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Conditioned Extreme, Determination and Visualization

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Abstract. In mathematics, and especially in the application of mathematics in a number of areas, problems and tasks are encountered in which extremes of function are required, with the variables being bound up with some additional conditions. Extremes of this kind are called conditional extremes. [2]. In this paper, we present the procedure for determining a conditional extremity of a function with two variables, we will present that the point is finding the curve in the space obtained as a cross section on two surfaces and by using free software we will have visualization of the conditional extremum.

Key words: Conditioned extreme, function, point, visualization.

I. Introduction

The determination of the extreme value of a function under a given condition is called the procedure for determining the Conditional extreme.

We will describe the procedure for determining conditional extremes of a function with two variables below. We will specify which conditions are necessary to be assigned so that we can determine a conditional extremity of a function in space. But we will also show that the point in which the conditional extremal is located is a point in space [5], and not a point in the plane xOy. (*as described in some literature sources* [1], [2], [3], [4])

Let the function $z = f(x, y)$ be given, where x and y are two independent variables that represent a surface in space D. And let a certain condition $\varphi(x,y) = 0$ be given. The equation $\varphi(x,y) = 0$ in the plane xOy is a curve, but represented in a space it is a surface C normal on the xOy plane. C surface intersects the surface D on a curve $\mathcal{L}$.

The solution to this problem is to find the point M (x_0, y_0) in the equation $\varphi(x,y) = 0$ in which the function $z = f(x, y)$ has an extreme value, [1], [2], [3], [4]. But if we visualize this problem in space then we will have to find the point M (x_0, y_0, z_0) in the curve L in which the curve L will have an extremity [5]. The point M (x_0, y_0, z_0) is called the point of the conditional extremity of the function $z = f(x, y)$ and in the point M (x_0, y_0, z_0) the function $z = f(x, y)$ does not have an extreme value.

II. Methods for determining conditional extremes

How to determine the extreme value of a function under a given condition?

Determining the extreme value of the contour (curve $\mathcal{L}$) is reduced to the determination of the extreme value of the function $z = f(x, y)$ which is achieved under given conditions for $\varphi(x,y) = 0$. There are two methods for finding a conditional extreme.

1. Method of direct elimination

The extreme value of the function $z = f(x, y)$ under the condition $\varphi(x,y) = 0$ can be found such that y is expressed from the equation $\varphi(x,y) = 0$ in the explicit form $y = y(x)$ and is replaced in the function $z = f(x, y)$. In this way a function with one variable $z = f(x, y(x))$ is obtained, and the task is reduced to determining the extreme value of a function of an independent variable. The derivation of the function after the variable x at the point of the conditional extremity should be $\frac{dz}{dx} = 0$. Then the procedure for determining the extremity of a function with one variable is used.

The second procedure for determining a conditional extremal is used when the elimination of y is complicated and it is desirable to remain on the functions $z = f(x, y)$ and $\varphi(x,y) = 0$.

2. Lagrange method for multipliers

How is the Lagrange function (= Lagrangian expression) defined for a real function of two variables given by the equation $z = f(x, y)$ in the search of a conditional extremal given by the conditional equation $\varphi(x,y) = 0$?

First is calculate the derivative of z,

$$dz = \frac{\partial z}{\partial x} dx + \frac{\partial z}{\partial y} dy$$

By dividing with dx and we get

$$\frac{dz}{dx} = \frac{\partial z}{\partial x} + \frac{\partial z}{\partial y}\frac{dy}{dx} \text{ and because } \frac{dz}{dx} = 0 \text{ is getting}$$

$$\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y}\frac{dy}{dx} = 0 \tag{1}$$

The derivative $\frac{dy}{dx}$ is obtained by differentiating the conditional equation $\varphi(x,y)=0$ by x, i.e.

$$\frac{\partial \varphi}{\partial x} + \frac{\partial \varphi}{\partial y} \cdot \frac{dy}{dx} = 0 \tag{2}$$

From what we get

$$\frac{\partial \varphi}{\partial y} \cdot \frac{dy}{dx} = -\frac{\partial \varphi}{\partial x}$$

$$\frac{dy}{dx} = -\frac{\frac{\partial \varphi}{\partial x}}{\frac{\partial \varphi}{\partial y}}$$

