ENERGY APPROACH IN COMPARISON OF THE EFFECTIVNESS OF THE STACEY'S AND CLAYTON AND ENGQUIST'S PARAXIAL BOUNDARIES

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ABSTRACT

The paraxial boundaries are artefacts used to simulate wave propagation out of the physical model to infinity. Their imperfection results with spurious reflection of the energy which is travelling towards the boundaries. To quantify the error due to the artificial reflection of the boundary, we compute and monitor energy emitted from the source and energy exiting the model. The analyses are done with numerical model constructed of two parts, central part and periphery formed from the boundaries. The wave motion is generated from the source located in the central part. In the vicinity of the source we chose a group of points symmetrically deployed with respect to the both axis. The sum of the energy calculated in each of those points is the energy generated from the source. In same manner using the points of the artificial boundaries we calculate the outgoing energy. By quantification of the input and outgoing energy, one can get impression of the effectiveness of the paraxial boundaries.

I. INTRODUCTION

The numerical analysis of half space always begins with reduction of the infinity to finite numerical model. The reduction is done with two types of boundaries. First one simulates the interface between the air and the medium of the half space, and the other three are artificial boundaries. These boundaries must satisfy the physical laws and simulate the infinity such that they are allowing the wave to go through them and leave the numerical model. Because these boundaries are approximations of the wave equation, one part of the wave will be reflected and some amount of the energy will be trapped inside the model.

In general, the artificial boundaries are separated in two groups: global and local. The global boundary formulation creates boundaries which are perfect absorbers. Creating numerical models with global boundaries is very complex and can be very expensive. The local boundaries are offering easy implementation, but their approximate formulation produces errors in the results.

Both boundaries compared in this article are using same type of approximation. The paraxial approximation uses the idea to separate the wave in two parts, one which is leaving the model and the other which is entering the model. The artificial boundaries which are using the paraxial idea are dealing only with the wave part which takes the energy out of the model.

Clayton and Engquist (CE) [2] presented one of the first paraxial boundaries. Their implementations in numerical simulation have emphasized their weak points. The numerical stability is conditioned by the ratio of the SV- and P-wave velocity. If this ratio exceed 0.46 it generates numerical instability. Much more concerning fact is that they produce numerical instability after the wavefront have passed through them. The interaction between the inner part of the numerical model and the periphery-paraxial boundaries makes entire numerical model to lose the stability and creates unreal results.

R. Stacey [5] has presented modified formulation of the paraxial boundaries. He is claiming that this borders are numerically stable for wither margins of the velocity ratio and that they are remaining stable for the complete numerical simulation not depending on the time of duration.

In this article, according to the law of conservation of energy, we are proving the instability of the first boundaries and confirming Stacey's statements. The energy generated in the source and the energy which deserts the model must be equal.

II. NUMERICAL MODEL

The numerical model should simulate half space as truncated part of ground basis of arbitrary construction. All calculations are done by using finite difference method. The points are forming square with size 140 m. The distance between the points is 1m in both directions.

The material properties are defined indirectly using the values of the velocities of P- and SV- waves. According to the expressions of both velocities one can express the demanding material property

$$v_p = \sqrt{\frac{(\lambda+2\mu)}{\rho}}; \qquad v_s = \sqrt{\frac{\mu}{\rho}} \tag{1}$$

Both models, the one bounded with CE's boundaries and the other bounded with Stacey's boundaries, have same material compositions. The compressional velocity is vp=250$\sqrt{3}$ m/s, and shear velocity vs=250 m/s. These values are making ratio which is within the stability margins for CE's boundaries.

The numerical models are divided in two parts. The central part is the interior of the model where the source of the wave motion is placed. For simulating source of spherical wave four points are used. Those points are surrounding the point on half length in both directions. The displacements at those points are given explicitly by the following law:

$$u = A\sin(\pi t/t_d) \tag{2}$$

with amplitude A=50 and t_d=0,05s. Each of these points generates radial displacement oriented towards the angles of the model. In order to reduce the grid dispersion effect, this pulse is filtered with Fourier transformations.

