Anything goes theorem, incomplete markets and Ricardian equivalence hypothesis

Dushko Josheski[1]

In this paper, Anything Goes: The Sonnenschein-Mantel-Debreu theorem (following the work of Sonnenchein (1972, 1973), Mantel (1974), and Debreu (1974)), has been applied to incomplete markets, Bottazzi, J.-M. and T. Hens (1996), in order to test the Ricardian equivalence hypothesis. In the naïve economic environment where public debt has a perfect substitute in lump-sum taxes the RET fails if one allows payoff matrix of economic agents to vary. If the law of one price does not apply (if any two portfolios have equal payoffs than their prices should be equal too) and that the payoffs are risk free.

Keywords: Anything Goes, RET hypothesis, incomplete markets, and equilibrium.

JEL classification: C62, D50

[1] Dushko Josheski, PhD Assistant professor, University Goce Delcev-Shtip, dushkojosheski@gmail.com

Incomplete markets and excess demand functions

First, it is assumed n-commodity, m-consumers in a pure trade economy with a set of $\{u_i\}$, strictly quasi concave (the negative of quasiconvex) , monotone utility functions. $f: \mathbb{R}^l \to R$ is strictly quasi concave if $f(\lambda x_1 + (1-\lambda)x_2) > \min\{f(x_1), f(x_2)\}$,holds for all x_1 , $x_2 \in \mathbb{R}^l$, with $x_1 \neq x_2$ and all $\lambda \in (0,1)$. [2] For each n+1 commodity in the Walrasian equilibrium model it is generated excess demand correspondence $(f^1, f^2 \dots \dots \dots . f^{l+1})$, Sonnenschein (1973), and the price functions and their domain is given as $P = \{p_1, p_2 \dots \dots p_n\}: p_i > 0 , \forall\, i$. For the purpose of analysis, it is assumed two-commodity economy, and that there are no restrictions on the demand functions, i.e. demand depends on total income, the only restriction is $p_2x_2 = y - p_1x_1$ where y is income. The condition for continuity, Robbin, Joel W. (2010), here states that f is said to be continuous on $\mathbb{R}^l$ if :

equation 1

$$\forall x_0 \in \mathbb{R}^l \forall \epsilon > 0\ \exists \delta > 0\ \forall x \in \mathbb{R}^l [|x - x_0| < \delta\ | \Rightarrow f(x) - f(x_0) < \epsilon|]$$

In previous condition ϵ is trimmed price space [3], x_0 is vector parameter , hence why the PDF is of a form $f_{x_0}(x) = (x - x_0)$ .One such function as in equation 1 would be $f(x) = x$. And the previous function is not continuous is:

equation 2

$$\forall x_0 \in \mathbb{R}^l \forall \epsilon > 0\ \exists \delta > 0\ \forall x \in \mathbb{R}^l [|x - x_0| < \delta \text{ and } |f(x) - f(x_0) \geq \epsilon|].$$

Uniformly continuous function is given as :

[2] Quasi concave function would be $f(\lambda x + (1-\lambda)y) \geq \min\{f(x), f(y)\}$

[3] Trimmed space as a location parameter class of probability functions that is parametrized by scalar or vector valued parameter x_0 which determines distributions or shift of the distribution.

equation 3

$$\forall \epsilon > 0 \ \exists \delta > 0 \forall x_0 \in \mathbb{R}^l \forall x \in \mathbb{R}^l [|x - x_0| < \delta \ | \Rightarrow f(x) - f(x_0) < \epsilon |]$$

And is not strictly uniformly continuous function if :

equation 4

$$\forall \epsilon > 0 \ \exists \delta > 0 \forall x_0 \in \mathbb{R}^l \forall x \in \mathbb{R}^l [|x - x_0| < \delta \ | \text{and } f(x) - f(x_0) \geq \epsilon |]$$

Now if $f: [a, b] \to R$ and for some constant K which is called Lipschitz constant for $\forall x_1, x_2 \in [a, b]$, then :

equation 5

$$|f(x_1) - f(x_2)| \leq K|x_1 - x_2|$$

Or:

Equation 6

$$d_R(f(x_1) - f(x_2)) \leq K d_{\mathbb{R}^l}(x_1 - x_2)^4$$

Then the functions is called Lipschitz function and one can write $\in \text{Lip}(a, b)$. Inverse mapping of Lipschitz is $f^{-1}: f(x) = x$. This continuous function $f: \mathbb{R}^n \to R$ is an excess demand function of an n-commodity if for every p in $\mathbb{R}^l$, Walras' Law with Equality $pf(p) = 0$, Debreu, (1974).But first let $\mathbb{R}^n = \{p \in R | \ p \gg 0, ||p|| = 1\}$, which is the set of positive price vectors in Euclidean form. Debreu (1974), defines also consumer as a pair $(\lesssim . \omega)$, where e is endowment vector in R_+. Than a function $f: \mathbb{R}^l \to R$ is said to be individual excess demand function of the consumer $(\lesssim . \omega)$ if for every price p in $\mathbb{R}^l$, $\omega + fp$ is the greatest element for $\lesssim$ of $\{x \in \mathbb{R}^l{}_+ | px \leq p\omega\}$.Now, as in Mantel (1974), the domain of utility functions of consumers $\{u^i\}$ is his trade set Θ^l, is $\Theta^l - b^i$, where $i = 1,2.., l$. Where b^i is in the nonnegative