If $\frac{dy}{dx}$ is replaced in (1) is obtained

$$\frac{\partial z}{\partial x}-\frac{\partial z}{\partial y}\frac{\frac{\partial \varphi}{\partial x}}{\frac{\partial \varphi}{\partial y}}=0 \qquad \text{or} \qquad \frac{\partial z}{\partial x}\cdot\frac{\partial \varphi}{\partial y}=\frac{\partial z}{\partial y}\cdot\frac{\partial \varphi}{\partial x}$$

If we mark the proportions with -λ, we get

$$\frac{\frac{\partial z}{\partial x}}{\frac{\partial \varphi}{\partial x}}=\frac{\frac{\partial z}{\partial y}}{\frac{\partial \varphi}{\partial y}}=-\lambda$$

where λ is a constant called the Lagrange multiplier.

The values of x, y, λ for which the function has an extremum will be found by solving the system obtained from the three equations

$$\frac{\frac{\partial z}{\partial x}}{\frac{\partial \varphi}{\partial x}}=-\lambda\,, \qquad \frac{\frac{\partial z}{\partial y}}{\frac{\partial \varphi}{\partial y}}=-\lambda\,, \qquad \varphi(x,y)=0$$

i.e. system equations are

$$\begin{cases} \frac{\partial z}{\partial x}+\lambda\frac{\partial \varphi}{\partial x}=0 \\ \frac{\partial z}{\partial y}+\lambda\frac{\partial \varphi}{\partial y}=0 \\ \varphi(x,y)=0 \end{cases} \qquad (3)$$

System solutions give the values for x, y and λ for which the function can have an extremal value at the given condition.

With additional tests or on the basis of the nature of the task, it is concluded whether that point is a minimum or a maximum.

III. Calculation of conditional extremes with the Lagrange method

What is the procedure for determining the conditional extremes with Lagrange method of undetermined multipliers

$$F(x,y,\lambda)=f(x,y)+\lambda\cdot\varphi(x,y)$$

From the functions $z=f(x,y)$ and the condition $\varphi(x,y) = 0$ by introducing a multiplier λ a new function is formed, so called the **Lagrange function** with the equation

$$F(x,y,\lambda)=f(x,y)+\lambda\cdot\varphi(x,y)$$

It determines the values of x, y and λ that satisfy the system equations (3) where the left sides are partial derivatives of the function $F(x,y,\lambda)$

$$\begin{cases} F_x{}' = 0 \\ F_y{}' = 0 \\ F_\lambda{}' = 0 \end{cases}$$

$$d^2F(x_k, y_k, \lambda_k) = F''_{xx}dx^2 + 2F''_{xy}dxdy + F''_{yy}dy^2$$

For obtained values (x_k, y_k, λ_k) the values of the expression are determined,

$$d^2F(x_k, y_k, \lambda_k) = F''_{xx}dx^2 + 2F''_{xy}dxdy + F''_{yy}dy^2$$

Provided that $\varphi(x, y) = 0$.

If

- $d^2F > 0$, then the function $z = f(x,y)$ has a minimum in the point $M_k(x_k, y_k, z_k)$ of the curve $\mathcal{L}$,
- $d^2F < 0$, then the function $z = f(x,y)$ has a maximum in the point $M_k(x_k, y_k, z_k)$ of the curve $\mathcal{L}$.

Example:

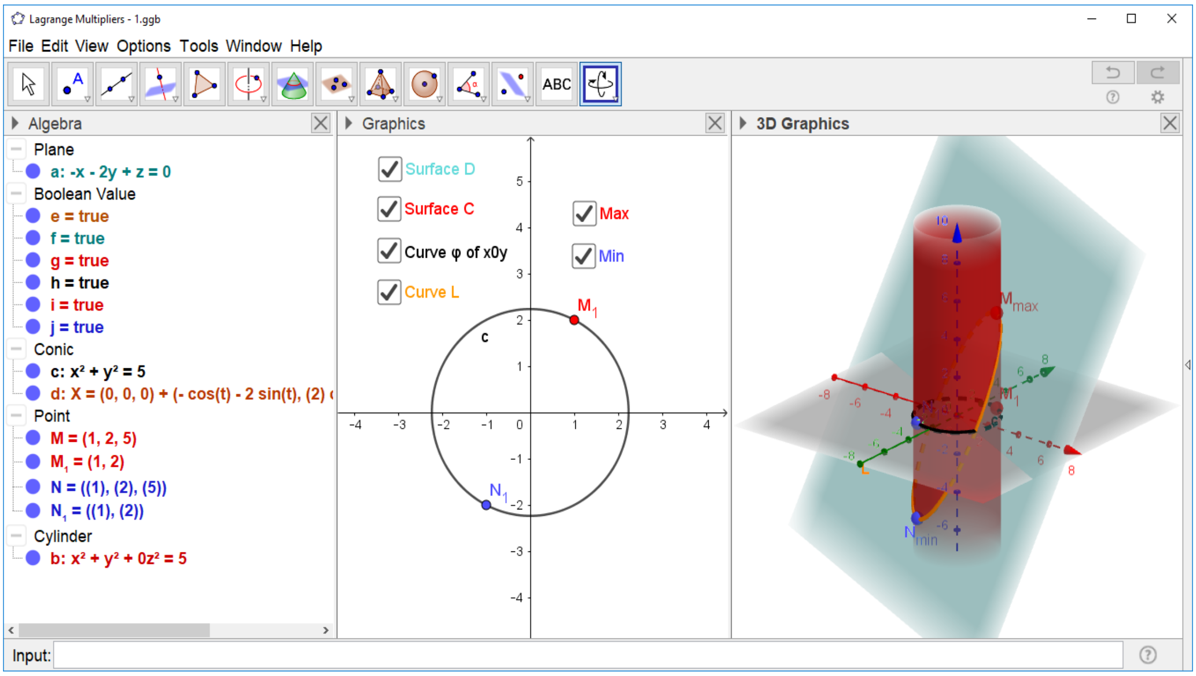
Determine the extreme value of the function $z = f(x, y) = x + 2y$ on condition $x^2 + y^2 = 5$ i.e $\varphi(x, y) = x^2 + y^2 - 5$.

Visualization in GeoGebra

$z = f(x, y) = x + 2y$ is the surface D

$\varphi(x, y) = x^2 + y^2 - 5$ is the surface C

L is the curve in the space that is the intersection of D and C.



Solution:

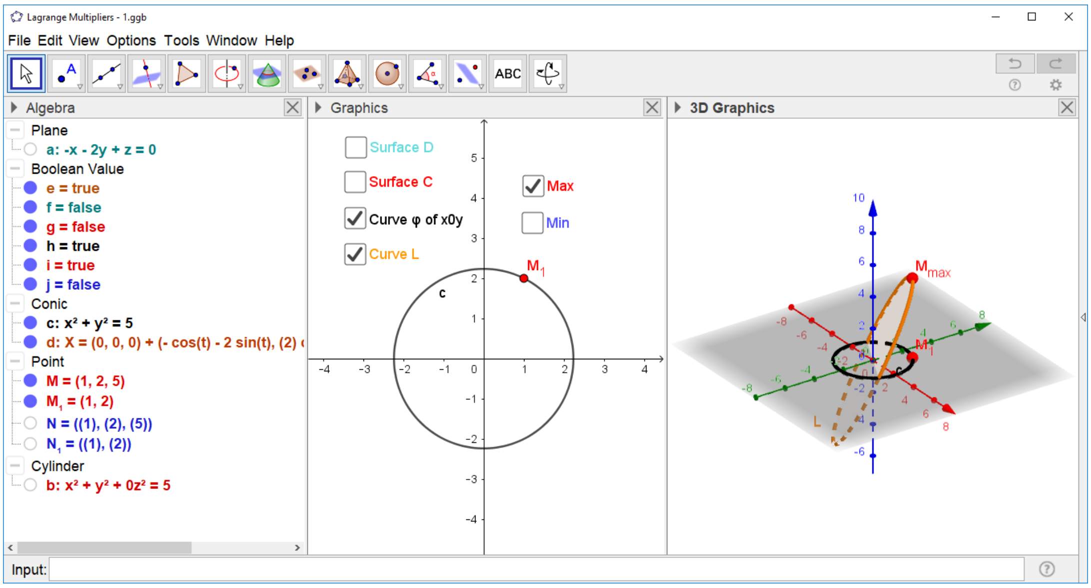
With the Lagrange method of multipliers, we form the function

