RELATIONSHIP BETWEEN THE EXTREMES OF A FUNCTIONAL AND ITS VARIATION

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Abstract

Methods to solve variational problems, the tasks for the study for maximum and minimum of functionals are very similar methods to study the maximum and minimum functions. Therefore, appropriate to outline briefly the theory of maximum and minimum functions and in parallel we will introduce similar concepts and prove similar theorems for functional.
The variation of functional is primary and linear with respect to a part of the increase of functional. In the study of functional, variation plays the same role played by the differential in the study of functions.
We will show that if there is a variation in the basic sense of the linear increase of functional, then there is a variation in the derivative sense of the parameter with initial value, and that both definitions are equivalent and we will give another definition of functional variation. The above we will show in several examples and we will prove a theorem which is a link between the extremes of a functional and its variation.

Key words: variation, extremes, functional.

We will appropriate to outline briefly the theory of maximum and minimum of functions and in parallel we will introduce similar concepts and prove similar theorem for functional.

<u>Definition 1</u> Variable y is called a function of a variable x which is indicated as follows: $y = f(x)$, where each value x in a particular area changes correspond to y i.e the number x corresponds to the number y .

<u>Definition 2</u> Image $v: X \to R$ is called functional.

We will consider functional v defined on a multitude of functions and use the designation $v[y(x)]$.

<u>Definition 3</u> Functional of n variables call an image of each ordered n variables $x_1 \in X_1, x_2 \in X_2, \ldots, x_n \in X_n$ where $X = (X_1 \times X_2 \times \ldots \times X_n)$, collate single value $F(x_1, x_2, \ldots, x_n) \in R$.

We will consider a special case of functional three variables, when $X_1 \equiv R, X_2 \equiv C^1_{[a,b]}, X_3 \equiv C_{[a,b]}$ and $F = F(x, y, y')$.

Consider space $X = R \times C^1_{[a,b]} \times C_{[a,b]}$ equipped with metrics

$$\rho(x_1, y_1, y_1') = |x_1 - x_2| + \max_x |y_1(x) - y_2(x)| + \max_x |y_1'(x) - y_2'(x)| .$$

<u>Definition 4</u> The increase Δx of the argument x of the function $f(x)$ at the point x_0 is called the difference between the two figures on this variable $\Delta x = x - x_0$.

If x is the independent variable, then the differential of x match with the increase, i.e $dx = \Delta x$.

<u>Definition 5</u> Increase or variation δy of the argument $y(x)$ of the functional $v[y(x)]$ in the point $y_0(x)$ is called the difference between the two functions $\delta y = y(x) - y_0(x)$.

The function $y(x)$ varies randomly in a certain class of functions (e.g the class of continuous functions $C_{[a,b]}$ or classes of functions with continuous first derivative $C^{(1)}_{[a,b]}$).

Let's look at the function $F = F(x, y, y', \ldots, y^{(k)})$ and functional $v: C^{(k)}_{[x_0, x_1]} \to R$ equal to

$$v[y(x)] = \int_{x_0}^{x_1} F(x, y, y', y'', \ldots, y^{(k)})\, dx .$$

In this connection must be established following definitions for the distance between the curves $y = y(x)$ and $y = y_1(x)$:

- the curves $y = y(x)$ and $y = y_1(x)$ are the zero order if the distance between them $\max_x |y(x) - y_1(x)|$, if small.

- the curves $y = y(x)$ and $y = y_1(x)$ are the first order if the distance between them $\max_x |y(x) - y_1(x)| + \max_x |y'(x) - y_1'(x)|$, if small.

- the curves $y = y(x)$ and $y = y_1(x)$ are the k-th order if the distance between them

$$\begin{gathered}\max_x |y(x) - y_1(x)| \\ + \max_x |y'(x) - y_1'(x)| \\ + \max_x |y''(x) - y_1''(x)| \\ \cdots\cdots\cdots\cdots\cdots \\ + \max_x \left|y^{(k)}(x) - y_1^{(k)}(x)\right|\end{gathered}$$

is small.

From these definitions, it follows that if the curves are close in terms of distance from the k-th order, they are even close in terms of distance from each-mean order. Now it can clarify the concept of functional continuity.