[4] Distance d from x_1 to x_2 is $|x_1 - x_2|$

orthant (like quadrant) Θ^l, of the n-dimensional Euclidean space $\mathbb{R}^n . E^n$ are n-trade commodity economies with a finite number of consumers. In addition, F^n is a set of all functions f-on a ϵ- trimmed price space $\Theta^n(\epsilon) = \{p \in \Theta^l | \epsilon \leq p_n \leq 1/\epsilon\}$. In addition, these functions are continuous as previously shown, homogenous $f(\lambda p) = f(p)$. Cardinal number f of F^n is $> c$ [5], Hobson(1907). First, F-has a part which is equivalent tot the continuum. This is straightforward since functions such as $f(x) = c$, where c is any number of the continuum , constitute such a part. It follows that $f \geq c$. Now, we assume that F is equivalent to a part of the continuum, and it is proven that such a part cannot have a cardinal number$< c$. This F –function can be ordered in the same type as the continuum, so that any assigned number in the continuum ξ , there corresponds a definite set of rules R_ξ, which defines a function f_ξ. The aggregate function $F(x)$ must contain every definable function of a real variable. We may consider $f_\xi(\xi)$ as a function of ξ, because its value can be arithmetically determined and therefore is an element of F aggregate of all functions. Now to define a norm[6], we choose a fixed number, let us say unity, we choose one, then the functions $\phi(\xi) = f_\xi(\xi) + 1$, so now at each point ξ , value of unity is added. New definable function now is $\phi(x)$, but this cannot possibly belong to aggregate F, $\phi(x)$ cannot be identical with $f_{\xi_1}(x)$, because $\phi(\xi_1)$ and $f_{\xi_1}(\xi_1)$, they differ by unity. So now F cannot be equivalent to the continuum and thus theorem $f > c$ is put in practices, meaning that the aggregate of all functions of a real variable has a cardinal number f higher than c.About the initial endowment of the consumer we assume that $\sum_i \omega \gg 0$. Initial endowment vector $\omega_i \in \mathbb{R}^l$. Ownership share of

[5] The cardinal number of the aggregate of all functions defined for the rational points only is the cardinal number of the ways of distributing on the aggregate of rational numbers the aggregate number of continuum.

[6] A norm is a real valued function $\nu \longmapsto ||\nu||$

firms is $\theta \geq 0$, $\forall\, j, j = 1, \ldots, J$. Vector of prices is $p = p_1, p_2, \ldots\ldots, p_l$ is said to be Walrasian if it "clears " all of the markets, that is only if it solves L equations in L unknowns, Mas-Colell, A, and Whinston, M.,and Green, J.,, (1995) :$z(p) = 0$. This vector of prices is the inverse demand function, in our case is F- function . And for n-consumers , their preferences are $x_i = \mathbb{R}_+^l$ and $\gtrsim_l$ strictly convex[7] and strictly monotone. Now, a note on properties of preferences in terms of utilities, Obara.,I,(2012),:

a) $\gtrsim on\ \ X$ [8]is locally nonsatiated if $\forall x \in X\,, \epsilon > 0, \exists y \in X\,, \|y - x\| < \epsilon, \wedge\, u(y) > u(x)$
b) $\gtrsim on\ \ X$ is monotone $\Leftrightarrow x \gg y, \wedge\, u(y) > u(x)\,, \forall x, y \in X\,, x > y \Rightarrow x \gtrsim y$
c) $\gtrsim on\ \ X$ is convex $\Leftrightarrow u$ is quasi concave $u(y) \geq u(x), u(z) \geq u(x), u(ay + (1-a)z) \geq x, \forall a \in [0,1]$
d) $\gtrsim on\ \ X$ is strictly convex $\Leftrightarrow u$ is strictly quasi concave $u(y) \geq u(x), u(z) \geq u(x), y \neq z,\ u(ay + (1-a)z) > x, \forall a \in [0,1]$
e) Proposition : $\gtrsim$ is differentially strictly convex if as in Mas-Colell, A. (1986)[9] , Jacobian determinant is nonsinugaler i.e. non zero:

$$\begin{vmatrix} \partial^2 u(x) & \partial u(x) \\ [\partial u(x)]^T & O \end{vmatrix} \neq 0$$

---

[7] Second welfare theorem: If all agents have convex preferences, then there always will be set prices such that Pareto Efficient allocation is a market equilibrium allocation for an appropriate assignment of endowments.

[8] Preference relation $\gtrsim$ is a relation $\gtrsim \subset \mathbb{R}_+^l \times \mathbb{R}_+^l$.With properties $x \gtrsim x, \forall x \in \mathbb{R}_+^l$ (reflexivity), $x \gtrsim y, y \gtrsim z \Rightarrow x \gtrsim z$ (transitivity), $\gtrsim$ is a closed set (continuity), $\forall (x \gtrsim y), \exists (y \gtrsim x)$ (completeness) ,given $\gtrsim, \forall\ (x \gg 0)$ the at least good set $\{\mathrm{y}: \mathrm{y} \gtrsim \mathrm{x}\ \}$is closed relative to R^l (boundary condition), A is convex, $if\ \{\mathrm{y}: \mathrm{y} \gtrsim \mathrm{x}\ \}$is convex set for every y , $ay + (1-\lambda)x \gtrsim x$, whenever $\mathrm{y} \gtrsim \mathrm{x}\ \ and\ 0 < a < 1$, Mas-Colell, A. (1986).

[9] Big O here may represent infinitesimal asymptotics or a product of two matrices

Also as previous we assume that $\sum_i \omega \gg 0$ (level of income or wealth), now the inverse aggregate demand function , $\varphi(p, \omega)$ defined for all price vectors $p \gg 0$, satisfies the following properties:

a) $\varphi(\cdot)$ is continuous function on $\mathbb{R}_{++}^{l} \times \mathbb{R}_{+}^{l}$
b) $\varphi(\cdot)$ is homogenous of degree zero , $\varphi(\alpha p, \alpha\omega) = \varphi(p, \omega), \forall(p, \omega), *> 0$
c) $p\varphi(p) = 0$ $\forall p$ (Walras law)
d) If $p^l \to p, p \neq 0, p_l = 0$ for some l so, $\max\{\varphi_1(p^l), \dots.., \varphi_l(p^l)\} \to \infty$
e) About the set of prices… $S_\omega = \{p \in S | \forall i, p_i \geqq \epsilon \}$, then $\varphi_l(p) > -s$ for every commodity l and every p.
f) Budget function can be given as : $\beta(p, \omega) = \{x \in R_{++}^{l}: p * x \leq \omega\}$
g) Expenditure function is $\bar{u} \in u(R_{++}^{l})$ is defined as $e_{\bar{u}} \in u(R_{++}^{l})$, and is homogenous of degreeone, concave, and of class , C^1 (continuous differentiable whose derivative is continuous ,i.e. continuously differentiable, $\partial e_{\bar{u}}(p) =) = hu(p)$[10].
h) *Proposition*: $\forall(p, \omega), the\ subst.matrix\ \partial_p\varphi(p, \omega) + \partial_\omega\varphi(p, \omega)\big(\varphi(p, \omega)\big)^T < 0, Hermitian\ matrix\ \alpha_{ij} = \bar{\alpha}_{ji}\ , on\ T_p\colon \{v\colon p \cdot v = 0\}$

