In ΔABC:

$$\sum a^2 \geq \sum (b-c)^2 + 4S\sqrt{\frac{2(2R-r)}{R}}$$

Proposed by Martin Lukarevski-Stip-Macedonia

Solution 1 by Kevin Soto Palacios – Huarmey – Peru

Solution 2 by Soumava Chakraborty-Kolkata-India

Solution 1 by Kevin Soto Palacios – Huarmey – Peru

Probar en un triángulo ABC

$$\sum a^2 \geq \sum (b-c)^2 + 4S\sqrt{\frac{2(2R-r)}{R}}$$

Tener en cuenta las siguientes identidades

$$ab = 2S\csc A,$$

$$bc = 2s\csc A,$$

$$ca = 2S\csc B,$$

$$a^2 + b^2 + c^2 = 4S(\cot A + \cot B + \cot C)$$

Si $\rightarrow \frac{A}{2} + \frac{B}{2} + \frac{C}{2} = \frac{\pi}{2} \leftrightarrow xy + yz + zx = 1 \leftrightarrow x = \tan\frac{A}{2}, y = \tan\frac{B}{2}, z = \tan\frac{C}{2}$

La desigualdad es equivalente

$$2(ab + bc + ca) - (a^2 + b^2 + c^2) \geq 4S\sqrt{4 - \frac{2r}{R}}$$

$$4S\big((\csc A - \cot A) + (\csc B - \cot B) + (\csc C - \cot C)\big) \geq 4S\sqrt{4 - \frac{2r}{R}}$$

$$\Leftrightarrow \tan\frac{A}{2}+\tan\frac{B}{2}+\tan\frac{C}{2}\geq\sqrt{4-\frac{2r}{R}}$$

$$\Leftrightarrow \tan^2\frac{A}{2}+\tan^2\frac{B}{2}+\tan^2\frac{C}{2}+\frac{2r}{R}=\tan^2\frac{A}{2}+\tan^2\frac{B}{2}+\tan^2\frac{C}{2}+8\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}\geq 2$$

1) Si: $m,n,p\in\mathbb{R}$ en un triángulo xyz, se cumple la siguiente desigualdad:

$$m^2+n^2+p^2\geq 2np\cos x+2pm\cos y+2mn\cos z$$

Siendo: $m=\tan\frac{x}{2}>0, y=\tan\frac{y}{2}>0, z=\tan\frac{z}{2}>0$, ***resulta:***

$$\tan^2\frac{x}{2}+\tan^2\frac{y}{2}+\tan^2\frac{z}{2}\geq 2\tan\frac{y}{2}\tan\frac{z}{2}\cos x+2\tan\frac{z}{2}\tan\frac{x}{2}\cos y+2\tan\frac{x}{2}\tan\frac{y}{2}\cos z$$

$$\tan^2\frac{x}{2}+\tan^2\frac{y}{2}+\tan^2\frac{z}{2}\geq 2\tan\frac{y}{2}\tan\frac{z}{2}\cos x+2\tan\frac{z}{2}\tan\frac{x}{2}\cos y+2\tan\frac{x}{2}\tan\frac{y}{2}\cos z$$

Probaremos que

$$2\tan\frac{y}{2}\tan\frac{z}{2}\cos x+2\tan\frac{z}{2}\tan\frac{x}{2}\cos y+2\tan\frac{x}{2}\tan\frac{y}{2}\cos z=2-8\sin\frac{x}{2}\sin\frac{y}{2}\sin\frac{z}{2}$$

$$\Leftrightarrow 2\left(\tan\frac{y}{2}\tan\frac{z}{2}-1\right)\cos x+2\left(\tan\frac{z}{2}\tan\frac{x}{2}-1\right)\cos y+2\left(\tan\frac{x}{2}\tan\frac{y}{2}-1\right)\cos z+$$
$$+2(\cos x+\cos y+\cos z)$$