Definition 6. The functional $v[y(x)]$ is continuous at $y = y_0(x)$ in terms of the distance from the k-th order, if for any positive integer ε exists integer $\delta > 0$ such that $|v[y(x)] - v[y_0(x)]| < \varepsilon$ at

$$\begin{gathered}\rho_\infty(y, y_0) = \max_x |y(x) - y_0(x)| < \delta, \\ \rho_\infty(y', y_0') = \max_x |y'(x) - y_0'(x)| < \delta, \\ \rho_\infty(y'', y_0'') = \max_x |y''(x) - y_0''(x)| < \delta, \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \rho_\infty(y^{(k)}, y_0^{(k)}) = \max_x \left|y^{(k)}(x) - y_0^{(k)}(x)\right| < \delta.\end{gathered}$$

This implies that the function $y(x)$ is taken from class of functions, for which the functional $v[y(x)]$ is defined.

Definition 7. A linear functional is called a functional $L[y(x)]$ satisfying the following condition:

$L[cy(x)] = cL[y(x)]$, where c is an arbitrary constant, and

$$L[y_1(x) + y_2(x)] = L[y_1(x)] + L[y_2(x)].$$

An example of a linear functional is $L[y(x)] = \int_{x_0}^{x_1} (p(x)y(x) + q(x)y'(x))\,dx$.

Example 1. Let us consider the functional

$v[x] = \int_0^1 x(t) sin\left(\frac{\pi}{2}t\right) dt: C_{[0,1]}^{(1)} \to R$ and $\mu(x) = \int_0^1 (x(t) + x'(t)) sin\left(\frac{\pi}{2}t\right) dt: C_{[0,1]}^{(1)} \to R.$

The functional $\mu(x) = \int_0^1 x(t)\sin\left(\frac{\pi}{2}t\right)dt + \int_0^1 x'(t)\sin\left(\frac{\pi}{2}t\right)dt$ is linear and continuous. Really $v(x+y) = v(x) + v(y)$, $v(\alpha x) = \alpha v(x)$.

$$\left|\int_0^1 y(t)\sin\left(\frac{\pi}{2}t\right)dt - \int_0^1 y_0(t)\sin\left(\frac{\pi}{2}t\right)dt\right| \le \int_0^1 \sin\left(\frac{\pi}{2}t\right)dt\left|y(t) - y_0(t)\right|dt \le$$

$$\le \max_{t\in[0,1]}\left|y(t) - y_0(t)\right|\int_0^1 \sin\left(\frac{\pi}{2}t\right)dt = \rho_\infty(y, y_0)\frac{2}{\pi}\cos\frac{\pi}{2}t\Big|_0^1 =$$

$$= \rho_\infty(y, y_0)(-\cos\frac{\pi}{2} + \cos 0) = \rho_\infty(x, y)\frac{2}{\pi}$$

Let $\delta = \varepsilon\frac{\pi}{2}$.

Therefore, when y_0 is fixed, for x we receive $\left|y - y_0\right| < \delta = \varepsilon\frac{\pi}{2} \Rightarrow \left|f(x) - f(y_0)\right| < \varepsilon\frac{\pi}{2}\frac{2}{\pi} = \varepsilon$, and therefore v is a continuous functional on $\forall x_0 \in [0,1]$.

Similarly we can check that μ is continuous functional.

Example 2. The functional $v[y(x)] = \int_0^1 y^2(t)\sin\left(\frac{\pi}{2}t\right)dt$, is not linear, but is continuous.

$$v(y)-v(y_0)=v[y(x)]=\int_0^1\left(y^2-y_0{}^2\right)\sin\left(\frac{\pi}{2}t\right)dt=$$

$$=\int_0^1(y-y_0)(y+y_0)\sin\left(\frac{\pi}{2}t\right)dt\le\max_x|y-y_0|\int_0^1(y+y_0)\sin\left(\frac{\pi}{2}t\right)dt\le$$

$$\le\max_x|y-y_0|\max_x|y+y_0|\int_0^1\sin\left(\frac{\pi}{2}t\right)dt$$

Let $M=\max_x|y_0(x)|$ and $\delta=\max_x|y(t)+y_0(t)|<3M$ for $\forall y\in B_{1/2M}(y_0)$.

