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Extension of Some Results of Inequality Relations Involving Multivalent Functions

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Abstract. In this paper, we extend our results of some inequality relations in which we include multivalent functions in order to give sufficient conditions (unfortunately not sharp) when the following implication holds:

$$\left|\arg\left[1+\frac{zf^{(p+1)}(z)}{f^{(p)}(z)}\right]\right| < \frac{\alpha\pi}{2} \quad (z\in\mathbb{D}) \quad \Rightarrow \quad \left|\arg\frac{zf'(z)}{f(z)}\right| < \frac{\beta\pi}{2} \quad (z\in\mathbb{D}).$$

Here $f(z)$ is a multivalent function, i.e., analytic on the unit disk and of the form $f(z) = z^p + a_{p+1}z^{p+1} + \cdots, p = 2, 3 \ldots$.

Keywords: Multivalent function; Implication; Iteration.

1. Introduction

Let for $n \in \mathbb{N}$ and $a \in \mathbb{C}$, $\mathcal{H}[a,n] = \big\{f \in \mathcal{H}(\mathbb{D}) : f(z) = a + a_n z^n + a_{n+1}z^{n+1} + \cdots\big\}$, where $\mathcal{H}(\mathbb{D})$ is the class of all functions that are analytic in the open unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$. Also, let for a positive integer p, $\mathcal{A}_p$ be the subclass of $H(\mathbb{D})$ consisting of functions of the form $f(z) = z^p + a_{p+1}z^{p+1} + \cdots$ and $\mathcal{A} \equiv \mathcal{A}_1$, so that $\mathcal{A}$ is the class of functions $f(z)$ which are analytic in $\mathbb{D}$ with normalization $f(0) = 0$ and $f'(0) = 1$. More details can be found in [7, 10, 15].

A function f is *multivalent* or *p-valent* in $\mathbb{D}$ if it takes no value more than p times in $\mathbb{D}$ and there is some ω_0 such that $f(z) = \omega_0$ has exactly p solutions in $\mathbb{D}$, when roots are counted in accordance with their multiplicities.

N. Xu and D.G. Yang given some interest results on multivalent functions in [17]. In this work, the idea is to extend inequality results for multivalent functions obtained in our previous paper [9]:

$$\left|\arg\left[1+\frac{zf^{(p+1)}(z)}{f^{(p)}(z)}\right]\right| < \frac{\alpha\pi}{2} \quad (z\in\mathbb{D}) \quad \Rightarrow \quad \left|\arg\frac{zf^{(p)}(z)}{f^{(p-1)}(z)}\right| < \frac{\beta_1\pi}{2} \quad (z\in\mathbb{D})$$

and

$$\left|\arg\frac{zf^{(p)}(z)}{f^{(p-1)}(z)}\right| < \frac{\beta_1\pi}{2} \quad (z\in\mathbb{D}) \quad \Rightarrow \quad \left|\arg\frac{zf^{(p-1)}(z)}{f^{(p-2)}(z)}\right| < \frac{\beta_2\pi}{2} \quad (z\in\mathbb{D}),$$

with an aim to give sufficient conditions when

$$\left|\arg\left[1+\frac{zf^{(p+1)}(z)}{f^{(p)}(z)}\right]\right| < \frac{\alpha\pi}{2} \quad (z\in\mathbb{D}) \tag{1}$$

implies

$$\left|\arg\frac{zf'(z)}{f(z)}\right| < \frac{\beta\pi}{2} \quad (z\in\mathbb{D}). \tag{2}$$

Results related to this can be found in the work of Cho et al. (see [3] - [6]). In [16], a linear combination of the analytical expressions of starlikeness and convexity is studied as a necessary and sufficient condition for starlikeness of an analytic function. In [14], the author consider a class of analytic and multivalent functions to investigate some sufficient conditions for that class.

We will use method from the theory of differential subordinations to get our result. Comprehensive references on this topic are [10] and [2]. Here are basic definitions and notations.

