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Wavelets and Continuous Wavelet Transform

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Abstract

Wavelet transform or wavelet analysis is a recently developed mathematical tool for signal analysis. In this paper are shown some relation for the wavelet transform and using Wolfram Mathematica 10 is given the wavelet transform of the signal $f(x)=\sin x$. A prove that multiplication of real number with wavelet and sum of two wavelets are wavelets is also provided.

Introduction

Wavelets have been around since the late 1980s, and have found many applications in signal processing, numerical analysis, operator theory, and other fields. In 1982, the French geophysicist Jean Morlet introduced the concept of a "wavelet", which means a small wave and studied wavelet transform as a new tool for seismic signal analysis. The wavelet transform is a tool that cuts up data or functions or operators into different frequency components, and then studies each component with a resolution matched to its scale. Wavelets have generated significant interest from both theoretical and applied researchers over the last few decades. In areas such as time-series analysis, approximation theory and numerical solutions of differential equations, wavelets are recognized as powerful weapons not just tools.

A. *Spaces* of functions

- Hilbert space $L^2(\mathbb{R})$ with $\langle f,g\rangle=\int_{\mathbb{R}} f(t)\overline{g}(t)dt$.
- Fourier transform of $f\in L^2(\mathbb{R})$ is $\hat{f}(\omega)=\int_{-\infty}^{\infty} f(t)e^{-i\omega t}dt.$
- $\mathbb{C}^2([a,b])$ - the space of functions with continuous derivatives up to order 2.
- Convolution of $f,g\in L^1(R)$ is $(f*g)(x)=\int_R f(y)g(x-y)dy$.

B. Wavelets

- $\psi\in L^2(\mathbb{R})$ is a wavelet if $\int_{-\infty}^{+\infty}\psi(t)dt=0$.
- $\psi\in L^2(\mathbb{R})$ is a wavelet if satisfy the admissibility condition

$$C_\psi=\int_{-\infty}^{\infty}\frac{|\hat{\psi}(\omega)|^2}{|\omega|}d\omega<\infty .$$

Lemma 1. Let $\varphi(t)$ be a nonzero n-times ($n\geq1$) differentialble functions such that $\varphi^{(n)}(x)\in L^2(R)$. Then $\psi(x)=\varphi^{(n)}(x)$ is a wavelet.

Corollary 2. For every non-zero element $\psi(x)\in L^2(R)$ with compact support, the following statements are equivalent:

1. The function $\psi(t)$ is a wavelet
2. The admissibility condition is satisfied.

Theorem 3. Let ψ be a wavelet and φ a bounded integrable function then the convolution $\psi*\varphi$ is a wavelet.

Example1. (Haar Wavelet)

$$\psi(x)=\begin{cases}1, & 0\leq x<\frac{1}{2}\\ -1, & \frac{1}{2}\leq x\leq 1\\ 0, & x<0 \text{ or } x>1\end{cases}$$

Example 2. (Mexican Hat Wavelet)

$$\psi(x)=-\frac{d^2}{dx^2}e^{-\frac{x^2}{2}}=(1-x^2)e^{-\frac{x^2}{2}}$$

Example 3. (Convolution)

$$\psi(x)*e^{-x^2},$$

where ψ is a Haar wavelet

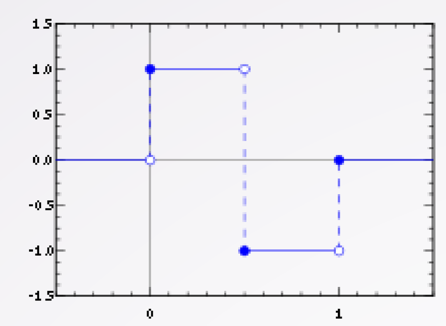


Figure 1. Haar Wavelet

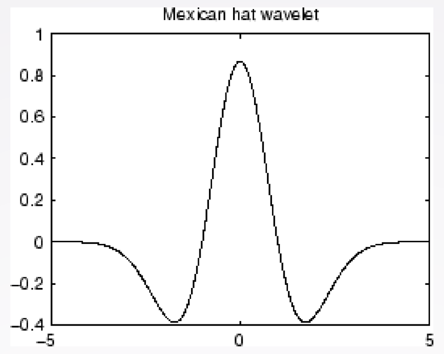


Figure 2. Mexican hat wavelet

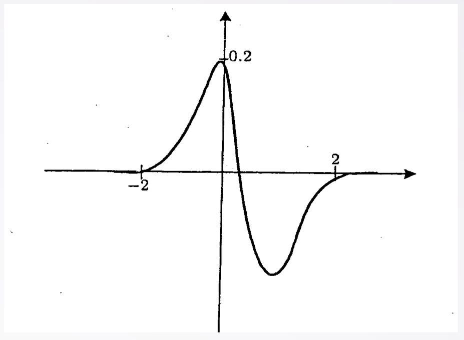


Figure 3. Convolution of the Haar wavelet and $\varphi(x)=e^{-x^2}$.

