$ and $p=0,1,2,\dots$ in $\mathcal{G}(\mathbf{R}^m)$ admits an associated distribution and it holds:

$$x_+^{-k}\cdot\delta^{(p)}(x)\approx\frac{(-1)^p k\cdot p!}{(p+k+1)!}\delta^{(k+p)}(x) \tag{37}$$

New results on products of distributions

- Marija Miteva, Biljana Jolevska-Tuneska, Tatjana Atanasova-Pacemska. *Results on Colombeau products of distribution* $x_+^{-r-1/2}$ *with Distributions* $x_-^{-k-1/2}$ *and* $x_-^{k-1/2}$. **Functional Analysis and its Application** (прифатен за публикување) **IF(2016)=0.450**
  - обопштување на (34)

- <u>Theorem4:</u> The product of generalized functions $x_+^{-r-1/2}$ and $x_-^{-k-1/2}$, for $r = 0,1,2,\dots$ and $k = 0,1,2,\dots$ in $\mathcal{G}(\mathbf{R})$ admits an associated distribution and it holds:

$$x_+^{-r-1/2} \cdot x_-^{-k-1/2} \approx \frac{\pi}{2(r+k)!} \delta^{(r+k)}(x) \qquad (41)$$

- Marija Miteva, Biljana Jolevska-Tuneska, Tatjana Atanasova-Pacemska. *Results on Colombeau products of distribution* $x_+^{-r-1/2}$ *with Distributions* $x_-^{-k-1/2}$ *and* $x_-^{k-1/2}$*.* **Functional Analysis and its Application** (прифатен за публикување) **IF(2016)=0.450**
  - обопштување на (32)

- Theorem 5: The product of generalized functions $x_+^{-r-1/2}$ and $x_-^{k-1/2}$, for $r = 0,1,2,\dots$, $k = 0,1,2,\dots$ and $r \geq k$ in $\mathcal{G}(\mathbf{R})$ admits an associated distribution and it holds:

$$x_+^{-r-1/2} \cdot x_-^{k-1/2} \approx C_{r,k}\delta^{(r-k)}(x) \qquad (42)$$

$$C_{r,k} = \frac{(-1)^r (2k-1)!!k!r!\pi}{2(4k-1)!!(2r-1)!!(r-k)!(r+k)!}\sum_{q=0}^{2k}(-1)^q \binom{2k}{q}\binom{r-k}{k-q}(2(r+q)-1)!!(2(k-q)-1)!!$$

БЛАГОДАРАМ ЗА ВНИМАНИЕТО

THANK YOU FOR YOUR ATTENTION