The aggregate production function in this economy would be :$Y = \{y \in \mathbb{R}^l; y \leq \sum_j \alpha_i a_{j\,,} (a_1, \dots., a_j) \geq 0\}$. In previous function $\alpha_i a_{j\,,}$ are basic activities of the firms , i.e the vector of level activities . Because preferences are strictly monotone, there can be no free goods at an equilibrium. Equilibrium exist with a pair(p, a), formed by a price vector $p \in S\ or\ p \in \mathbb{R}_{+}^{l}$, if and only if :$\varphi(p) - \sum_j \alpha_i a_{j\,,} = 0$. Differentiability of the excess demand

[10] Compensated demand function

function implies continuity. Now by differentiating both sides of $\varphi(\lambda p) = \varphi(p)$ with respect to $\lambda > 0$ and then evaluating at $\lambda = 1$, we obtain:

equation 7

a) $\sum_k \frac{\partial z_l(p)}{\partial p_k} p_k = 0 \quad \forall\, l \text{ and } p\ [D\varphi(p)p = 0]$

b) $\sum_k \frac{\partial z_k(p)}{\partial p_l} p_k = -\varphi_l(p)\ \forall\, l \text{ and } p\ D\varphi(p)p = -\varphi(p)$

These previous results come from Euler formula for homogenous functions. Formally, lets suppose that $f(x_1, \dots, x_l)$ is homogenous of degree r $(r = \cdots.., -1,0,1, \dots.)$ and it is differentiable. Then at any $(\bar{x}_1, \dots.., \bar{x}_l)$ there is :

equation 8

$\sum_{i=1}^{n} \frac{\partial f(\bar{x}_1, \dots, \bar{x}_n)}{\partial x_l} \bar{x}_l = rf(\bar{x}_1, \dots.., \bar{x}_l)$ or in matrix notation $\nabla f(\bar{x}) * \bar{x} = rf(\bar{x})$

By definition, we have: $f(t\bar{x}_1, \dots.., t\bar{x}_L) - t^r f(\bar{x}_1, \dots.., \bar{x}_l) = 0$, differentiating previous with respect to t , we have $\sum_{i=1}^{n} \frac{\partial f(t\bar{x}_1, \dots, t\bar{x}_N)}{\partial x_l} \bar{x}_l - rt^{r-1} f(\bar{x}_1, \dots.., \bar{x}_l) = 0$.For a function that is homogenous of degree zero Euler formula says : $\sum_{i=1}^{n} \frac{\partial f(\bar{x}_1, \dots, \bar{x}_l)}{\partial x_l} \bar{x}_l = 0$.

Pseudoequilibrium in incomplete markets

Pseudo equilibrium occurs if the initial share market is cleared and and share prices are positive at every date -event, provided that no consumer can be satiated at any event date pair. There exist two dates 0 and 1. This is a case of sequential trade under uncertainty. Equilibrium here is a set of prices at a first date, a set of common price expectations for the future, and a consistent set of individual plans for consumers and producers, so that each individual plan is optimal for the agent, given the appropriate budgetary constraints, Radner, R., (1972). So at date 1 trade exchange occurs and ℓ commodities are spot traded by the following prices $p_s \in \mathbb{R}^l_{++}$ Furthermore, $\sum_{i=1}^{l} x_{isc}$ is the commodity claimed by the individual if state s occurs, x_{sc} is commodity $c = 1,2 \ldots, C$ produced. So that, $\sum_{i=1}^{l} x_{isc} = x_{sc}$, Arrow , K., (1953), assuming absence of saturation of individuals. Economy is subject to restraint $\sum_{s=1}^{S} \sum_{c=1}^{C} \bar{p}_{sc} * x_{sc} = y_i$.Arrow sets theorem that is utility $V_i(x_1, \ldots\ldots, x_{sc})$ is quasiconcave for every individual i , then any optimal risk bearing allocation can be realized by a system of competitive markets in claims on commodities. There are precisely S types of securities[11]; unit of security of the sth type is a claim paying one monetary unit if state s occurs and zero otherwise. Now, q_s is the price of security $\bar{p}_{sc} = q_s p_{sc}$. Price of securities is more precisely $q_s = \frac{\sum_{s=1}^{S} \sum_{c=1}^{C} \bar{p}_{sc} * x_{sc}}{y}$, or $q_s = \frac{y_i}{y}$. Budget restraint divided by income. Or $\sum_{i=1}^{S} q_s = \frac{\sum_{i=1}^{I} \sum_{s=1}^{S} \sum_{c=1}^{C} \bar{p}_{sc} * x^{*}{}_{isc}}{y} = \frac{\sum_{i=1}^{I} y_i}{y} = 1$. At date 0 there is trade on $K \leq S$ assets, i.e. number of Arrow-Debreu securities is less or equal than the states of nature. Unit of assets k, delivers a return vector $\alpha_{ks} \in R_s^l$ of goods if state s occurs. We also denote that $y_i \in R^k$ and $x_i \in R_+^{ls}$, these are the trade and consumption plans of agent i.Now by definition: The plans $(\bar{y}, \bar{x})$ and prices $q \in R^k$ and $p \in R_+^{ls}$ constitute equilibrium if :

Equation 9

a) $\forall(\bar{y}_i, \bar{x}_i), \max u_i(x_i)$ subject to $q * y_i \leq 0, p_s x_{is} \leq p_s \omega_{is} + \sum_k y_{ik}(\alpha k_s * p_s), for\ \forall s$
b) $\sum_i y_i = 0 \sum_i (x_i - \omega_i) \leq 0$

The pair $(\bar{p}, \bar{x}) \in R_+^{ls} \times R_+^{lsN}$ constitutes equilibrium if :

a) $\forall x, \max u_i(x_i)$subject to $p x_i \leq p \omega_i$, $\wedge(p_i * (x_{i1} - \omega_{i1}), \ldots\ldots, (p_s * (x_{is} - \omega_{is}) \in L(p) \subset R^S$
b) $\sum_i (x_i - \omega_i) \leq 0$

[11] When security is sold, when s state occurs, money is transferred in a way determined by the securities, and the allocation of commodities occurs at market in a usual way, without further risk bearing.

