PRODUCTS OF DISTRIBUTIONS IN COLOMBEAU ALGEBRA

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Introduction

- Products of distributions in Colombeau algebra

- Large employment of Schwartz's distributions
- Two main problems for distributional theory:

  - **Product of distributions** (not any two distributions can always be multiplied)

  - **differentiating the product of distributions** (the product of distributions not always satisfy the Leibniz rule)

Introduction

- Two complementary points of view:
- Continuous linear functional f

$$\varphi \to \langle f, \varphi \rangle$$

- φ - smooth function with compact support (test function)

- sequential approach

$$\varphi_n \to \delta(x)$$

$$f_n(x) = (f * \varphi_n)(x) = \langle f(y), \varphi_n(x-y) \rangle$$

$$f_n \to f$$

- representatives of f

- nets of regularization $\varphi_\varepsilon = \frac{1}{\varepsilon}\varphi\left(\frac{x}{\varepsilon}\right)$

Introduction

- Non-linear structure is lost in a way identifying sequences with their limit
- All the operations then are done with the regularized functions (the sequences of smooth functions)
- With the inverse process starting from the result, the function is returned from the regularization

Introduction

- Jean-Francois Colombeau

- New generalized functions and multiplication of distributions, 1984
- Colombeau algebra $\mathcal{G}(\mathbf{R})$

- New theory of generalized functions - more general then the theory of Schwartz's distributions

Introduction

- Diferentiation - same properties as distributions
- Multiplication and nonlinear operations - properties different from distributions
- Any finite product of generalized functions is still generalized function
- The algebra of these generalized functions is closed with respect to many nonlinear operations
- Any finite product of distributions is a generalized function and not a distribution in general

Colombeau algebra

- Associative differential algebra of generalized functions, containing the algebra of smooth functions as a subalgebra
- The distribution space $\mathcal{D}'$ is linearly embedded in it as a subspace
- Multiplication is compatible with the operations of differentiation and products with C^∞ - differentiable functions

Colombeau algebra

- $\mathbf{N_0} = \mathbf{N} \cup \{0\}$ - set of non-negative integers
- $\mathcal{D}(\mathbf{R})$ - the space of all C^∞ - functions $\varphi : \mathbf{R} \to \mathbf{C}$ with compact support
- For $j \in \mathbf{N_0}$ and $q \in \mathbf{N_0}$ we denote

$$A_q(\mathbf{R}) = \left\{ \varphi(x) \in \mathcal{D}(\mathbf{R}) \middle| \int_{\mathbf{R}} \varphi(x)\, dx = 1; \int_{\mathbf{R}} x^j \varphi(x)\, dx = 0, j = 1, \dots, q \right\}$$

- $A_1 \supset A_2 \supset A_3 \dots$ and $A_k \neq \varnothing$, $k \in \mathbf{N}$

Colombeau algebra

- $$\varphi_{\varepsilon}(x) = \frac{1}{\varepsilon}\varphi\left(\frac{x}{\varepsilon}\right) \qquad \check{\varphi}(x) = \varphi(-x) \qquad \varepsilon > 0$$
- ***Deffinition***: $\mathcal{E}(\mathbf{R})$ - algebra of functions $f(\varphi, x)$
  $f(\varphi, x): A_0(\mathbf{R}) \times \mathbf{R} \to \mathbf{C}$ - infinitely differentiable for fixed 'parameter‘ φ
- Embedding of distributions - in a way that the embedding of C^{∞} - functions will be identity

Colombeau algebra

- f , g - smooth functions
- $\left(f*\varphi_{\varepsilon}\right)_{\varepsilon>0}$ - an embedding of f
- $\left(g*\varphi_{\varepsilon}\right)_{\varepsilon>0}$ - an embedding of g

$$\left(f*\varphi_{\varepsilon}\right)\left(g*\varphi_{\varepsilon}\right)\neq\left(fg\right)*\varphi_{\varepsilon}$$

Colombeau algebra

- The generalized functions of Colombeau are elements of the quotient algebra

$$\mathcal{G} \equiv \mathcal{G}\left(\mathbf{R}\right) = \frac{\mathcal{E}_M\left[\mathbf{R}\right]}{\mathcal{I}\left[\mathbf{R}\right]}$$

Colombeau algebra

- $\mathcal{E}_M[\mathbf{R}]$ - subalgebra of 'moderate' functions such that for each compact subset K of $\mathbf{R}$ and any $p \in \mathbf{N}_0$ there is $q \in \mathbf{N}$ such that for each $\varphi \in A_q(\mathbf{R})$ there are $c > 0, \eta > 0$ and it holds

$$\sup_{x \in K} \left| \partial^p f(\varphi_\varepsilon, x) \right| \leq c\varepsilon^{-q}$$

for $0 < \varepsilon < \eta$.

