# Mutual Information of Unitary Isotropically Distributed Inputs in the Non-coherent MAC

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*Abstract*— **We study the sum-rate of the non-coherent, block Rayleigh fading MAC, with $K$ single-antenna users and a receiver which employs $N \geq K$ antennas. We derive the mutual information $I(\mathbf{x}_1, \ldots, \mathbf{x}_K; \mathbf{Y})$ with independent unitary isotropically distributed (i. d.) input signals, a setup motivated by the capacity analysis of point-to-point MIMO channels. The results are derived in a semi-analytical form and are valid in the whole SNR region (not only the high-SNR regime).**

## I. Introduction

We study the multiple-access channel (MAC), where $K$ users transmit their data to a common receiver with $N \geq K$ receive antennas. The channel model is the classical Rayleigh block fading model, initiated by Hochwald and Marzetta [2]. We focus on the *non-coherent* communication scenario where the users (terminals) are aware of the statistics of the fading but not of its realization. Instead of using pilot symbols to estimate the channel, we adopt the subspace-based approach where no channel estimation is performed and the information is conveyed by subspaces invariant to fading [1].

*Contribution:*The contribution of this work is a step towards the non-coherent MAC capacity characterization, since we study the mutual information between the users and the receiver (sum-rate), achievable when the individual users employ independent, unitary *isotropically distributed* (i. d.) input signals. The use of these input signals is motivated by the capacity analysis of point-to-point MIMO channels undergoing block Rayleigh fading [2], [1].

The main result of the work is a closed form expression of the receive signal density, which yields a method that allows evaluating the mutual information of the non-coherent multiple access channel with independent, unitary, isotropically distributed user inputs. The derivation is motivated by recent results from the literature which present expressions for the mutual information achieved by IID complex Gaussian inputs in the point-to-point MIMO setting [4], and by the method described in [3] for evaluation the receive signal density of unitary in the same setting. To recall, obtaining closed form solution of the mutual information of unitary i. d. input signals is a difficult task, since it is associated with assessing differential entropies which are difficult to characterize. Moreover, straight Monte-Carlo computation is not feasible because it would entail large-dimensional histograms.

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The main value of the obtained results is that they offer the opportunity to address the performance loss associated with having no CSI (neither transmit nor receive), and having no cooperation between the users, compared to the case with full channel knowledge and full cooperation (coherent MIMO).

## II. System Model

The system model can be written as follows:

$$\mathbf{Y} = \mathbf{S}\mathbf{X} + \mathbf{W}, \tag{1}$$

where the rows of $\mathbf{X} \in \mathbb{C}^{KxT}$ represent the users' transmit vectors, $\mathbf{Y} \in \mathbb{C}^{NxT}$ is the receive matrix, $\mathbf{S} \in \mathbb{C}^{NxK}$ is the channel matrix, and $\mathbf{W} \in \mathbb{C}^{NxT}$ the noise matrix. Both $\mathbf{S}$ and $\mathbf{W}$ have IID zero-mean unit-variance complex Gaussian entries. According to our setup, the $l$-th row vector of $\mathbf{X}$, $l = 1, \ldots, K$, is given by $\mathbf{x_l} = \sqrt{\frac{\rho T}{K}}\mathbf{v_l}$, where each $\mathbf{v_l}$ is independent and uniformly distributed (thus i. d.) on the unit sphere in $\mathbb{C}^T$. $\rho$ indicates the average SNR per receive antenna.

## III. Computation of the Mutual information

The mutual information between the $K$ users and the receiver is $I(\mathbf{X}; \mathbf{Y}) = h(\mathbf{Y}) - h(\mathbf{Y} \mid \mathbf{X})$. In the following we will sketch the derivation of $h(\mathbf{Y} \mid \mathbf{X})$ and $h(\mathbf{Y})$.

*Derivation of $h(\mathbf{Y} \mid \mathbf{X})$:* For given $\mathbf{X}$, the rows of $\mathbf{Y}$ are independent Gaussian vectors, with identical covariance matrices

$$\mathbb{E}\left[\mathbf{y}_l^{\mathrm{H}}\mathbf{y}_l \mid \mathbf{X}\right] = \mathbf{I}_T + \mathbf{X}^{\mathrm{H}}\mathbf{X}, \tag{2}$$

where $l = 1, \ldots, K$. By using this observation, and after some linear algebra manipulation, we obtain

$$h(\mathbf{Y} \mid \mathbf{X}) = N\mathbb{E}\Delta + TN\log(\pi e), \tag{3}$$

where

$$\Delta = \mathbb{E}\log\det\left(\mathbf{I}_K + \frac{\rho T}{K}\mathbf{V}\mathbf{V}^{\mathrm{H}}\right). \tag{4}$$