Here we denote that $G^{S,K} = \{L(p) \subset R^S: L\ is\ a\ K\ dimensional\ subspace\}$, this is Grassman manifold[12] of K planes in R^S .And for $\forall p \wedge L \in \tilde{G}^{S,K} = \cup_{k' \leq k} G^{S,K}$, and $f(p,L) \in R^{lS}$ is the excess aggregate demand vector and the consumption set for every consumer is $\{x_i \in R_+^{ls}: (p_i)(x_i - \omega_i), \ldots\ldots, (p_s)(x_s - \omega_s) \in L\}$. And the pair$(p,L) \in R_{++}^{ls} \times \tilde{G}^{S,K}$, constitutes equilibrium if: $f(p,L) = 0$, $L = asset\ span\ \left(g_1(p), \ldots, g_k(p)\right)$. Return vector is $g_k(p) = p_i\alpha_{ki}, \ldots, p_s\alpha_{ks}$.And now if $L = R^S$, it is ok, with the result this is complete market case. And in such a case beyond Budget constraint there will be no restrictions on transfer of purchasing power across states of nature. However, since $K < S$ this is incomplete market case. So , in general pair $(p,L) \in R_{++}^{ls} \times \tilde{G}^{S,K}$, constitutes pseudo equilibrium if :

a) $f(p,L) = 0$
b) $g_k(p) \in L, for\ \forall\ (k = 1, \ldots, K$

(p,L) is an equilibrium if $\{g_1(p), \ldots\ldots\ldots, g_k(p)\}$ are linearly independent. About the questions of existence of pseudo equilibrium, first one can define unit norms such as: $f(p(L),L) = 0$ and $g_i(L) = g_iP(L)$. So given K functions $g_i: G^{S,K} \to R^S$ there is an L such that $g_i(L) \in L, \forall i$, also each function is continuous. One can define a section of the vector bundle σ, and L yields equilibrium if and only if $\sigma(L) = \sigma_0(L)$, and σ_0 is zero intersection and a reason for establishing pseudo equilibrium, because every bundle must intersect at zero intersection. In Radner, R., (1972), setting stock price –share price system is (p,ϑ) , safe asset is $r = (c_1, \ldots., c_r)$, S Arrow assets are : $r_{ss} = 1$, or $r_{ks} = 0$,the strike price c, at which put or call option can be exercised on primary asset r_k is given by: $\begin{cases} r_{call\ option} = \max(0, r_{ks} - c)_{1 \leq s \leq S} \\ r_{put\ option} = \max(0, c - r_{ks})_{1 \leq s \leq S} \end{cases}$. $\sum_{i=1}^{K} z_i \in R^k$ is allowable contract

---

[12] A Grassmann manifold is a certain collection of vector subspaces of a vector space.

(allowable portfolio of assets) , a commodity bundle is : $\sum_{is=1}^{LS} x_{is} \in R_{+}^{LS}$. In equilibrium utility, function:$max\, \vartheta_0(\bar{f} - f_0)U_i(x_i = \tilde{x}_i, z_{imm} = \bar{\omega}_{im} - \tilde{x}_{im} = 0, y_j = \sum_i(\tilde{x}_{im} - \bar{\omega}_{im})\ i \in I, j \in J$, Here m are all such markets in the economy, $\tilde{x}_i$ is pseudo value of x_i, y_j is the production function in economy, this function is maximized subject to: $\sum_k \vartheta_{ik} z_{ik} \leq 0$, and $\forall s, \sum_I p_{ils} x_{ils} = \sum_I p_{ils}\omega_{ils} + \vartheta_{is}\sum_k r_{ks} z_{ik}$. In pseudo equilibrium allocation of resources and prices is:$(z, x) \in R^{kI} \times R_{+}^{LSI}, (p, \vartheta) \in R_{+}^{K+LS}$.In market clearing in pseudo equilibrium $\sum_{i=1}^{I} \tilde{x}_{im} - \sum_{i=1}^{I} \bar{\omega}_{im} \gg 0$, so that $\sum_{i=1}^{I} \tilde{x}_{im} \ll \sum_{i=1}^{I} \bar{\omega}_{im}$. Bottazzi, J.-M. and T. Hens (1996), obtain first SMD incomplete markets result. They use Radner, R., (1972), but without short sale constraints. There are $k \leq S$ real assets, each of these assets can be purchased or sold at price, q^k at date 0. $L \rightarrow z(L)$ is a continuous homogenous function, $z(L)$ is excess demand function, also $L \in G_{++}^{k}(R^n) = \{L \in G^k(R^n), L \cap R_{+}^{n} = \{0\}\}$, and $z(L) \in L$,there is an economy such that for every L in a compact subset of $G_{++}^{k}(R^n)$, $z(L)$ is the aggregate excess demand function when consumers maximize their utility in their net trade span $L - R^n$. Goods space dimension in this model is given as: $m = (S+1)l$, excess demand is given as :$z = (z_o, \ldots., z_s) \in \mathbb{R}^{(S+1)l}$,also $L \rightarrow \beta(L)e(L)$, where e is vector, and $e(L)$ is perpendicular or orthogonal projection of a Euclidean vector, β is positive real valued function, and $\beta(L)e(L)$ is individual demand function. Budget set for every agent is given as: $\beta(p, q) = \{z \in \mathbb{R}^{(S+1)l} | \theta \in R^k, \bar{p}z_0 + \theta q = 0, p_s z_s = p_s(A_s\theta_s), s = 1, \ldots, S\}$. In previous expression θ is portfolio consumption, q is share price or security price, A_s represents real assets. Now, proposition is set that $\pi \in \mathbb{R}^{(S+1)l} \cap L(p)^{\perp} = L(\lambda(p)^{\perp}: L(p) = L(z)$, where π is property of real assets, now since vector $e - e(p) \in L(p)^{\perp}$.Now, since $e - e(p) \in L(p)^{\perp} > 0, e - e(p) \in L(p)^{\perp} \cap L(p')^{\perp}$.So, now $e - e(p)$ is magnitude of incompleteness or the price of $(S - k)$ which is the magnitude of incompleteness. Now if $L(\bar{p})$ (maximal dimension of goods prices)[13], has a maximal dimension, than every homogenous continuous function z which satisfies Walras' law i.e.$z(p) \in L(p)$ is the excess demand function of an m-consumer economy in an equilibrium (or not equilibrium) price $\bar{p}$. Now, m number of agents should be minimal. In a, one good finance models, with zero portfolio consumption $\theta = 0$, following utility of agents $\tilde{u}(\theta) = u(A * \theta + \omega)$ applies, where ω is resource endowment, and this utility function is maximized s.t. $q * \theta = 0$. So that now number of goods minus magnitude of incompleteness is $m = (S+1)l - (S-k)$,is $m = (S+0)1 - (S-k) = k$.Minimal number of consumers should be $m-1$.So that aggregate excess demand function is a sum from 1 to m of all individual excess demand functions, in local of $\bar{p}$,i.e. $z(p) = \sum_{i=1}^{m} \beta_i(p) e_i(p)$, this is generated in environment of strictly convex preferences. In previous expression, $\beta_i(p) \in R_{++}, e_i(p) = proj_{L(p)}(e_i)$.There exist m points $(\hat{e}_i, \dots., \hat{e}_m) \in L(\bar{p})$ in the interior of nonempty convex set (C) $relin(C) := \{p \in C: \forall q \in C\ \exists q > 1: \lambda p + (1-\lambda) q \epsilon C\}$.Direct sum of this convex preferences $(M = (C(e_i, \dots., e_m) = e_m + (e_1 - e_m, \dots, e_{m-1} - e_m$ and orthogonal $L(p)^{\perp}$ goods space should equal goods space dimension, i.e. $M \oplus L(p)^{\perp} = \mathbb{R}^{(S+1)l}$. For every function z for which Walras' law, continuity and homogeneity applies, there exists neighbourhood of no arbitrage prices$(\bar{p}, \bar{q})$, and $\bar{p}$ is such that the return matrix has max rank, z then is the aggregate excess demand function of m economy. This result applies for $m-1$ consumers only. So now, law of one price applies only in the case $z(p) = \sum_{i=1}^{m-1} \beta_i(p) e_i(p)$.This is also critical proof for the RET theorem in the last section. Since, individual excess demand functions can be identified locally, the last proof applies to them also. Norm