Let $f(z), g(z) \in \mathcal{A}$. We say that $f(z)$ is *subordinate* to $g(z)$, and write $f(z) \prec g(z)$, if there exists a function $\omega(z)$, analytic in the unit disc $\mathbb{D}$, such that $\omega(0) = 0$, $|\omega(z)| < 1$ and $f(z) = g(\omega(z))$ for all $z \in \mathbb{D}$. Also, if $g(z)$ is univalent in $\mathbb{D}$ then $f(z) \prec g(z)$ if and only if $f(0) = g(0)$ and $f(\mathbb{D}) \subseteq g(\mathbb{D})$.

The general theory of differential subordinations, as well as the theory of first-order differential subordinations, was introduced by Miller and Mocanu in [11] and [12]. In fact, if $\phi : \mathbb{C}^2 \to \mathbb{C}$, $\mathbb{C}$ complex plane, is analytic in a domain D, if $h(z)$ is univalent in $\mathbb{D}$, and if $p(z)$ is analytic in $\mathbb{D}$ with $(p(z), zp'(z)) \in D$ when $z \in \mathbb{D}$, then $p(z)$ is said to satisfy a first-order differential subordination if

$$\phi(p(z), zp'(z)) \prec h(z). \tag{3}$$

A univalent function $q(z)$ is called a *dominant* of the differential subordination (3) if $p(z) \prec q(z)$ for all $p(z)$ satisfying (3). If $\widetilde{q}(z)$ is a dominant of (3) and $\widetilde{q}(z) \prec q(z)$ for all dominants of (3), then we say that $\widetilde{q}(z)$ is the *best dominant* of the differential subordination (3).

In [8] strong differential subordination and superordination results are obtained for analytic functions in the open unit disk, which are associated with an integral operator. The object of the [13] is to find a subclass of p-valent starlike (or $p-$ valent convex) functions in the unit disk which are mapped by certain integral operator onto $p-$valent starlike (or $p-$ valent convex) functions.

To obtain conditions when (1) implies (2) we will use the following lemma from the theory of differential subordinations.

Lemma 1.1. [10, Theorem 2.3i(i), p. 35] *Let* $\Omega \subset \mathbb{C}$ *and suppose that the function* $\psi : \mathbb{C}^2 \times \mathbb{D} \to \mathbb{C}$ *satisfies* $\psi(ix, y; z) \notin \Omega$ *for all* $x \in \mathbb{R}$, $y \leq -(1+x^2)/2$, *and* $z \in \mathbb{D}$. *If* $q \in H[1,1]$ *and* $\psi(q(z), zq'(z); z) \in \Omega$ *for all* $z \in \mathbb{D}$, *then* $\operatorname{Re} q(z) > 0$, $z \in \mathbb{D}$.

2. Main Results

Theorem 2.1. *Let* $f \in \mathcal{A}_p$, $p \geq 2$, $0 < \beta_{k-1} \leq 1$, $2 \leq k \leq p$, *k integer, and suppose that* $f^{(m)}(z) \neq 0$ *for all* $z \in \mathbb{D} \setminus \{0\}$ *and for all positive integer m. Now, let define a sequence* $\beta_k, (k = 1, 2, ..., p)$ *such that* $\beta_1 = \beta$ *and*

$$\beta_k = \beta_k(\beta_{k-1}) \equiv \operatorname{arcctg} \frac{-1 + x_*^{\beta_{k-1}} \cos \frac{\beta_{k-1}\pi}{2}}{\beta_{k-1}\frac{1+x_*^2}{2x_*} + x_*^{\beta_{k-1}} \sin \frac{\beta_{k-1}\pi}{2}}, \quad k = 2, 3, ..., p,$$

where x_* is the bigger, of the only two positive solutions of the equation

$$2x^{\beta_{k-1}+1} \sin \frac{\beta_{k-1}\pi}{2} + \left(\beta_{k-1}x^2 + \beta_{k-1} - x^2 + 1\right) x^{\beta_{k-1}} \cos \frac{\beta_{k-1}\pi}{2} + x^2 - 1 = 0.$$

Finally, let

$$\alpha \equiv \alpha(\beta_p) = \operatorname{arctg}\left[\frac{\beta_p}{1-\beta_p} \cdot \left(\frac{1-\beta_p}{1+\beta_p}\right)^{(1+\beta_p)/2} + \operatorname{tg}\frac{\beta_p\pi}{2}\right].$$