C. Basic operations

- Translation operator - $T_x f(t)=f(t-x)$
- Dilatation operator - $D_a f(x)=|a|^{-1/2}f(a^{-1}x)$.

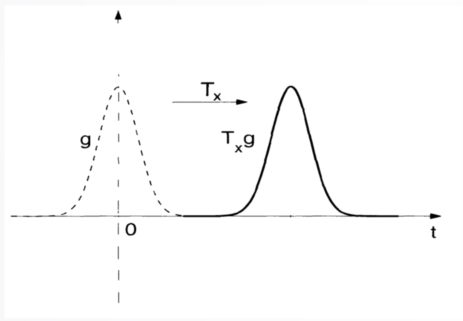
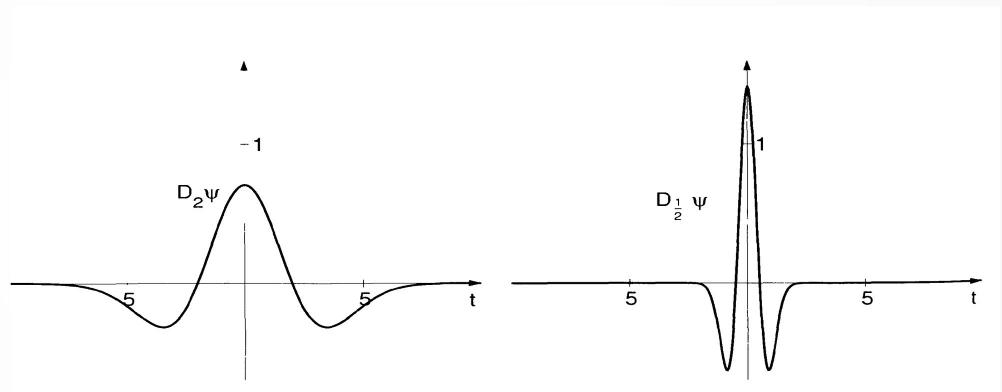


Figure 4. Translated function



a) b)

Figure 5. Dilatation of the function a) by factor 2 b) by factor 1/2

D. The continuous wavelet transform

- The continuous wavelet transform W_ψ of $f\in L^2(R)$, where $a\in R\setminus\{0\}$, $b\in R$ and $\overline{\psi}$ denotes complex conjugate is defined as

$$W_\psi f(a,b)=|a|^{-1/2}\int_R f(t)\,\overline{\psi}\left(\frac{t-b}{a}\right)dt \quad .$$

- Family of wavelets $\psi_{a,b}(t)=|a|^{-1/2}\psi\left(\frac{t-b}{a}\right), a\in R\setminus\{0\}, b\in R$
- Other forms - $W_\psi f(a,b)=\langle f,\psi_{a,b}\rangle=\langle f,T_bD_a\psi\rangle$.

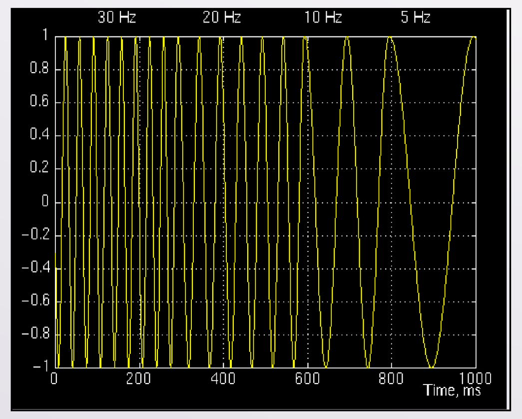


Figure 6. Signal with four frequencies

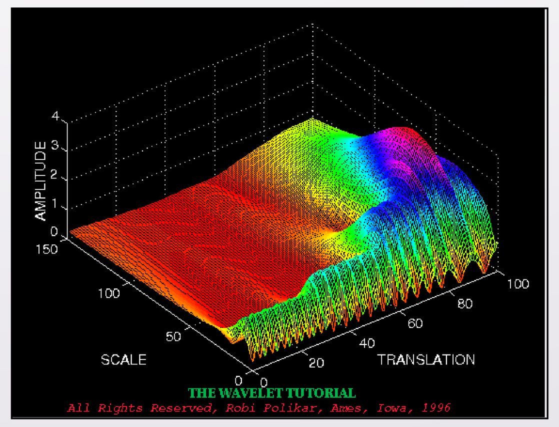


Figure 7. Wavelet transform of the signal

Main results

Problem 1. If $\psi_1(x)$ and $\psi_2(x)$ are two wavelets then for $a\in R$, $a\psi_1(x)$ and $\psi_1(x)+\psi_2(x)$ are wavelets.

