Some random matrix results with application to the multiple access channel

Marija Miteva #, Tome Eftimov* and Zoran Utkovski #*
* - Labaratory for Complex Systems and Networks, Macedonian Academy of Sciences and Arts
# - Faculty of Computer Science, University Goce Delcev Stip, R. Macedonia

The study of the capacity bounds for the K-user multiple access channel (MAC), reveals that certain random matrix theory results are of practical relevance to the problem. The result form the point-to-point block Rayleigh fading channel indicate that isotorpically distibuted input signals are capacity achieving in the high-SNR regime. The derivation of the mutual information obtained with these input signals in the MAC case requires analysis of the eigenvalues of Gramm matrices of the type $\mathbf{VV}^H$, where $\mathbf{V} \in \mathbb{C}^{K\times T}$ and the rows of V are random vectors which are uniformly distributed on a unit sphere in $\mathbb{C}^T$.

System model

- K single-antenna users;
- one receiver with $N \geq K$ receive antennas;

The system model is the following:

$$\mathbf{Y} = \mathbf{SX} + \mathbf{W},$$

where
$\mathbf{X} \in \mathbb{C}^{K\times T}$, $\mathbf{Y} \in \mathbb{C}^{N\times T}$, $\mathbf{S} \in \mathbb{C}^{N\times K}$ with i.i.d $CN(0,1)$ entries and $\mathbf{W} \in \mathbb{C}^{N\times T}$ with i.i.d $CN(0,1)$ entries.

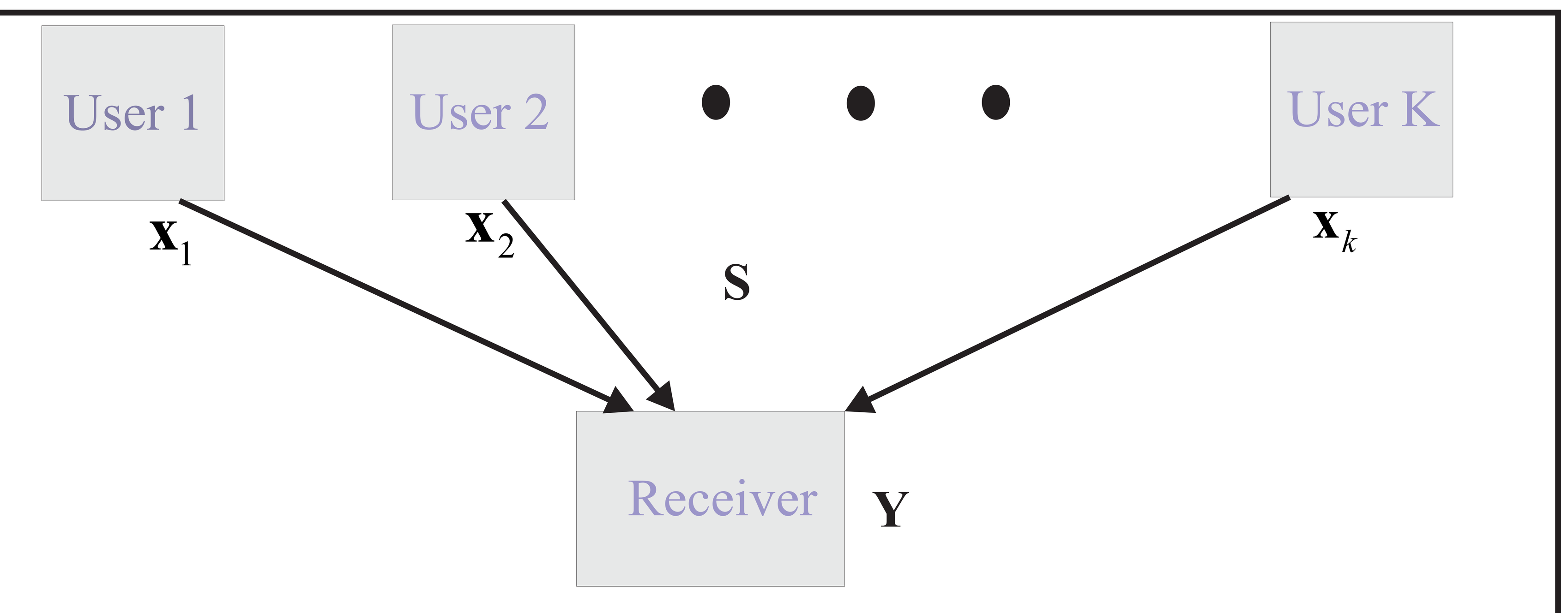

Figure 1: System model

Non-coherent communication

- The channel **S** is unknown;
- Communication based on subspaces;
- Conjecture: The capacity achieving inputs are of the form:

$$\mathbf{x}_l = \sqrt{\frac{rT}{K}}\mathbf{v}_l,$$

where the vectors $\mathbf{v}_l$ are independent and uniformly distributed on the unit sphere $\mathbb{C}^T$.
- r - SNR per receive antenna.

Computation of mutual information

We are interested in the mutual information of non-coherent MAC:

$$I(\mathbf{X};\mathbf{Y}) = \mathrm{h}(\mathbf{Y}) - \mathrm{h}(\mathbf{Y}\,|\,\mathbf{X}),$$

in high-SNR regime.