Then the following implication holds:

$$\left|\arg\left[1 + \frac{zf^{(p+1)}(z)}{f^{(p)}(z)}\right]\right| < \frac{\alpha\pi}{2} \quad (z \in \mathbb{D}) \quad \Rightarrow \quad \left|\arg\frac{zf'(z)}{f(z)}\right| < \frac{\beta\pi}{2} \quad (z \in \mathbb{D}).$$

Proof. First, we will show that

$$\left|\arg\frac{zf''(z)}{f'(z)}\right| < \frac{\beta_2\pi}{2} \quad \Rightarrow \quad \left|\arg\frac{zf'(z)}{f(z)}\right| < \frac{\beta_1\pi}{2}.$$

For that purpose we choose $q^{\beta_1}(z) = \frac{zf'(z)}{f(z)}$. So, we have

$$\frac{z[q^{\beta_1}(z)]'}{q^{\beta_1}(z)} = \frac{z\beta_1 q^{\beta_1-1}(z)q'(z)}{q^{\beta_1}(z)} = 1 + \frac{zf''(z)}{f'(z)} - q^{\beta_1}(z)$$

i.e.

$$\frac{zf''(z)}{f'(z)} = \frac{z\beta_1 q'(z)}{q(z)} + q^{\beta_1}(z) - 1.$$

Further, for the function $\psi(r,s;z)$ so that $\psi(r,s;z) = \beta_1 \frac{s}{r} + r^{\beta_1} - 1$, we have

$$\psi(q(z), zq'(z); z) = \beta_1 \frac{zq'(z)}{q(z)} + q^{\beta_1}(z) - 1 \in \Omega \equiv \{\omega : |\arg \omega| < \frac{\beta_2 \pi}{2}\},$$

i.e.

$$|\arg \psi(q(z), zq'(z); z)| < \frac{\beta_2 \pi}{2} \quad (z \in \mathbb{D}).$$

From Lemma 1.1 to prove

$$\left|\arg \frac{zf'(z)}{f(z)}\right| < \frac{\beta_1 \pi}{2} \quad (z \in \mathbb{D}),$$

we get that it is enough to show that

$$\psi(ix, y; z) = -\beta_1 \cdot \frac{y}{x} \cdot i + (ix)^{\beta_1} - 1 \notin \Omega$$

for all real $x, y \leq -\frac{1+x^2}{2}$ ($n = 1$ in the Lemma 1.1) and for all $z \in \mathbb{D}$.

In the case $x > 0$ we have

$$\text{ctg } [\arg \psi(ix, y; z)] = \frac{-1 + x^{\beta_1} \cos \frac{\beta_1 \pi}{2}}{-\beta_1 \frac{y}{x} + x^{\beta_1} \sin \frac{\beta_1 \pi}{2}} \leq h(x),$$

$$h(x) \equiv \frac{-1 + x^{\beta_1} \cos \frac{\beta_1 \pi}{2}}{\beta_1 \frac{1+x^2}{2x} + x^{\beta_1} \sin \frac{\beta_1 \pi}{2}}\,.$$

Similarly, in the case $x < 0$,

$$|\text{ctg}\, [\arg \psi(ix, y; z)]| = \left|\text{ctg } \left[\arg \left(-\beta_1 \cdot \frac{y}{|x|} \cdot i + (i|x|)^{\beta_1} - 1\right)\right]\right| \leq h(|x|).$$

$h(x)$ is continuous on $(0, +\infty)$, $h(0) = 0$, $\lim\limits_{x \to +\infty} h(x) > 0$, $h'(0) < 0$ and $\lim\limits_{x \to +\infty} h'(x) > 0$. Furthermore, $h(x)$ has exactly one local minimum (at point x_{**}) and exactly one local maximum (at point $x_* > x_{**}$) on $(0, +\infty)$ (explained in [9]). So,

$$\sup \left\{ |\arg \psi(ix, y; z)| : x > 0, y \leq -\frac{1+x^2}{2} \right\} = \text{arcctg}[h(x_*)] = \beta_2(\beta_1),$$

where $\beta_1 \equiv \beta$.

In a similar way we can show that the same is true also for $x < 0$.