$$F(x,y,\lambda) = x + 2y + \lambda \cdot (x^2 + y^2 - 5)\,.$$

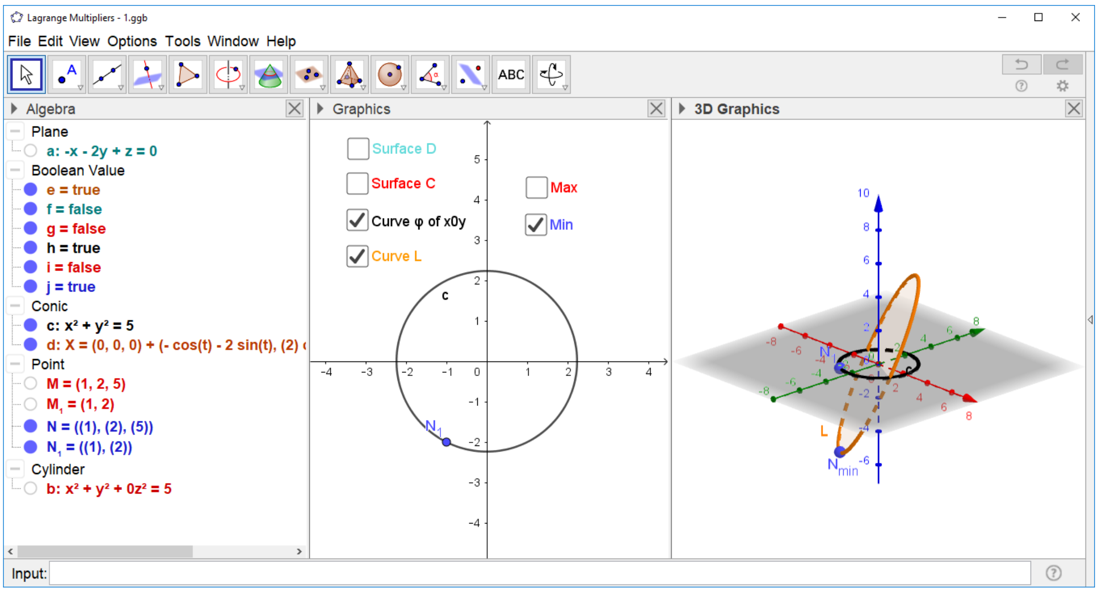
Partial derivatives are determined, equated to 0, and the system equations are obtained:

$$\begin{cases} F_x{}' = 0 \\ F_y{}' = 0 \\ F_\lambda{}' = 0 \end{cases} \text{т.е.} \begin{cases} 1 + 2\lambda x = 0 \\ 2 + 2\lambda y = 0 \\ x^2 + y^2 - 5 = 0 \end{cases}$$

We get a solution to the system $\lambda_1 = -\dfrac{1}{2}$ and $x_1 = 1, y_1 = 2$ i.e. point $M_1(1,2)$. If we replace it in the equation $z = f(x,y) = x + 2y$, then we get $z_1 = 5$. We got a stationary point $M(1,2,5)$ from L in which the function can have a conditioned extremity.



We get a solution to the system $\lambda_2 = \dfrac{1}{2}$ и $x_2 = -1, y_2 = -2$ i.e. point $N_1(-1,-2)$. If we replace it in the equation $z = f(x,y) = x + 2y$, then we get $z_2 = -5$. We got another stationary point $N(-1,-2,-5)$ from L in which the function can have a conditioned extremity.