[13] $\bar{p}$ is such that the $S \times k$ matrix or $p_s * A_s$, has rank K. In the simplest case if it is 2×2 matrix with two states 0 and 1 rank will be one i.e. $k = 1$

defined here is $p_e(z) = \frac{e-\bar{z}}{\|e-\bar{z}\|}$, where $\bar{z}$ is the closest point from z to e. And, $p_e\big(\beta(\bar{p})e(\bar{p})\big) = \lambda\bar{p}, \beta(\bar{p}) = 1, \lambda > 0$.Also defined is $L(p_e(z))$, and $z = \beta\left(p_e\big(e(\bar{p})\big)\right)e(\bar{p})$, andL$L\left(p_e\big(e(\bar{p})\big)\right)e(\bar{p}) = L(p)$, this applies if $p = \bar{p}$, which is true in $L(p)^{\perp}$.

Nonexistence of equilibrium in incomplete markets

Hart, (1975), has shown that under standard assumptions equilibrium may not exist in incomplete markets. In this model there are two assets and tow states ($\ell = 2$ and $S = 2$ or 3), in matrix form:$a_{11} = (1{,}0), a_{12} = (1{,}0), a_{21} = (0{,}1), a_{22} = (0{,}1)$. Markets are complete when goods equals consumption set i.e. $L = R^2$, and if prices are not collinear, it means that the relative prices are not same. Otherwise $L > 0$ and contains some positive vector,$f(p, L(p)) = f(p, 0)$, it is as if no transfer was possible across the two states. Optimal portfolios are given as: $\varphi_i^* \in R^K$, optimal consumption plans are $x_i^* = \{x_{1i}^*, \ldots.., x_{1S}^*\}$ is our case with two states and two dates : $x_1^i = R_+^2$ and $x_2^i = R_+^2$ are consumption of agent i, at date 1 and date 2. Spot markets open at date 1 or 2 and futures market are costly to organize and futures are not permitted. There are no borrowing no lending possibilities and goods cannot be stored. Utility function here is: $U^i\big(x_1^i, x_2^i\big) = V^i(x_1^i) + \beta^i W^i\big(x_2^i\big)$. Endowments of agents at date 1 and 2 are given as : $\omega_1^i \in R_+^2$ and $\omega_2^i \in R_+^2$. And utility $U^i\big(x_1^i, x_2^i\big)$ is maximized subject to: $p_1 x_1^i \leq p_1\omega_1^i$ and $p_2 x_2^i \leq p_2\omega_2^i$ and $x_1^i, x_2^i \geq 0$. Price system also is ordered pair:(p, q) , p is a goods price system and q is security price system. Securities indexation is: $f = F + 1, \ldots., F + G$, and security f is represented by a functions a_2^f, a_3^f that map S into R_+^H which is set of goods in the economy. a_2^f, a_3^f we consider like dividends which security pays. If consumer holds φ units of security f and if state s occurs, the consumer will receive vector of goods φa_3^f at date 3. At this date futures market close, and goods markets open. Security trading plan for agent i is given as: $\varphi^i = \varphi_1^i, \varphi_2^i$. At first date (1) budget constraint is: $p_1(s)x_1^i(s) + q_1(s)\varphi_1^i(s) \leq p_1(s)\omega_1^i(s)$. At second date budget constrain is: $p_2(s)x_2^i(s) + q_2(s)\varphi_2^i(s) \leq p_2(s)\omega_2^i(s) + p_2(s)\left(\sum_{f=1}^{F} \varphi_1^f(s) + a_2^f\right) + \sum_{f=1}^{F} q_{2f}(s)\varphi_1^f(s)$ or $p_3(s)x_3^i(s) \leq p_3(s)\omega_3^i(s) + p_3(s)\left(\sum_{f=1}^{F+G} \varphi_{2f}^i(s) + a_3^f(s)\right)$.And therefore for a consumption allocation is said to be Pareto optimal if it is not Pareto dominated by other array of consumption plans, that is : $U^i\left(x'^i\right) = U^i\left(x^i\right), \forall i$.But because the goods in economy are gross substitutes, following rule applies : $\downarrow p \Rightarrow \downarrow x$, and if $\uparrow p \Rightarrow \uparrow x$, also $\frac{\partial x}{\partial p} > 0$.Therefore, $\frac{q_1(s)}{q_2(s)} = \frac{p_1(s)}{p_2(s)}$ for $s = 1,2$. And in equilibrium,$\frac{p_1(1)}{p_2(1)} = \frac{p_1(2)}{p_2(2)} = 1$, this contradicts assumption in equilibrium that p_1 and p_2 are linearly independent. Second case in equilibrium is if p_1 and p_2 are linearly dependent. There is only one security at date 1 and therefore there is no way of transferring wealth from state 1 to state 2, and we are assuming that in equilibrium there is no trading at date 1. Prices are normalized to unity meaning that $p(1) = (\frac{1}{3}, \frac{2}{3})$ and $p(2) = \left(\frac{2}{3}, \frac{1}{3}\right)$, so they are linearly independent vectors .And since two cases are being ruled out means that there is no possible equilibrium for this economy.