Colombeau algebra

- $\mathcal{I}[\mathbf{R}]$ is an ideal of $\mathcal{E}_M[\mathbf{R}]$ consisting of all functions $f(\varphi, x)$ such that for each compact subset K of $\mathbf{R}$ and any $p \in \mathbf{N}_0$ there is $q \in \mathbf{N}$ such that for every $r \geq q$ and each $\varphi \in A_r(\mathbf{R})$ there are $c > 0, \eta > 0$ and it holds

$$\sup_{x \in K} \left| \partial^p f(\varphi_\varepsilon, x) \right| \leq c\varepsilon^{r-q}$$

for $0 < \varepsilon < \eta$.

Colombeau algebra

- $\mathcal{G}(\mathbf{R})$ contains the distributions on $\mathbf{R}$ canonically embedded by the map

$$i:\mathcal{D}'(\mathbf{R})\to\mathcal{G}(\mathbf{R}):\ u\to\ \tilde{u}=\left\{\tilde{u}(\varphi,x)=\left(u*\overset{\vee}{\varphi}\right)(x):\varphi\in A_q(\mathbf{R})\right\}$$

- $$(f*g)(x)=\int_{\mathbf{R}} f(y)g(x-y)dy$$

- $$\tilde{u}(\varphi,x)=\langle u(y),\varphi(y-x)\rangle$$

Colombeau algebra

- Generalized functions $f, g \in \mathcal{G}(\mathbf{R})$ are said to be ***associated*** $\left(f \approx g \right)$ if for some representatives $f\left(\varphi_\varepsilon, x\right)$ and $g\left(\varphi_\varepsilon, x\right)$ and arbitrary $\psi(x) \in \mathcal{D}(\mathbf{R})$ there is a $q \in \mathbf{N_0}$ such that for any $\varphi(x) \in A_q(\mathbf{R})$

$$\lim_{\varepsilon \to 0_+} \int_{\mathbf{R}} \left| f\left(\varphi_\varepsilon, x\right) - g\left(\varphi_\varepsilon, x\right) \right| \psi\left(x\right) dx = 0$$

Colombeau algebra

- A generalized function $f \in \mathcal{G}$ is said to *admit some* $u \in \mathcal{D}'(\mathbf{R})$ *as 'associated distribution'* $(f \approx u)$ if for some representative $f(\varphi_\varepsilon, x)$ and any $\psi(x) \in \mathcal{D}(\mathbf{R})$ there is a $q \in \mathbf{N_0}$ such that for any $\varphi(x) \in A_q(\mathbf{R})$

$$\lim_{\varepsilon \to 0_+} \int_{\mathbf{R}} f(\varphi_\varepsilon, x)\psi(x)\,dx = \langle u, \psi \rangle$$

Colombeau algebra

- ✓ Above definitions are independent of the representatives chosen
- ✓ The distribution associated, if it exists, is unique
- ✓ The association is a faithful generalization of the equality of distributions

Colombeau algebra

- Product of two distributions in $\mathcal{G}$ is in general Colombeau generalized function which may not always be associated to the third distribution
- *By* ***Colombeau product of distributions*** *is meant the product of their embeddings in* $\mathcal{G}$ *whenever the result admits an associated distribution*
- If the regularized model product of two distributions exists, then their Colombeau product also exists and it is same with the first one

Colombeau products of distributions

- **Theorem1**: The product of the generalized functions $\widetilde{(\cos x - \sin x)}$ and $\widetilde{\delta^{(r)}(x)}$ for $r = 0, 1, 2, \ldots$ in $\mathcal{G}(\mathbf{R})$ admits associated distribution and it holds:

$$\widetilde{(\cos x - \sin x)} \cdot \widetilde{\delta^{(r)}(x)} \approx \sum_{i=0}^{r} \binom{r}{i} (-1)^{b_i} \delta^{(r-i)}(x)$$

where $b_0 = 1;\ b_1 = 3,\ b_n = b_{n-2} + 5$ for $n \geq 2$.

Colombeau products of distributions

- **Theorem2**: The product of the generalized functions $\widetilde{(\sin x + \cos x)}$ and $\widetilde{\delta^{(r)}}(x)$ for $r = 0, 1, 2, \ldots$ in $\mathcal{G}(\mathbf{R})$ admits associated distribution and it holds:

$$\widetilde{(\sin x + \cos x)} \cdot \widetilde{\delta^{(r)}}(x) \approx \sum_{i=0}^{r} \binom{r}{i} (-1)^{b_i} \delta^{(r-i)}(x)$$

where $b_0 = 1;\ b_1 = 4,\ b_n = b_{n-2} + 5$ for $n \geq 2$.

Colombeau products of distributions

- **Theorem3**: The product of the generalized functions $\widetilde{e^x}$ and $\widetilde{\delta^{(r)}(x)}$ for $r = 0, 1, 2, \ldots$ in $\mathcal{G}(\mathbf{R})$ admits associated distribution and it holds:

$$\widetilde{e^x} \cdot \widetilde{\delta^{(r)}}(x) \approx \sum_{i=0}^{r} \binom{r}{i} (-1)^{1+i} \delta^{(r-i)}(x)$$

THANK YOU FOR YOUR ATTENTION