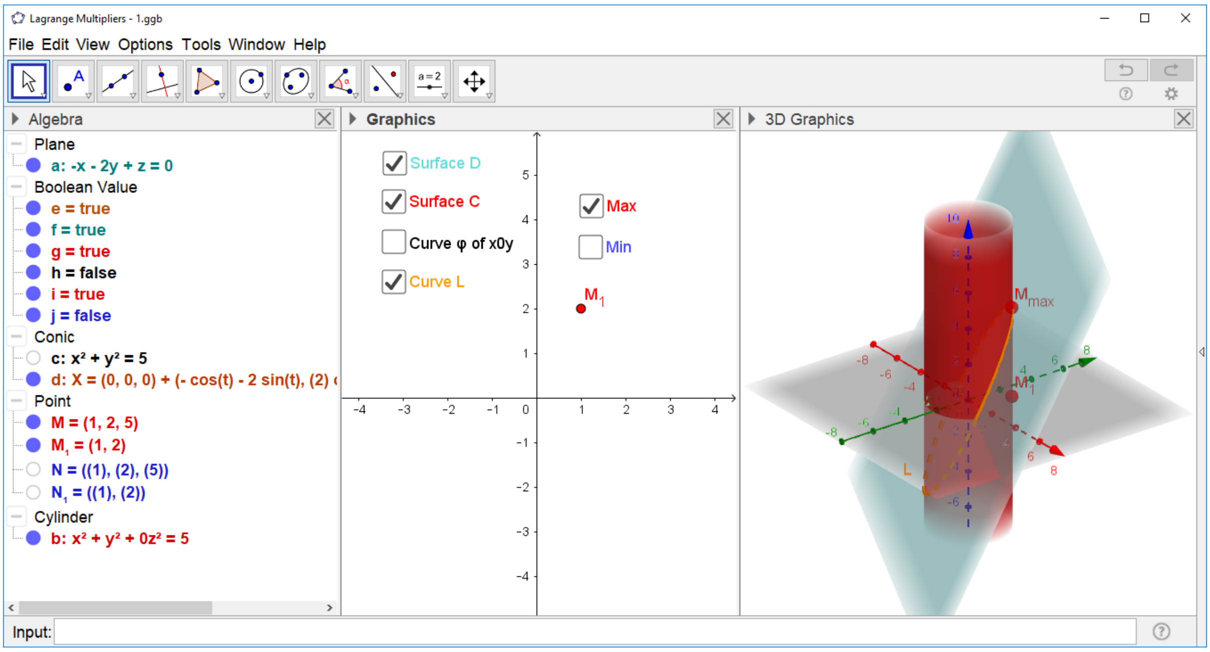
The second derivatives of the function $F(x,y,\lambda)=x+2y+\lambda\cdot(x^2+y^2-5)$ are:

$$F^{"}_{xx}=2\lambda \quad F^{"}_{yy}=2\lambda \quad F^{"}_{xy}=0$$

For $\lambda_1=-\dfrac{1}{2}$ and $x_1=1, y_1=2$, we get $F^{"}_{xx}=-1 \quad F^{"}_{yy}=-1 \quad F^{"}_{xy}=0$. The value of the expression

$$d^2F=F^{"}_{xx}dx^2+2F^{"}_{xy}dxdy+F^{"}_{yy}dy^2=-1dx^2-1dy^2<0.$$

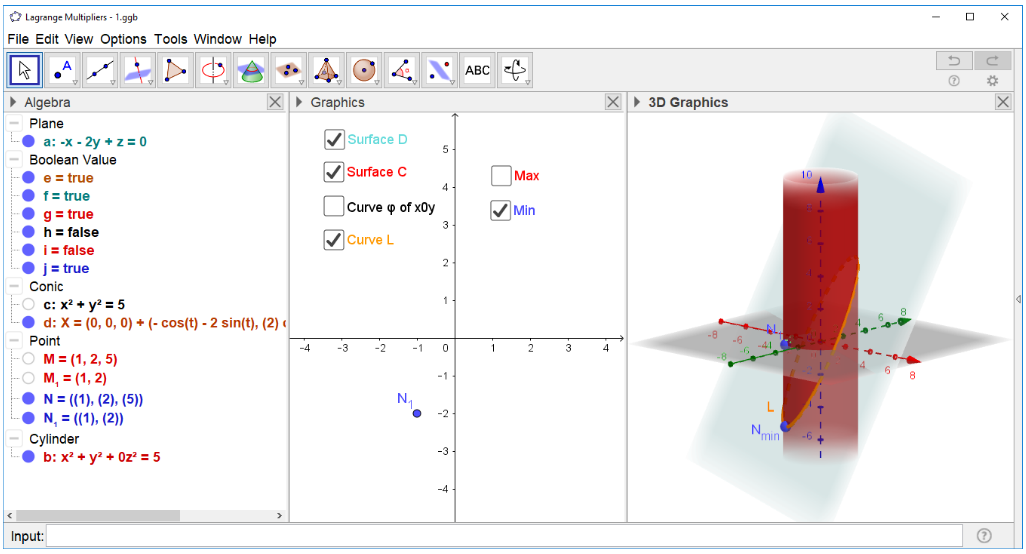
Follows the function has a conditioned maximum in the point $M(1,2,5)=M_{max}$