Indeed, from the inequality $-\delta<y(t)-y_0(t)<\delta,\quad за\,\forall t$, where $\delta=\frac{1}{2}M$, and $\frac{1}{2}M<y(t)<\frac{3}{2}M,\quad за\,\forall t$ should $|y(t)-y_0(t)|<|y(t)|+|y_0(t)|<3M$

Then, for $\forall y\in B_{\frac{\pi}{6M}\varepsilon}(y_0)$ we

obtain $v(y)-v(y_0)\le\max_x|y-y_0|\max_x|y+y_0|\int_0^1\sin\left(\frac{\pi}{2}t\right)dt\le\frac{\pi}{6M}\varepsilon\frac{2}{\pi}3M<\varepsilon$.

If the increase of the function $\Delta f=f(x+\Delta x)-f(x)$ can be represented in the form $\Delta f=A(x)\Delta x+\beta(x,\Delta x)\Delta x$ where $A(x)$ is independent of Δx, and $\beta(x,\Delta x)\to 0$ at $\Delta x\to 0$, then the function is called differentiable, and the linear part with respect to the increase Δx, $A(x)\Delta x$ is called a differential of the function and means df . Dividing to Δx and make a border transition at $\Delta x\to 0$, we obtain that $\lim_{\Delta x\to 0}\frac{\Delta f}{\Delta x}=A(x)+\lim_{\Delta x\to 0}\rho(x,\Delta x)=A(x)=f'(x)$ and therefore $df=f'(x)\Delta x$.

If the increase of the functional $\Delta v=v[y(x)+\delta y]-v[y(x)]$ can be written as $\Delta v=L[y(x),\delta y]+\beta(y(x),\delta y)max|\delta y|$ where $\max|\delta y|$ is the maximum value of $|\delta y|$, and $\beta(y(x),\delta y)\to 0$ at $\max|\delta y|\to 0$,it is linear with respect to the increase of functional δy, i.e. $L[y(x),\delta y]$ is called variation of functional and means δv.

So, the variation of the functional is primary and linear with respect to δy ,a part of the increase of the functional.

In the study of functional, variation plays the same role played by the differential in the study of functions.

Example 3. We will look the functional $v[y(x)]=\int\limits_0^1 y(t)\sin\left(\frac{\pi}{2}t\right)dt, \qquad x\in[a,b]$.

The increase of this functional can be expressed in the form $\Delta v = v(y+\delta y)-v(y)$ where $\delta y = y_0(t)-y(t)$. As applicable, in this functional, we have

$$\Delta v[y(x)]=\int\limits_0^1 \left(y_0(t)-y(t)-y_0(t)\right)\sin\left(\frac{\pi}{2}t\right)dt-\int\limits_0^1 y_0(t)\sin\left(\frac{\pi}{2}t\right)dt=$$

$$=\int\limits_0^1 \left(y(t)-y_0(t)\right)\sin\left(\frac{\pi}{2}t\right)dt=\int\limits_0^1 \delta y\sin\left(\frac{\pi}{2}t\right)dt$$

$$v(y+\alpha\delta y)-v(y)=\alpha\Delta v$$

$$v(y+\delta y_1+\delta y_2)-v(y)=\Delta v(\delta y_1)+\Delta v(\delta y_2)$$

Therefore, the variation of functional $v[y(x)]$ is linear in terms of the increase δy and $\Delta v[y(x)]=\int\limits_0^1 \delta y\sin\left(\frac{\pi}{2}t\right)dt$.

Example 4. We will look the functional $v[y(x)]=\int\limits_0^1 y^2(t)\sin\left(\frac{\pi}{2}t\right)dt$.

$$\Delta v=\int_0^1\left(y_0(t)+y(t)-y_0(t)\right)^2\sin\left(\frac{\pi}{2}t\right)dt-\int_0^1 y_0{}^2(t)\sin\left(\frac{\pi}{2}t\right)dt=$$

$$=\int_0^1\left(y^2(t)-y_0{}^2(t)\right)\sin\left(\frac{\pi}{2}t\right)dt=\int_0^1\left(y(t)-y_0(t)\right)\left(y(t)+y_0(t)\right)\sin\left(\frac{\pi}{2}t\right)dt=$$

$$=\int_0^1 y(t)\delta y\sin\left(\frac{\pi}{2}t\right)dt+\int_0^1 y_0(t)\delta y\sin\left(\frac{\pi}{2}t\right)dt=$$