Prove: $C_\psi=\int_{-\infty}^{\infty}\frac{|\alpha\hat{\psi}_1(\omega)|^2}{|\omega|}d\omega=\alpha^2\int_{-\infty}^{\infty}\frac{|\hat{\psi}_1(\omega)|^2}{|\omega|}d\omega<\infty$

$$C_\psi=\int_{-\infty}^{\infty}\frac{|\hat{\psi}(\omega)|^2}{|\omega|}d\omega\leq\int_{-\infty}^{\infty}\frac{|\hat{\psi}_1(\omega)|^2}{|\omega|}d\omega+\int_{-\infty}^{\infty}\frac{|\hat{\psi}_2(\omega)|^2}{|\omega|}d\omega+\int_{-\infty}^{\infty}\frac{2\int_{-\infty}^{\infty}\left|\psi_1(\omega)e^{-2\pi i\omega t}\right|dt\int_{-\infty}^{\infty}\left|\psi_2(\omega)e^{-2\pi i\omega t}\right|dt}{|\omega|}d\omega<\infty$$

Problem 2. Let ψ and ϕ are wavelets and let $f,g\in L^2(R)$, for $\forall\alpha,\beta\in C$. Prove the relations:

a) $W_\psi(\alpha f+\beta g)(a,b)=\alpha W_\psi f(a,b)+\beta W_\psi g(a,b)$,

b) $W_\psi(T_x f)(a,b)=W_\psi f(a,b-x)$, $c\in R$.

c) $W_\psi(D_s f)(a,b)=W_\psi f(a/s,b/s)$, $s>0$.

d) $W_\psi f(a,b)=W_f\psi(1/a,-b/a),\ a\neq0$.

e) $W_{\alpha\psi+\beta\phi}f(a,b)=\overline{\alpha}\,W_\psi f(a,b)+\overline{\beta}W_\phi f(a,b)$,

f) $W_{T_x\psi}f(a,b)=W_\psi f(a,b+xa)$, $x\in R$.

g) $W_{D_x\psi}f(a,b)=1/\sqrt{x}\,W_\psi f(ac,b),\ x>0$.

Problem 3. Compute the wavelet transform of the signal $f(x)=\sin x$ using the Haar wavelet.

Solution: $W_\psi f(a,b)=|a|^{-1/2}\int_R \sin x\,\overline{\psi}(\frac{x-b}{a})dx=|a|^{-1/2}\left(\cos b+\cos(a+b)-2\cos\left(\frac{a}{2}+b\right)\right)$

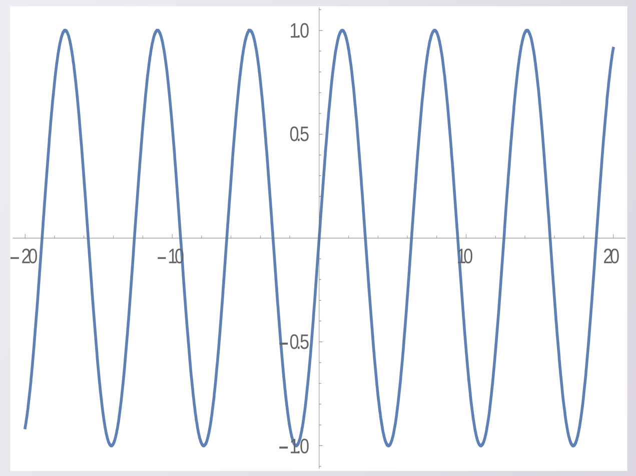
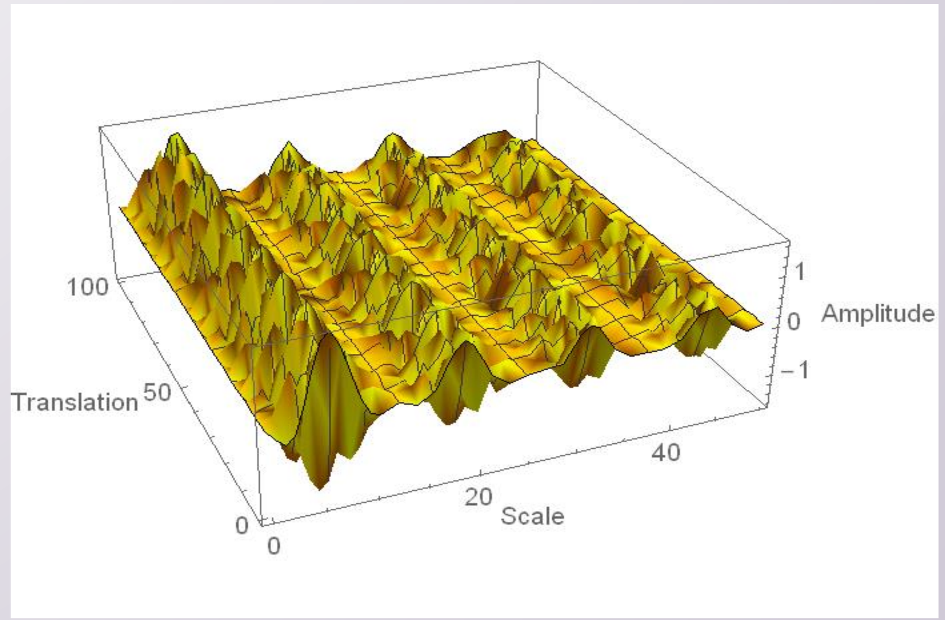


Figure 8. a) Signal $f(x)=\sin x$,



b) Wavelet transform of the signal