For $\lambda_2=-\dfrac{1}{2}$ and $x_2=-1, y_2=-2$, we get $F^{"}_{xx}=1 \quad F^{"}_{yy}=1 \quad F^{"}_{xy}=0$. The value of the expression

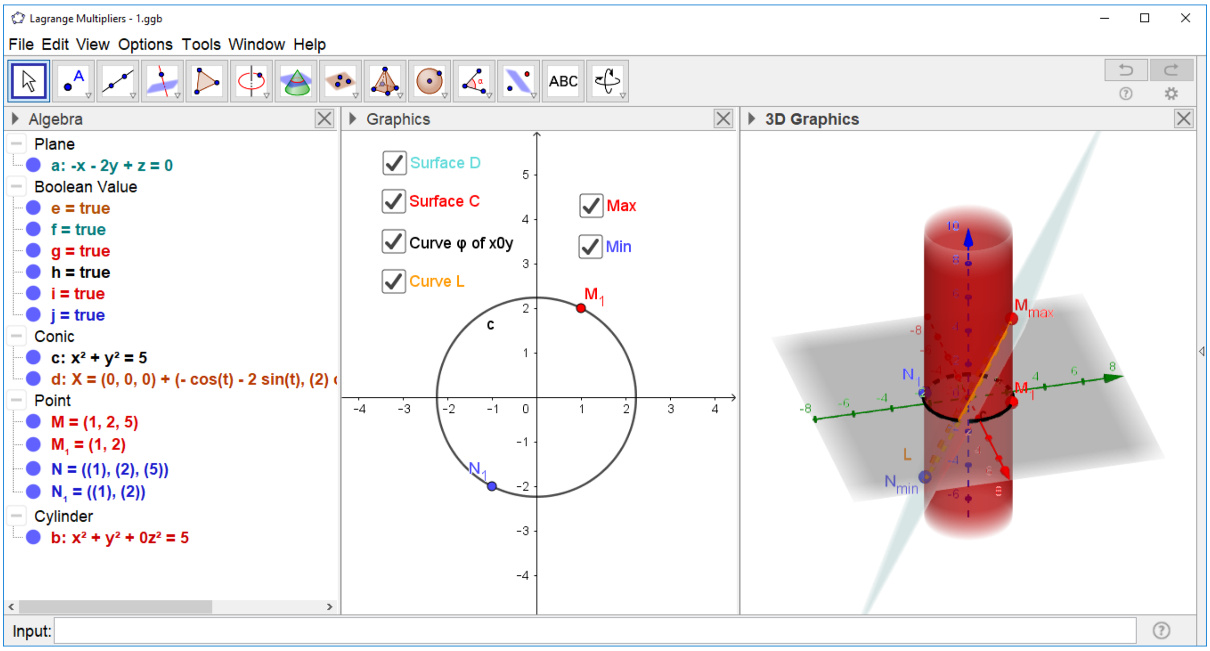
$$d^2F=F^{"}_{xx}dx^2+2F^{"}_{xy}dxdy+F^{"}_{yy}dy^2=1dx^2+1dy^2>0.$$

Follows the function has a conditioned minimum in the point $N(-1,-2,-5)=N_{min.}$



Visualization of the example in GeoGebra.

The functions are entered through the input bar in GeoGebra in the drawing pad and the 3D graph. The intersection of the objects is determined the curve L is obtained for which the extreme value is determined. After the algebraic determination of the extreme values of the value bar, the points are entered and visually represented in 3D. Checkboxes specify what will be displayed.



IV Conclusion

A conditioned extreme is a small problem of mathematics that has been applied in other areas. To understand the laws that apply to solving these problems, and to have greater application, it is necessary to visualize them. Visualization should use software that visually poses a problem. In this visualization problem we introduced the point of the conditional extremum and showed that it is a point that lies on the curve in the space L, not the curve in *xOy*.

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