The wave equation which is used in the central part simulates the propagation of the P-SV waves:

$$\rho\frac{\partial^2 u}{\partial t^2} = (\lambda+2\mu)\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 w}{\partial x\partial z}\right)+\mu\left(\frac{\partial^2 u}{\partial z^2}-\frac{\partial^2 w}{\partial x\partial z}\right)$$
$$\rho\frac{\partial^2 w}{\partial t^2} = (\lambda+2\mu)\left(\frac{\partial^2 u}{\partial x\partial z}+\frac{\partial^2 w}{\partial z^2}\right)+\mu\left(\frac{\partial^2 w}{\partial x^2}-\frac{\partial^2 u}{\partial x\partial z}\right) \quad (3)$$

Kelly et al.[4] presented a finite difference formulation for the wave equation used in this model. The numerical stability of the central part is depending on time step and point distances. In [4] the condition for numerical stability which combines these values is defined.

The quantification of the source energy is done with 36 points from the net. These points are surrounding the four source points forming a small square, κ, with size 9m. The location of the square is such that intersection point of its diagonals is the fictive source point. Using the points of the edges of κ we calculated the displacements in both directions. The derivatives of the displacements are used for the calculation of the input energy (Aki и Richards [1]):

$$E = \rho\alpha\Delta \sum_{k=0}^{(t_p+t_d)/\Delta t} \sum_{i=1}^{36} v_{i,k}^2 \quad \Delta t \quad (4)$$

where $v_i^k = \sqrt{\dot{u}_i^2+\dot{w}_i^2}$ is the radial velocity of the medium fraction at point i and time step k, ρ is the density of the medium, and α is the velocity of the P-wave. Belonging area of the point i is $\Delta = h \cos\gamma$, where γ is the angel the horizontal axis and the line which goes through the source point and point i and $\frac{\pi}{4} \le \gamma \le \frac{\pi}{4}$. The time needed for the pulse to reach the furthest points of the square is $t_p = 9h/(\sqrt{2}\ \alpha)$.

The integral over the time can be approximated by making sum in all time steps, starting from t=0 since t=t_d. Because the equation contains another integral, it is necessary to introduce another sum. This sum calculates the energy in each point and collects for all points in every time step.

The difference of the models is in the second, the periphery. The first two are bounded with four CE's paraxial boundaries and the second one uses four Clayton's P3 boundaries. The intersections of the boundaries are 90^0 corners. These parts of the model can produce numerical instability. To avoid this, the corners and the first neighbouring points in both axis directions are replaced with rotated first order CE's and P3 Stacey's boundaries in each model respectively.

III. RESULTS

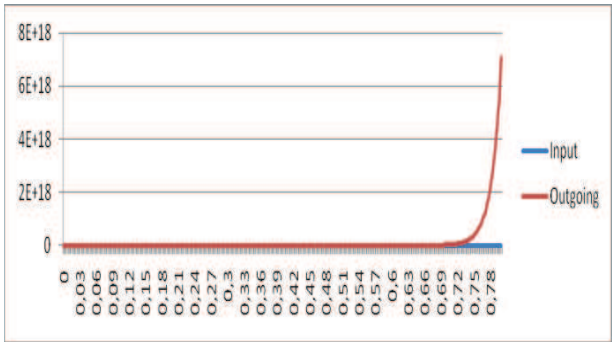

Figure 1: Instability of CE's boundary.

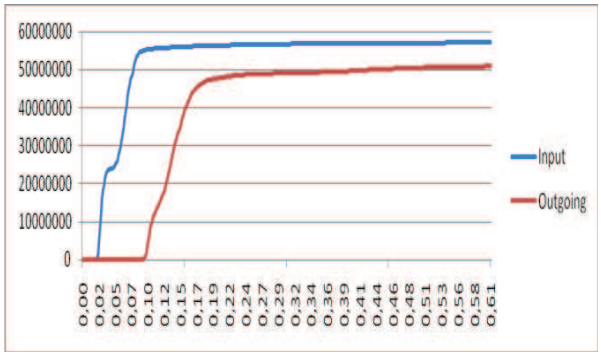

Figure 2: Numerically stable CE's boundary.

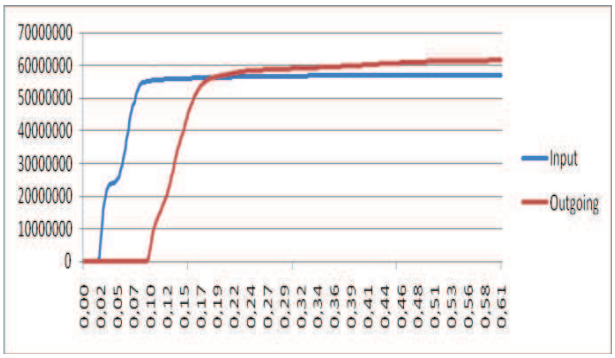

Figure 3: Numerically stable Stacey's boundary.