Taxes and incomplete markets

Government produces nothing, imposes tax and collects taxes, and distributes proceeds, and receives net income:$g_r = t * \bar{x} - \sum_H g^h$. In previous expression g^h are net lump sum government transfers to household h,[14] and

[14] h is a part fo the consumption vector $x^h = (x_1^h, \bar{x}^h)$, $\bar{x}^h$ is the consumption of the numeraire good. Utility is equal: $u^h(x^h, \vartheta^h)$, ϑ^h are other variables that affect utility such as: levels of pollution, average quality of a good consumed. $\vartheta^h = \beta^h t_1 x_1$, β^h is the share of tax received back by some household h, and $\sum_H \beta^h = 1$.Tax proceed redistributions are "externalities" to each household.

taxes are the difference between produced and consumed goods: $t \equiv (x^h - x^p)$. Now differentiating government revenue g_r with respect to t of the previous expression gives the following result: $\frac{dg_r}{dt} = \frac{d\bar{x}}{dt} * t - \sum_H g^h$. Also. $\bar{x} = \sum_H \bar{x}^h$ (sum of numeraire consumption). Stiglitz et al. (1986), further defines $\frac{dg^h}{dt}$ to be a change in lump sum income per unit change in tax, $E_\vartheta^h = \left[\frac{\partial E^h}{\partial \vartheta^h}\right] = \left[\frac{\partial u^h}{\partial \vartheta^h}\right]$, expenditure function of the household that gives minimum expenditure necessary to obtain level of utility u^h, $\pi_z^f = \frac{\partial Y_f}{\partial \vartheta^h}$, is the firms profit function. In the last expression of the derivative of government revenues to income this part is: $\sum_H g^h = \bar{x} - \left(\sum_F \pi_z^f \frac{dz^f}{dt} - \sum_H E_\vartheta^h \frac{dz^h}{dt}\right)$ .F and f's are firms. Or if $\Pi^t = \pi_z^f \frac{d\vartheta^f}{dt}$, $B^t = E_\vartheta^h \frac{d\vartheta^f}{dt}$,For the equilibrium to be optimal following condition should apply: $\frac{dg_r}{dt} = (\Pi^t - B^t) = 0$. As for the optimal taxes one has: $\frac{d\bar{x}}{dt} * t = -(\Pi^t - B^t) \Rightarrow t = -(\Pi^t - B^t)\left(\frac{d\bar{x}}{dt}\right)^{-1}$. Now if markets are incomplete as in previous definition in Arrow-Debreu sense, then changes in demand change market prices will change the nature of composite product, so externalities will exist and adverse selection will apply to the economy. In Stiglitz model there are two period only. In the $S(2) = k$ and $S = 0,1,2$, vector of prices is : $p = p_1, p_2, \dots \dots, p_k$.Family holdings in period zero are : W_0^h, and household utility is: $U^h(W_0^h; p) = \sum_k u_{2k}^h \left(x_k^{h*}; W_0^h, p_k\right)\Psi_k$, where u_{2k}^h is the utility of the consumption of good in period 2, x_k^{h*} is the vector of consumption that maximizes household utility function at period 2, Ψ_k is probability that state k materializes. This expression is subject to constraint: $p_k \tilde{x}_k^h \leq 0$, $\tilde{x}_k^h$ is the individuals (second period) net traded vector, for commodity zero: $\tilde{x}_{0k}^h = \tilde{x}_{0k}^{h*} - W_0^h$. Fotr other commodities: $\tilde{x}_{jk}^h = \tilde{x}_{jk}^{h*} - W_j^h$.Household two period utility is a sum of utilities in the two periods: $u^h(W_0^h;p) = u_1^h(\bar{W}^h - W_0^h) + U^h(W_0^h;p)$, where $\bar{W}^h - W_0^h$ denotes consumption in period 1 of store of value of good, $\bar{W}^h$ is the total initial endowment of the good. Now in therms of incomplete market with two period utilities we can rewrite the $\frac{dg_r}{dt}$ expression as: $\frac{dg_r}{dt} = \sum_H \sum_k \frac{dE^h dp_k}{dp_k dt} \Psi_k = \sum_k \left(\sum_h \tilde{x}_k^h * \frac{m_k^h}{U_1^h} \right) \frac{dp_k}{dt} \Psi_k$, where m_k^h is marginal utility of income to household h when $s = k$ state is realized .So in conclusion there will exist taxes that improve overall welfare.