$$=\int_0^1 y_0(t)\delta y\sin\left(\frac{\pi}{2}t\right)dt+\int_0^1 y(t)\delta y\sin\left(\frac{\pi}{2}t\right)dt=$$

$$=\int_0^1 y_0(t)\delta y\sin\left(\frac{\pi}{2}t\right)dt+\delta y\int_0^1 (y_0+\delta y)\sin\left(\frac{\pi}{2}t\right)dt=$$

$$=\int_0^1 y_0(t)\delta y\sin\left(\frac{\pi}{2}t\right)dt+\delta y\int_0^1 y_0(t)\delta y\sin\left(\frac{\pi}{2}t\right)dt+\delta y\int_0^1 \delta y\sin\left(\frac{\pi}{2}t\right)dt=$$

$$=2\int_0^1 y_0(t)\delta y\sin\left(\frac{\pi}{2}t\right)dt+\int_0^1 \delta y\sin\left(\frac{\pi}{2}t\right)dt\,\delta y$$

Therefore, the variation of functional $v[y(x)]$ is linear in terms of the increase δy and $L\left(y(x),\delta y\right)=\int_0^1 y_0(t)\delta y\sin\left(\frac{\pi}{2}t\right)dt$.

Now, we can give another, almost equivalent definition of the differential function and functional variation. Consider the value of the function $f(x+\alpha\Delta x)$ at fixed x and Δx and variable parameter value α. When $\alpha=1$ we get an increase $f(x+\Delta x)$ in the value of the function when $\alpha=0$ and receiving initial value of the function $f(x)$. It is easy to verify that the derivative of $f(x+\alpha\Delta x)$ of α at $\alpha=0$ is equal to the differential of the function at the point x.

Indeed, from the rule for differentiating a complex function we have

$$\frac{\partial}{\partial\alpha}f(x+\alpha\Delta x)\Big|_{\alpha=0}=f'(x+\alpha\Delta x)\big|_{\alpha=0}=f'(x)\Delta x=df(x)$$

i.e the differential function of $f(x)$ is equal to $\frac{\partial}{\partial \alpha} f(x+\alpha\Delta x)\Big|_{\alpha=0}$.

For functionals of the type $v[y(x)]$, the variable can be defined as a functional derivative of $v[y(x)+\alpha\delta y]$ of α at $\alpha=0$.

Indeed, if there is a functional variation in terms of basic linear part of the increase, then its increase is $\Delta v = v[y(x)+\alpha\delta y]-v[y(x)] = L(y,\alpha\delta y)+\beta(y,\alpha\delta y)|\alpha|\max|\delta y|$

The derivative $v[y(x)+\alpha\delta y]$ of α at $\alpha=0$ is equal to

$$\lim_{\Delta\alpha\to 0}\frac{\Delta v}{\alpha} = \lim_{\alpha\to 0}\frac{\Delta v}{\alpha}$$
$$= \lim_{\alpha\to 0}\frac{L(y,\alpha\delta y)+\beta[y(x),\alpha\delta y]|\alpha|max|\delta y|}{\alpha}$$
$$= \lim_{\alpha\to 0}\frac{L(y,\alpha\delta y)}{\alpha} + \lim_{\alpha\to 0}\frac{\beta[y(x),\alpha\delta y]|\alpha|max|\delta y|}{\alpha} = L(y,\delta y)$$

because there is the linearity $L(y,\alpha\delta y)=\alpha L(y,\delta y)$ and

$\lim_{\alpha\to 0}\frac{\beta[y(x),\alpha\delta y]|\alpha|\max|\delta y|}{\alpha} = \lim_{\alpha\to 0}\beta[y(x),\alpha\delta y]\max|\delta y| = 0$ because $\beta[y(x),\alpha\delta y]\to 0$ at $\alpha\to 0$.

Example 5. We will look functional $v[y(x)] = \int_0^1 y^2(t)\sin\left(\frac{\pi}{2}t\right)dt$.