The comparison is done on three models. The source has same location in all of them. It is positioned at the intersection of the diagonals. All three models are symmetrical in both directions for simplifying the result analysis.

Beside the velocity ratio, the numerical stability of the CE's boundaries is directly dependent to the time interval. The simulations done with the first model are with time step of Δt=0,002s. The time-energy diagram on Figure 1 clearly shows the weakness of the CE's boundaries. Since the wave front passes through the borders they are increasing

exponentially the displacements at the points of the boundaries. As the exiting energy is function of the displacements computed at those points, it is increased in time as well.

Doing several tests analyzing the time interval, the time step $\Delta t=0,0015(15)$s is recognized as the most favourable for the previously mentioned velocities. The numerical simulations from model 1 are repeated and the obtained results are shown on the time-energy diagram on Figure 2. There are two curves on the diagram, one is representing the increase of the input energy in time and the other one is for the change of the outgoing energy in time. The curve of the input energy shows that the increase of this energy is in two parts. It corresponds with the sine function of the source displacements. First part is the increase of the energy while the sine function is increasing and the second part correlates with the decreasing part of the sine function. At the peak of the sine function, the difference between the displacements in two successive time steps are almost zero, and hence the increase of the energy is almost zero. Because the wave needs more time to reach the points of the border, compared to the points where the input energy is measured, the increase of the values of the second curve starts later then the first one. The slope of this curve corresponds to the energy which is carried of the wave passing through the boundary. Some part of the energy will be reflected. This energy will travel towards the next border and by reaching it will produce increase of the outgoing energy. Thus this curves shows continuously rise of the values in time. Another remark can be stressed out from this curve. The slope of the outgoing energy is stabilized in time which is sign for numerical stability. The numerical analysis is done up to t=0,6s. The relative error, computed with the input and the outgoing energy at that time, is 10,9%. This will be the landmark for the comparison between the two paraxial boundaries.

Using the same material properties and time step, the numerical simulations from the previous two models are repeated using the third model. Figure 3. is the time-energy diagram related to the results of this simulation. Comparing the curves of the input energies from Figure 2 and Figure 3. it is easy to conclude that the input energy is same for both models. The main difference between the curves of the outgoing energy form Figure 2. and Figure 3. shows the different way of working of these two boundaries. While the CE's boundaries are producing outgoing energy which is smaller than the input, the Stacey's boundaries are generating more outgoing than input energy. The rise tendency of the values of the outgoing energy is kept in time, showing that Stacey's boundaries are producing reflections as well. Similar like the curve from Figure 2. this one is stabilizing in time as well. It proves that some part of the energy, travelling in the model as reflected wave, will leave the model next time it reaches one of the boundaries.

Knowing that the relative error includes absolute error, the comparison of the relative errors computed at the same time can be done. Stacey's P3 boundaries are producing relative error of 7,9% which is smaller comparing to the CE's boundaries. This comparison proves Stacey's claims[5] that his boundaries despite being more numerically stable, they are more accurate than CE's as well.

IV. CONCLUSIONS

This article gives some evaluations of the numerical stability and accuracy of the two most famous paraxial boundaries. The evaluation is performed by computing the input and outgoing energy on three models with same material properties and two different time intervals. Two of the models have CE's borders, while the third one is bounded with Stacey's P3 boundaries.

The first model stressed out the weakness of the CE's boundaries, their numerical instability in time. This instability is because of the strong dependency to the time interval. Making several test by changing only the time step, we found that CE's are stable with time step $\Delta t=0,0015(15)$s.

Calling the law of conservation of energy, we compared the relative error computed with the injected energy and exiting energy after 0,6s as information about the reflections which are generated by the boundaries. These computations proved that the Stacey's P3 boundaries, although are first order paraxial approximations, are more accurate than the second order CE's paraxial boundaries for short simulations.

The energy analysis in this article pointed out the different way of working of these boundaries. Stacey's boundaries are generating bigger exiting energy compared with the injected, while CE's exiting energy is smaller than the inputted. This is important considering longer simulations. Because in both cases the outgoing energy has tendency to rise in time, Stacey's relative error will be increased, while CE's error will be decreased if their stability is kept.

V. REFERENCES

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