1. Derivation of $\mathrm{h}(\mathbf{Y})$

In the high-SNR approximation, for $h(\mathbf{Y})$ we obtain:

$$\begin{aligned} h(\mathbf{Y}) \geq\ & (TK - K^2 + NK)\log_2\left(\frac{rT}{K}\right) + \log_2 |G(T,K)| \\ & + (T-K)\mathbb{E}\left[\log_2 \det\left(\mathbf{S}^H\mathbf{S}\right)\right] \\ & + (T-K+N)\mathbb{E}\left[\log_2 \det\left(\boldsymbol{\Sigma\Sigma}^{\mathrm{H}}\right)\right] \\ & + (NT - KT + K^2)\log_2(pe), \end{aligned}$$

where $\mathbf{V} = \mathbf{U\Sigma V}_1^{\mathrm{H}}$ is the SVD decomposition.

2. Derivation of $\mathrm{h}(\mathbf{Y}\,|\,\mathbf{X})$

For $\mathrm{h}(\mathbf{Y}\,|\,\mathbf{X})$ we obtain:

$$\begin{aligned} h(\mathbf{Y}\,|\,\mathbf{X}) = NK\log_2\left(\frac{rT}{K}\right) + N\mathbb{E}[\log_2 \det(\Sigma\Sigma^H)] \\ + TN\log_2(pe) \end{aligned}$$

Mutual information

For $I(\mathbf{X};\mathbf{Y})$ we have:

$$\begin{aligned} \mathrm{I}(\mathbf{X};\mathbf{Y}) \geq\ & K\left(1-\frac{K}{T}\right)\log_2\frac{Tr}{K} + \frac{1}{T}\log_2|G(T,K)| \\ & - K\left(1-\frac{K}{T}\right)\log_2(pe) + \left(1-\frac{K}{T}\right)\mathbb{E}\left[\log_2\det(\mathbf{S}^{\mathrm{H}}\mathbf{S})\right] \\ & + \left(1-\frac{K}{T}\right)\mathbb{E}\left[\log_2\det(\boldsymbol{\Sigma\Sigma}^{\mathrm{H}})\right]. \end{aligned}$$

Of interest is to evaluate

$$\Delta = \mathbb{E}[\log_2 \det(\boldsymbol{\Sigma\Sigma}^H)],$$

where $\mathbf{V} = \mathbf{U\Sigma V}_1^{\mathrm{H}}$ is the SVD decomposition.

First, we note that $\det(\boldsymbol{\Sigma\Sigma}^H) = \det(\mathbf{VV}^H)$. We recall that the rows of $\mathbf{V} \in \mathbb{C}^{K\times T}$ are independent and uniformly distributed on the unit sphere in $\mathbb{C}^T$.

Using the classical QR decomposition of **V**, we get

$$\det(\mathbf{VV}^H) = \prod_{k=1}^{K} R_{kk}^2.$$

In the Wishart case, when the row vectors of **V** are choosen independently from normal distribution, the variables R_{kk}^2 are independent and chi-square distributed with respective parameters $b'(T-k+1), k=1,\ldots,K$.

In the Gram case, when the row vectors of **V** are choosen independently and uniformly distributed on the unit sphere in $\mathbb{C}^T$, the variables R_{kk}^2 are independent and beta distributed with respective parameters $(b'(T-k+1), b'(k-1)), k=2\ldots,K$.

$b' = \frac{b}{2}$, where b=1,2, or 4, which corresponds to the classical matrix models (real, complex and quaternionic).

Let we denote

$$G_{T,K} = \ln\det(\mathbf{VV}^H) = \ln\prod_{k=1}^{K} R_{kk}^2 = \sum_{k=2}^{K}\ln R_{kk}^2$$

Our case corresponds to the Gram case. Using the result from the Gram case, we obtain

$$\mathbb{E}[G_{T,K}] = -\sum_{k=1}^{K}\sum_{i=1}^{\infty}\frac{\Gamma(T)}{\Gamma(T-k+1)\Gamma(k-1)}\cdot\frac{\boldsymbol{\beta}(k-1+i, N-k+1)}{i},$$

where $\boldsymbol{\beta}(.,.)$ is beta function.

When $N \to \infty$, and the ratio K/N is fixed, $K/N = c, c \in (0,1)$,

$$\lim_n \sup_{K\leq T}\left|\frac{1}{T}\mathbb{E}[G_{T,K}] + J\left(1-\frac{K}{T}\right)\log_2 e\right| = 0,$$

where

$$\begin{aligned} & J(u) = u\log u - u + 1,\ for\ u > 0 \\ & J(u) = 1,\ for\ u = 0 \\ & J(u) = +\infty,\ for\ u < 0 \end{aligned}$$

Finally we have

$$\Delta = \log_2 e\,\mathbb{E}[G_{T,K}].$$

We derive some results for the eigenvalues of Gram matrices of the type $\mathbf{VV}^{\mathrm{H}}$, where $\mathbf{V} \in \mathbb{C}^{K\times T}$ and the rows of **V** are random vectors which are uniformly distributed on a unit sphere $\mathbb{C}^T$. These matrices are of relevance to the derivation of the mutual informaton of the K-user MAC, obtained with isotropically distributed unitary input signals. As result, we pave the way for the capacity characterization of the non-coherent K-user MAC in the high-SNR regime.