Further, let us denote $\mathbf{G} = \mathbf{V}\mathbf{V}^{\mathrm{H}}$ and $\mu = \frac{\rho T}{K}$. In order to evaluate $\Delta$, we need to evaluate the product of the eigenvalues of the Gramm matrix $\mathbf{G} = \mathbf{V}\mathbf{V}^{\mathrm{H}}$, where the rows of $\mathbf{V} \in \mathbb{C}^{K\times T}$ are independent and uniformly distributed on the unit sphere in $\mathbb{C}^T$. Since no general results for the eigenvalue distribution of this type of matrices are found in the literature, we proceed by taking the QR decomposition (rather than the SVD), $\mathbf{V} = \mathbf{Q}\mathbf{R}$, where $\mathbf{Q} \in \mathbb{C}^{T\times K}$ is unitary, and $\mathbf{R} \in \mathbb{C}^{T\times K}$ is upper-triangular with non-negative diagonal elements. From the Bartlett decomposition we have that $\det(\mathbf{G}) = \prod_{k=1}^{K} r_{kk}^2$,where $r_{kk}^2$ are the diagonal elements of $\mathbf{R}$. From [5] we have that $r_{kk}^2$ , $k = 2, \ldots, K$ are independent and beta distributed with respective parameters $(\alpha_k, \beta_k) = (T-k+1, k-1)$, and $r_{11}^2 = 1$. Hence, for $\Delta$ we obtain

$$\Delta = \mathbb{E} \log \prod_{k=1}^{K} \lambda_k, \tag{5}$$

where, $\lambda_k = 1 + \mu r_{kk}^2$, $k = 1, \ldots, K$. With this, $\Delta$ becomes

$$\Delta = \log(1+\mu) + \sum_{k=2}^{K} \int_1^{\mu+1} f(\lambda_k) \log \lambda_k \mathrm{d}\lambda_k, \tag{6}$$

where the pdf of $\lambda_k$ is given by

$$f(\lambda_k) = \frac{1}{\mu} \frac{\Gamma(\alpha_k + \beta_k)}{\Gamma(\alpha_k)\Gamma(\beta_k)} \left(\frac{\lambda_k - 1}{\mu}\right)^{\alpha_k - 1} \left(1 - \frac{\lambda_k - 1}{\mu}\right)^{\beta_k - 1}. \tag{7}$$

After some manipulation we obtain

$$\Delta = \log(1+\mu) + \sum_{k=2}^{K} \sum_{i=1}^{\infty} (-1)^{i-1} \frac{\mu^i}{i} \frac{\Gamma(\alpha_k + \beta_k)}{\Gamma(\alpha_k)\Gamma(\beta_k)} \mathrm{B}(i + \alpha_k, \beta_k), \tag{8}$$

where

$$\mathrm{B}(i + \alpha_k, \beta_k) = \int_0^1 x_k^{i+\alpha_k - 1} (1 - x_k)^{\beta_k - 1} \mathrm{d}x_k \tag{9}$$

is the B-function. Finally for $h(\mathbf{Y} \mid \mathbf{X})$ we obtain

$$\begin{aligned} h(\mathbf{Y}|\mathbf{X}) &= N\Delta + TN \log(\pi e) \\ &= N \log(1+\mu) \\ &\quad + \sum_{k=2}^{K} \sum_{i=1}^{\infty} (-1)^{i-1} \frac{\mu^i}{i} \frac{\Gamma(T)}{\Gamma(T-k+1+i)\Gamma(k-1)} \\ &\quad \cdot \mathrm{B}(i + T - k + 1, k - 1)) + TN \log(\pi e). \end{aligned} \tag{10}$$

*Derivation of* $h(\mathbf{Y})$*:* The evaluation of $h(\mathbf{Y})$ requires a closed form solution for the received signal density $p(\mathbf{Y})$. We start by writing

$$p(\mathbf{Y}|\mathbf{V}) = \frac{1}{\pi^{TN}} \frac{\exp\left\{-\mathrm{tr}\left[\mathbf{Y}(\mathbf{I}_T + \mathbf{X}^{\mathrm{H}}\mathbf{X})^{-1}\mathbf{Y}^{\mathrm{H}}\right]\right\}}{\det\left(\mathbf{I}_T + \mathbf{X}^{\mathrm{H}}\mathbf{X}\right)^N}. \tag{11}$$

By taking the LQ decomposition $\mathbf{X} = \mathbf{L_X Q_X}$, and the eigenvalue decomposition $\mathbf{L}^{\mathrm{H}}\mathbf{L} = \mathbf{U_L \Lambda U_L}^{\mathrm{H}}$, we obtain

$$p(\mathbf{Y}|\mathbf{V}) = \frac{\exp\left\{-\mathrm{tr}\left[\mathbf{Y}\mathbf{Y}^{\mathrm{H}} - \mathbf{Y}\mathbf{\Phi}\left(\mathbf{I}_K + \frac{K}{\rho T}\mathbf{\Lambda}^{-1}\right)^{-1}\mathbf{\Phi}^{\mathrm{H}}\mathbf{Y}^{\mathrm{H}}\right]\right\}}{\pi^{TN}\det\left(\mathbf{I}_K + \frac{\rho T}{K}\mathbf{V}\mathbf{V}^{\mathrm{H}}\right)^N}, \tag{12}$$