We find the derivative of this functional at $\alpha=0$.

$$v[y(x)] = \left(\int_0^1 (y+\alpha\delta y)^2\sin\left(\frac{\pi}{2}t\right)dt\right)'_{\alpha=0} = 2\left(\int_0^1 (y+\alpha\delta y)\sin\left(\frac{\pi}{2}t\right)\delta y\,dt\right)\Bigg|_{\alpha=0} =$$

$$= 2\int_0^1 y\delta y\sin\left(\frac{\pi}{2}t\right)dt$$

Therefore, the linear part with respect to the increase is $2\int_0^1 y\delta y\sin\left(\frac{\pi}{2}t\right)dt$.

Therefore, if there is a variation in the basic sense of linear increase of the functional, then there is variation in the derivative sense of a parameter at initial value of the parameter, both definitions are equivalent. Thus, we can give another definition of functional variation.

Definition 8. Variation of functional $v[y(x)]$ is equal to $\frac{\partial}{\partial\alpha}v[y(x)+\alpha\delta y]|_{\alpha=0}$.

Definition 9. The functional $v[y(x)]$ reaches a maximum of the curve $y=y_0(x)$, if for the values of each functional $v[y(x)]$ near k, the curve $y=y_0(x)$ is fewer than $v[y_0(x)]$ i.e $\Delta v = v[y(x)] - v[y_0(x)] \leq 0$.

If $\Delta v \leq 0$, the equality $\Delta v = 0$ is fulfilled at $y(x)=y_0(x)$, indicating that the curve $y=y_0(x)$ is reached strict maximum.

Analogously we can determined that the curve $y=y_0(x)$ reaches a minimum. In this case, for all curves $\Delta v \geq 0$ in the vicinity of the curve $y=y_0(x)$.

Prior to formulate the following theorem for functional, we will remember the following:

If differentiable function $f(x)$ reaches a maximum or minimum in an internal point $x=x_0$ in the domain of the function, at this point the diferential of the function is equal to zero, i.e $df=0$.

Theorem 1. If the functional $v[y(x)]$, which has a variation reaches a maximum or minimum at $y=y_0(x)$, where $y(x)$ are internal points of the domain of functional, its variation is zero at $y=y_0(x)$, i.e $\delta v=0$.

Proof of Theorem 1

In fixed $y_0(x)$ and δy, $v[y_0(x)+\alpha\delta y]=\varphi(\alpha)$ is a function of α , that at $\alpha=0$, supposedly reaches maximum or minimum, hence its derivative at $\alpha=0$ is zero, i.e

$\varphi'(0)=0$ *или* $\frac{\partial}{\partial\alpha}v[y_0(x)+\alpha\delta y]|_{\alpha=0}=0$.

Therefore, the curve reaches an extreme of the functional, then its variance is equal to zero. (α can adopt values around the point $\alpha=0$, both positive and negative values, as extreme concept of $y_0(x)$ are internal points of the domain of functional.)

The concept of extreme functional needs clarification. Multiband compression has been noted above, the closeness of the curves may be interpreted differently, so that, when the maximum or minimum is necessary to specify the order in terms of distance. If the functional reaches a maximum or minimum of the curve, with respect to all curves, which is small, i.e. with respect to the curves in near-proximity sense the zero line, then the maximum or minimum is called strong. If the same functional $v[y(x)]$ reaches a maximum or minimum of the curve $y = y_0(x)$, only in respect of the curve close to the point in the proximity of the first order, i.e with respect to curves closer to not only the Y axis but also in the direction of the tangent, the maximum or minimum It called weak. Obviously, if the curve is reached strong maximum (or minimum), it is like reaching weak, as if the curve is near, meaning the proximity of the first order, it is close to and sense the proximity of zero order. Also, it is possible to curve reaches weak maximum (or minimum) and at the same time not reached a strong maximum (or minimum), ie between curves near as ordinate and the direction of the tangent may not exist, which (in the case when miminuma) and between curves close ordinate, but not in the direction of the tangent, you can find those for which (in the case when the minimum).

Conclusion

Methods to solve variational problems are very similar methods to study the maximum and minimum functions, so the theory of maximum and minimum functions is close to the concepts and theorems for functionals.

The variation of functional is primary and linear with respect to a part of the increase of functional and in the study of functional, variation plays the same role played by the differential in the study of functions.

If there is a variation in the basic sense of the linear increase of functional, then there is a variation in the derivative sense of the parameter with initial value, and that both definitions are equivalent which permit to give another definition of functional variation and a theorem, link between the extremes of a functional and its variation.

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