$$\Leftrightarrow \frac{2\cos\left(\frac{y+z}{2}\right)}{\cos\frac{y}{2}\cos\frac{z}{2}}\cos x-\frac{2\cos\left(\frac{z+x}{2}\right)}{\cos\frac{z}{2}\cos\frac{x}{2}}\cos y-\frac{2\cos\left(\frac{x+y}{2}\right)}{\cos\frac{x}{2}\cos\frac{y}{2}}\cos z+2\left(1+4\sin\frac{x}{2}\sin\frac{y}{2}\sin\frac{z}{2}\right)$$

$$\Leftrightarrow -\frac{(\sin 2x+\sin 2y+\sin 2z)}{2\cos\frac{x}{2}\cos\frac{y}{2}\cos\frac{z}{2}}+2\left(1+4\sin\frac{x}{2}\sin\frac{y}{2}\sin\frac{z}{2}\right)=$$

$$=-16\sin\frac{x}{2}\sin\frac{y}{2}\sin\frac{z}{2}+2\left(1+4\sin\frac{x}{2}\sin\frac{y}{2}\sin\frac{z}{2}\right)$$

Por la tanto

$$\tan^2\frac{x}{2}+\tan^2\frac{y}{2}+\tan^2\frac{z}{2}\geq 2-8\sin\frac{x}{2}\sin\frac{y}{2}\sin\frac{z}{2}$$

$$\Leftrightarrow \tan^2\frac{x}{2}+\tan^2\frac{y}{2}+\tan^2\frac{z}{2}+8\sin\frac{x}{2}\sin\frac{y}{2}\sin\frac{z}{2}\geq 2 \quad \textit{(LQQD)}$$

Solution 2 by Soumava Chakraborty-Kolkata-India

$$\sum a^2 \overset{(1)}{\underset{?}{\geq}} \sum (b-c)^2 + 4s\sqrt{\frac{2(2R-r)}{R}}$$

$$\textbf{(1)} \quad \Leftrightarrow 2\sum ab - \sum a^2 \geq 4rs\sqrt{\frac{2(2R-r)}{R}}$$

$$\Leftrightarrow 2(s^2+4Rr+r^2)-2(s^2-4Rr-r^2)\geq 4rs\sqrt{\frac{2(2R-r)}{R}}$$

$$\Leftrightarrow 4r(4R+r)\geq 4rs\sqrt{\frac{2(2R-r)}{R}}$$

$$\Leftrightarrow R(4R+r)^2\geq 2s^2(2R-r)=s^2(4R-2r) \quad \textbf{(2)}$$

Using Rouche's inequality,

$$s^2(4R-2r)\leq (4R-2r)(2R^2+10Rr-r^2)+$$

$$+2(4R-2r)(R-2r)\sqrt{R^2-2Rr}\overset{?}{\leq} R(4R+r)^2\sqrt{R^2-2Rr}$$

$$\Leftrightarrow 8R^3-28R^2r+25Rr^2-2r^3\overset{?}{\geq} 4(R-2r)(2R-r)$$

$$\Leftrightarrow (R-2r)(8R^2-12Rr+r^2)\overset{?}{\geq} 4(R-2r)(2R-r)\sqrt{R^2-2Rr}$$

$$\Leftrightarrow 8R^2-12Rr+r^2\overset{?}{\geq} 4(2R-r)\sqrt{R^2-2Rr}\ (\because R-2r\geq 0) \quad \textbf{(3)}$$

Now, $8R^2-12Rr+r^2=r^2(8t^2-12t+1)$ ***(where*** $t=\frac{R}{r}$***)***

$$=r^2\{8(t^2-4)-12(t-2)+32-24+1\}$$

$$= r^2\{(t-2)(8t+16-12)+9\} = r^2\{(t-2)(8t+4)+9\}$$

$$> 0, \because t = \frac{R}{r} \geq 2 \text{ (Euler)}$$

$$\therefore \textit{(3)} \Leftrightarrow (8R^2 - 12Rr + r^2)^2 - 16(2R-r)^2(R^2 - 2Rr) \geq 0$$

$$\Leftrightarrow 16R^2r^2 + 8Rr^3 + r^4 \overset{?}{\geq} 0 \rightarrow \textit{true}$$

$$\Rightarrow \textit{(3) is true} \Rightarrow \textit{(2) is true} \Rightarrow \textit{(1) is true (Proved)}$$