where $\mathbf{\Phi} = \mathbf{Q}_{\mathbf{X}}^{\mathrm{H}}\mathbf{U_L}$. This implies that $p(\mathbf{Y}|\mathbf{V}) = p(\mathbf{Y}|\mathbf{\Lambda})$, where

$$p(\mathbf{Y}|\mathbf{\Lambda}) = \frac{\exp\left\{-\mathrm{tr}\mathbf{Y}\mathbf{Y}^{\mathrm{H}}\right\}}{\pi^{TN}\det\left(\mathbf{I}_K + \frac{\rho T}{K}\mathbf{V}\mathbf{V}^{\mathrm{H}}\right)^N} \cdot \Xi, \tag{13}$$

and

$$\Xi = \mathbb{E}_{|\Lambda} \exp\left\{\mathrm{tr}\left[\mathbf{Y}\mathbf{\Phi}(\mathbf{I}_K + \frac{K}{\rho T}\mathbf{\Lambda}^{-1})\mathbf{\Phi}^{\mathrm{H}}\mathbf{Y}^{\mathrm{H}}\right]\right\}. \tag{14}$$

The expectation is taken over the i. d. unitary matrix $\Phi$. If we introduce the SVD $\mathbf{Y}^{\mathrm{H}}\mathbf{Y} = \mathbf{U_Y A U_Y}$, after some manipulation for $\Xi$ we obtain

$$\Xi = \mathbb{E}_{|\mathbf{\Lambda}} \exp\left\{\mathrm{tr}\left[\mathbf{A\Phi B\Phi}^{\mathrm{H}}\right]\right\}, \tag{15}$$

where $\mathbf{B} = \left(\mathbf{I}_K + \frac{K}{\rho T}\mathbf{\Lambda}^{-1}\right)^{-1}$. The derivation is based on the fact that, since $\mathbf{\Phi}$ is unitary and i. d, $\mathbf{U_Y}^{\mathrm{H}}\mathbf{\Phi}$ is also unitary and i. d.

It will be convenient to transform the non-negative diagonal matrix $\mathbf{A} \geq 0$ into a negative-definite matrix by choosing the scalar $a > 0$ such that $a\mathbf{I}_T > \mathbf{A}$. With this one can show that $\Xi$ becomes

$$\Xi = \exp\left\{\mathrm{tr}\left[a\mathbf{B}\right]\right\} \mathbb{E}_{|\mathbf{\Lambda}} \exp\left\{-\mathrm{tr}\left[a\mathbf{I}_T - \mathbf{A\Phi B\Phi}^{\mathrm{H}}\right]\right\} \tag{16}$$

Using the results presented in [3](pp.1476-1477 eq.(13,16,21)) for $\Xi$ we obtain

$$\begin{aligned} \Xi =& \frac{\Gamma(T)\cdots\Gamma(T+1-K)(-1)^{K(K-1)/2}}{\Gamma(K+1)\cdots\Gamma(1)(2\pi)^K} \cdot \int d\lambda_1 \cdots \int d\lambda_K \\ &\cdot \prod_{k=1}^{K} \left[\frac{e^{-i\lambda_k}}{(-b_k\sigma_1 - j\lambda_k)\cdots(-b_k\sigma_M - j\lambda_k)(-j\lambda_k)^{T-M}}\right] \\ &\cdot \prod_{l<k} ((\lambda_k - jb_k a) - (\lambda_l - jb_l a))^2. \end{aligned} \tag{17}$$

Finally, for $p(\mathbf{Y})$ we have

$$\begin{aligned} p(\mathbf{Y}) =& \frac{(-1)^{K(K-1)/2}}{\pi^{TN}} \frac{\exp\{-\mathrm{tr}\mathbf{Y}\mathbf{Y}^{\mathrm{H}}\}}{\det(\mathbf{I}_K + \frac{\rho T}{K}\mathbf{V}\mathbf{V}^{\mathrm{H}})^N} \\ &\cdot \frac{\Gamma(T)\cdots\Gamma(T+1-K)}{\Gamma(K)\cdots\Gamma(1)} \det\mathbf{F}, \end{aligned} \tag{18}$$

where F is a $K \times K$ Hankel matrix whose $(p, n)$ entry is

$$\begin{aligned} F_{pn} =& j[\sum_{m=1}^{M} \frac{\exp\{b_n\sigma_m\}(jb_n\sigma_m - jb_n a)^{p+n-2}}{(b_n\sigma_m)^{T-M}\prod_{l\neq m}(b_n\sigma_m - b_l\sigma_l)} \\ &+ \left.\frac{f^{(T-M-1)}(\lambda)}{(T-M-1)!)}\right|_{\lambda=0}], \end{aligned} \tag{19}$$

and

$$f(\lambda) = \frac{\exp\{-j\lambda\}(\lambda - jba)^{p+n-2}}{(-b\sigma_1 - j\lambda)\cdots(-b\sigma_M - j\lambda)}. \tag{20}$$

Having obtained the received signal density, $h(\mathbf{Y})$ can be evaluated by a Monte-Carlo approach

$$h(\mathbf{Y}) = -\int p(\mathbf{Y}) \log p(\mathbf{Y}) d\mathbf{Y}. \tag{21}$$

## References


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