Law of one price on securities

The payoff matrix of all Arrow securities must be identity (unit) matrix, Lengwiler (2004). In the common example the payoff matrix, that is collection of all Arrow type securities (elementary assets),[15] S is unit matrix, i.e. integer matrix consisting of all $1s$, $e \coloneqq [1, \{m, n\}]$. And $\det(m, n) \neq 0$, and trace matrix should be $tr(e) = \sum_{s=1}^{S} a_{ii}$. In the same time Arrow prices are given as:$\alpha = [a_1, \dots, a_s]$. Each of these Arrow prices, are given as product of financial markets prices and A^{-1} ,or inverse matrix of the matrix of returns, (markets are only complete if returns matrix is invertible), or $\prod_{s=1}^{S} q \cdot A(e_i)^{-1}$.Now risk free rate of return (risk free interest rate)is equal to :$\rho - 1 = \frac{1}{\sum_{s=1}^{S} a_{1,s}}$, where $\sum_{i=1}^{n} a_{1,s} = \beta$, and now risk neutral probabilities $\tilde{a}_s$are given as: $\tilde{a}_s \coloneqq \rho a_s$. About the decomposition of the price of the option, one has: $q_j = ar^j$. For the law of one price to holds, transaction costs must not exists, and there must be no bid-ask spread. And if two portfolios, produce the same cash flow $r * z = r * z'$, than they must also cost the same $q * z = q * z'$.Risk neutral pricing is given as :$R_j^s \coloneqq \frac{r_s^j}{q_j}$,where R_j^s is the gross rate of return of

[15] Contingent claims denominated in dollars, derivative with a payout that is dependent on the realization of some uncertain future event.(options, swaps, future contracts etc.).

asset j .Expected rate of return of any asset ,evaluated with risk neutral probabilities is given as :$\hat{E}\{R^j\} = \rho$ .Here please note that: $\rho = \beta^{-1}$. Asset span in the economy is given as:$\mathcal{M} = \{z \in R^s \coloneqq z = hR \; for \; some \; h \in R^j\}$, $\mathcal{M} \subseteq R^s$, in previous case markets are incomplete, but for markets to be complete $\mathcal{M} = R^s$. In the asset span of the economy function h denotes holdings of security j, while hR denotes portfolio of payoffs $\sum_j h_j r_j$. Now, $1 \in \mathcal{M}$ is risk free payoff, riskless assets exists only when markets are complete. Let remember that,$\varphi(\lambda p) = \varphi(p)$, and $\lambda = 1$, for excess demand function to be homogenous so one can denote payoff as:$\lambda 1 \in R^S$, $\lambda \in R$ is state independent claim and $q_s \equiv q(e_s)$, and $\mathcal{M} = \{q\theta : q \in R^j\}$.If there exist two states s and s' $r_s^j \neq r_{s'}^j$.

Tax cuts and fiscal policy

Tax cut is defined as: $\tilde{\tau}_0 - \tau_0 = -d$, and it is financed through debt, $\tilde{\tau} - \tau = (1 + r)d = -(1 + r)\tilde{\tau}_0 - \tau_0$. Formally Ricardian equivalence to holds following applies: $\Phi(t) = d(t) - \int_t^\infty (\tau(\bar{t}) - G(\bar{t}) \cdot e^{-rA(t,\bar{t})} dt = 0$[16]. Government debt is equal to $\dot{d}(t) = r(t)d(t) + G(t) - \tau(t)$, Heijdra, B.J., F.Van Der Ploeg, (2002). In the first period tax cut is financed through debt, but in the second period taxes are increased, by the principal plus interest due on the issued debt. Tax cut should leave present value of government spending unchanged, but the risk free payoff paying $(1 + r)$, does not mean that the risk free payoff belongs to the asset span, since $r > 0$ is exogenously determined, and it might or might not belong to the asset span $\mathcal{M}$, Divino, Orillo,(2017). Asset prices are $q\theta \in R_+^j$, tax obligations are given as: $\tau = (\tau_0, t1) \in R_{++}^{1+S}$ $\forall h \in H$, and $\max_{(x_0,\tilde{x}_s) \in \beta^h(q,\mathcal{M},\tau)} U^h = (x_0, \tilde{x}_s)$, s.t. 2 period

[16] Actuarial revenue is $r^A(t) = r(t) + M(t)$, where $M(t)$ is instant probability for death.

constraint:$x_0 + q\theta = \omega_o^h - t_o$; $x_s = \omega_s^h + q_s\theta - t_s$,taxpayer budget set it is defined as: $\beta^h(q, \mathcal{M}, \tau) = \{x \in R_+^{1+S}: \exists z \in \mathcal{M}\}: x_0 - \omega_0^h + \tilde{\tau}_0 = -q\theta, \tilde{x}_s - \widetilde{\omega}_s^h + \tilde{t}_s 1 = z, q: \mathcal{M} \to R$. If the tax cut previously defined is enacted then we will have: $x_0 - \omega_0^h + \tilde{\tau}_0 = -q\theta - d$, $\tilde{x}_s - \widetilde{\omega}_s^h + \tilde{t}_s = q\theta + (1+r)d$.So , now question here is whether agents can neutralize $(1+r)d$, and not that government bonds are net wealth as in Barro,(1974). For the law of one price to apply here $\beta^h(q, \mathcal{M}, \tau) = \beta^h(q, \mathcal{M}, \tilde{\tau}), \exists z \in R^s, \exists z(s') = z + (1+r)d1 \in \mathcal{M}$. RET holds if and only if it does not affect the individual demand sets defined as: $\varphi^h(p, q, \tau) = \{x \in \beta^h(p, q, \tau): \neg\exists x' \in \beta^h(p, q, \tau): U^h(x') > U^h(x)\}$. The last expression is in line with the second welfare theorem where if economy is specified by:

Equation 10

$$\left(\{x \succsim_j\}_{j=1}^{j}, \{x'\}_{s=1}^{S}, \omega_o^h\right), for\ (x, x'), \exists p = (p_1, \dots p_s) \vee q = (q_1, \dots q_s \neq 0, \exists(\omega_1, \dots \omega_s), \textstyle\sum_s \omega_s = p\omega_s^h + \sum_i p_i x',$$