When $x = 0$ we have

$$\lim_{|x| \to 0} |\arg \psi(ix, y; z)| = \lim_{x \to 0^+} \text{arcctg}[h(x)] = \frac{\pi}{2} \geq \beta_2(\beta_1).$$

Then for all $z \in \mathbb{D}$,

$$\left|\arg \frac{zf''(z)}{f'(z)}\right| < \frac{\beta_2 \pi}{2} \quad \Rightarrow \quad \left|\arg \frac{zf'(z)}{f(z)}\right| < \frac{\beta_1 \pi}{2}.$$

Next, we choose as before

$$q^{\beta_{t-1}}(z) = \frac{zf^{t-1}(z)}{f^{t-2}(z)}, \quad 3 \le t \le p$$

to prove implications

$$\left|\arg \frac{zf^{(t)}(z)}{f^{(t-1)}(z)}\right| < \frac{\beta_t \pi}{2} \quad (z \in \mathbb{D}) \quad \Rightarrow$$

$$\left|\arg \frac{zf^{(t-1)}(z)}{f^{(t-2)}(z)}\right| < \frac{\beta_{t-1} \pi}{2} \quad (z \in \mathbb{D}), \quad 3 \le t \le p.$$

Applying the same proof as before by iteration we receive

$$\psi(ix, y; z) = -\beta_{t-1} \cdot \frac{y}{x} \cdot i + (ix)^{\beta_{t-1}} - 1 \notin \Omega, \quad t = 3, ..., p,$$

for all real $x, y \le -\frac{1+x^2}{2}$ ($n = 1$ in the Lemma 1.1), for all $z \in \mathbb{D}$, and

$$\beta_t = \beta_t(\beta_{t-1}) \equiv \operatorname{arcctg} \frac{-1 + x_*^{\beta_{t-1}} \cos \frac{\beta_{t-1}\pi}{2}}{\beta_{t-1} \frac{1+x_*^2}{2x_*} + x_*^{\beta_{t-1}} \sin \frac{\beta_{t-1}\pi}{2}},$$

$t = 3, ..., p$.

Consequently, the following holds

$$\left|\arg \frac{zf^{(p)}(z)}{f^{(p-1)}(z)}\right| < \frac{\beta_p \pi}{2} \quad (z \in \mathbb{D}) \quad \Rightarrow \quad \left|\arg \frac{zf'(z)}{f(z)}\right| < \frac{\beta \pi}{2} \quad (z \in \mathbb{D}).$$

Now, it remains to show that

$$\left|\arg \left[1 + \frac{zf^{(p+1)}(z)}{f^{(p)}(z)}\right]\right| < \frac{\alpha \pi}{2} \quad \Rightarrow \quad \left|\arg \frac{zf^{(p)}(z)}{f^{(p-1)}(z)}\right| < \frac{\beta_p \pi}{2}$$

for all $z \in \mathbb{D}$.

To prove that implication we use the proof of Theorem 2.1 from article [9], so that β_1 we replaced by β_p

This completes the proof of the theorem. ■

For $p = 2$ we receive.

Corollary 2.2. *Let $f \in \mathcal{A}_2$, $0 < \beta_1 \le 1$, and suppose that $f'(z) \ne 0$ for all $z \in \mathbb{D} \setminus \{0\}$. Now, let $\beta_1 = \beta$ and*

$$\beta_2 = \beta_2(\beta_1) \equiv \operatorname{arcctg} \frac{-1 + x_*^{\beta_1} \cos \frac{\beta_1\pi}{2}}{\beta_1 \frac{1+x_*^2}{2x_*} + x_*^{\beta_1} \sin \frac{\beta_1\pi}{2}},$$

where x_* is the bigger, of the only two positive solutions of the equation

$$2x^{\beta_1+1}\sin\frac{\beta_1\pi}{2}+\left(\beta_1x^2+\beta_1-x^2+1\right)x^{\beta_1}\cos\frac{\beta_1\pi}{2}+x^2-1=0.$$

Then the following implication holds:

$$\left|\arg\frac{zf''(z)}{f'(z)}\right|<\frac{\beta_2\pi}{2}\quad(z\in\mathbb{D})\quad\Rightarrow\quad\left|\arg\frac{zf'(z)}{f(z)}\right|<\frac{\beta_1\pi}{2}\quad(z\in\mathbb{D}).$$

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