Previous constitutes pseudoequilibrium with transfers and that is: $\forall s, x', \max px_s \leq px', \forall x \in R_+^S$, and $\forall j, if\ x_j \succ x_j' \parallel q_j x_j \geq \omega_j$ and $\sum_s x'_s = \omega_s^h + \sum_j x_j'$.In such a case $\lambda x + (1-\lambda)x' \in R_{++}^{1+s} \in \mathbb{R}^N$ is convex where $\lambda \in [0,1]$.This is also known as Separating hyperplanes theorem in other words, $R_{++}^{1+s} \subset \mathbb{R}^N$ is convex if it contains two vectors x and x' , and a segment that connects them. Now, law of one price holds if $1 \in \mathcal{M}$, but in the model public debt is not available for consumers to purchase. Only risky assets are available for them to try to replicate risk free payoff. RET does not hold if $1 \notin \mathcal{M}$. Now if we define:

Equation 11

$$\mathrm{Im}\mathcal{M} = \left\{q\left(e_1, \dots\dots, e_j\right) \in (R^S)^j: F(z) = q\ for\ some\ z \in V\right\}$$
$$\mathrm{Ker}\mathcal{M} = z \in V: F(z) = 0$$

Set of linear mapping function would be given as:

Equation 12

$$\mathcal{F} = \{(e_1, \ldots\ldots, e_j) \in (R^S)^j : q(e_1, \ldots\ldots, e_j) = 0\}$$

We note here that Fourier transform of a common function is given as:$\mathcal{F}_x[1](k) = \int_{-\infty}^{+\infty} e^{-2\pi ikx} dx = \delta(k)$, or Fourier transformation of a delta function is : $\mathcal{F}_x[\delta(x - x_0)(k)] = \int_{-\infty}^{+\infty} e^{-2\pi ikx} \delta(x - x_0) dx = e^{-2\pi ikx}$, $\mathcal{F}_x^{-1} = \delta[(x)k] = \int_{-\infty}^{+\infty} \delta(x) e^{2\pi ikx} dx = 1$. Now since, $(R^S)^j = 1$, in our case Fourier transform of one is 1, since Rank$(R^S) = 1$.If $Ker(\mathcal{M}) = 0$, than its dimension is given as: $\dim(Ker\mathcal{M}) = \dim(K^N) - \dim(Im\mathcal{M}) = n - rank(\mathcal{M}) = 1 - 1 = 0$.This is actually the distance $(1\mathcal{M})$. Now since delta function, is continuous and is close there exists complement of S which is an open set. In this open set one cannot expect to replicate risk free payoffs. This is because the complement set has its own limit points, and has its own set closure, has its own neighborhood, disjoint of S, Croft, Falconer, and Guy, K.(1991) $\mathcal{R} \subset R^{JS}, \forall(e_1, \ldots\ldots, e_j)$, Rank$\mathcal{M} = 0$, RET fails.Since the set of endogenous variables is :$\Theta \coloneqq \{\theta \in R^J : q\theta = 1\}$, since the rank of V is full (vectors are linearly dependent), and it is an injective transformation. Therefore, the Lebesque measure is:$\mu_L(S') = (b - a) - \sum_k (b_k - a_k) = 0$. Hence, agent cannot replicate risk free payoff.

Concluding remarks

This paper proved that if the law of one price holds than RET holds if agents are able to replicate risk free payoffs. The result is applied on the incomplete markets model. Under the law of one price agents are not affected by the changes of fiscal policy in terms of the risk free payoff as long as that payoff belongs to the fixed payoff matrix. But if columns J of payoff matrix vary law of one price will not hold, hence RET will not hold too.

References

1. Arrow, K. (1953), The Role of Securities in the Optimal Allocation of Risk Bearing, Review of Economic Studies, 1964, 31, 91-96.
2. Barro, R.,(1974), Are Government Bonds Net Wealth?, *Journal of Political Economy*, **82**, issue 6, p. 1095-1117.
3. Bottazzi, J.-M. and T. Hens (1996), Excess-demand functions and incomplete market, . *Journal of Economic Theory*. **68**: 49–63.
4. Croft, H. T.; Falconer, K. J.; and Guy, R. K.(1991), Unsolved Problems in Geometry. New York: Springer-Verlag, p. 2,
5. Debreu, G. (1974). Excess-demand functions. Journal of Mathematical Economics. 1: 15–21. doi:10.1016/0304-4068(74)90032-9
6. Divino, J.A.,Orrilo,J.,(2017), Failure of the Ricardian Equivalence Theory in Economies with Incomplete Markets, Brazilian review of econometrics, Vol 37, No 1 (2017)
7. Hart, O., (1975) ,On the Optimality of Equilibrium when the Market Structure is Incomplete. Journal of Economic Theory 11: 418-443.
8. Heijdra, B.J., F.Van Der Ploeg, (2002),The Foundations of modern macroeconomics, Oxford University Press, USA , ISBN: 0198776179, 784 pages
9. Hobson, E.W., (1907), The theory of functions of a real variable and the theory of Fourier's series, Cambridge, University Press
10. Lengwiler,Y., (2004), Microfoundations of Financial Economics: An Introduction to General Equilibrium Asset Pricing,
11. Mantel, R. (1974). On the characterization of aggregate excess-demand. *Journal of Economic Theory*. **7**: 348–353
12. Mas-Colell, A, and Whinston, M.,and Green, J.,, (1995), Microeconomic Theory, Oxford University Press
13. Mas-Colell, A. (1986). Four lectures on the differentiable approach to general equilibrium. Lecture Note in Mathematics. **1330**: 19–49.
14. Obara.,I,(2012), Preference and utility, Lecture slides at UCLA

15. Radner, R., (1972), Existence of equilibrium of plans, prices, and price expectations in a sequence of markets, Econometrica, Vol. 40, No. 2 (March, 1972)
16. Robbin, Joel W. (2010), Continuity and Uniform Continuity , UW-Madison Department of Mathematics
17. Sonnenschein, H. (1973). Do Walras' identity and continuity characterize the class of community excess-demand functions?. Journal of Economic Theory. 6: 345–354.
18. Stiglitz, Joseph E.; Greenwald, Bruce C. (1986). Externalities in economies with imperfect information and incomplete markets. Quarterly Journal of Economics. Oxford University Press via JSTOR. 101 (2): 229–64.