Non-coherent two-way relaying with decode-and-forward: An achievable rate region

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***Abstract*— We study the two-user MIMO block fading two-way relay channel in the non-coherent setting, where neither the terminals nor the relay have transmit or receive knowledge of the channel realizations. We present a lower bound on the achievable sum-rate with decode-and-forward (DF) at the relay node. As a byproduct we present an achievable pre-log region of the DF scheme, defined as the limiting ratio of the rate region to the logarithm of the signal-to-noise ratio (SNR) as the SNR tends to infinity.**

I. Introduction

We consider a three-node network where one node acts as a relay to enable bidirectional communication between two other nodes (terminals). We assume that no direct link is available between the terminals, a setup often denoted as the separated two-way relay channel (sTWRC). The system is assumed to operate in the half-duplex mode where the nodes do not transmit and receive signals simultaneously. Such half-duplex relay systems suffer from a substantial loss in terms of spectral efficiency due to the pre-log factor $1/2$, which dominates the capacity at high signal-to-noise ratio (SNR).

A two-way relaying protocol has been proposed to overcome such a spectral efficiency loss in the half-duplex one-way system [1], [2]. Also, the analog network coding (ANC) based on self interference cancelling has been employed for improving the performance of the two-way system in [2]–[4].

There have been substantial recent efforts to characterize the performance bounds of the two-way relay channel, and finding the optimal transmission strategy (capacity region) for the two-way relay with a single relay node has lately attracted a lot of attention. Results for the achievable rate regions of different relaying strategies including *amplify-and-forward (AF)*, *decode-and-forward (DF)*, *compress-and-forward* (CF), etc., have been reported in [5], [6] and [2], [3], [7]–[9].

These works address the so called *coherent* setup when some amount of channel knowledge at the terminals and/or at the relay is assumed. In contrast to these approaches, we focus on the *non-coherent* communication scenario where the terminals and the relay are aware of the statistics of the fading but not of its realization, i.e. they have *neither* transmit *nor* receive channel knowledge. We note that this setup is different from the one analyzed in [10] where the authors address the case with multiple relays, and denote as "non-coherent" the setup when the relays do not have any knowledge of the channel realizations, but the terminals have receive channel knowledge.

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Studying the capacity in the non-coherent setting is fundamental to the characterization of the performance loss incurred by the lack of *a priori* channel knowledge at the receiver, compared to the *coherent* case when a genie provides the receiver with perfect channel state information. Further, it gives a fundamental assessment of the cost associated with obtaining and distributing channel knowledge in the wireless network.

The exact characterization of the capacity region for two-way relaying channels in the non-coherent regime is an open problem, even under the high signal-to-noise-ratio (high-SNR) assumption. As a step towards the characterization of the capacity region in the high-SNR regime, we will concentrate on the performance of the decode-and-forward (DF) strategy and derive a lower bound on the achievable rate region. As a byproduct of the analysis, we will present an achievable pre-log region of the DF scheme, defined as the limiting ratio of the rate region to the logarithm of the SNR as the SNR tends to infinity. The motivation to study the pre-log region is the fact that it is the main indicator of the performance of a particular relaying strategy in the high-SNR regime.

Notation: Uppercase boldface letters denote matrices and lowercase boldface letters designate vectors. Uppercase calligraphic letters denote sets. The superscript H stands for Hermitian transposition. We denote by $p(\mathbf{R})$ the distribution of a random matrix $\mathbf{R}$. Expectation is denoted by $\mathbb{E}[\cdot]$ and trace by $\mathrm{tr}(\cdot)$. We denote by $\mathbf{I}_N$ the $N \times N$ identity matrix. Furthermore, $\mathcal{CN}(0, \sigma^2)$ stands for the distribution of a circularly-symmetric complex Gaussian random variable with covariance σ^2. For two functions $f(x)$ and $g(x)$, the notation $f(x) = o(g(x))$, $x \to \infty$, means that $\lim_{x\to\infty} |f(x)/g(x)| = 0$. Finally, $\log(\cdot)$ indicates the natural logarithm.

II. Preliminaries

A. Capacity of the MIMO Point-to-point Channel

The non-coherent MIMO point-to-point channel is a starting point for the analysis of the non-coherent two-way relay channel. The system equation is given as

$$\mathbf{Y} = \mathbf{H}\mathbf{X} + \mathbf{W}, \tag{1}$$

where $\mathbf{X} \in \mathbb{C}^{M\times T}$ is the transmit matrix with power constraint $\mathbb{E}[\mathrm{tr}\left(\mathbf{X}^{\mathrm{H}}\mathbf{X}\right)] \leq PT$, $\mathbf{H} \in \mathbb{C}^{N\times M}$ is the channel matrix, with

i. i. d $\mathcal{CN}(0,1)$ entries and $\mathbf{W} \in \mathbb{C}^{N\times T}$ is the noise matrix, with i. i. d. $\mathcal{CN}(0,\sigma^2)$ entries. The SNR per receive antenna is $\frac{P}{\sigma^2}$. When $N \geq M$ and $T \geq M+N$, the high-SNR capacity of this channel is given by [11]

$$C_{M,N} = M\left(1-\frac{M}{T}\right)\log_2\frac{P}{\sigma^2} + c_{M,N} + o(1), \quad (2)$$

where $c_{M,N}$ is a term which depends only on M, N and T, but does not depend on the SNR and $o(1)$ is a term which vanishes at high SNR.

The key element exploited in [11] to establish (2) is the optimality of *isotropically distributed* unitary input signals in the high-SNR regime [12].

Definition 1: We say that a random matrix $\mathbf{R} \in \mathbb{C}^{M\times T}$, for $T \geq M$, is isotropically distributed (i. d.) if its distribution is invariant under rotation

$$p(\mathbf{R}) = p(\mathbf{RQ}), \quad (3)$$

for any deterministic unitary matrix $\mathbf{Q} \in \mathbb{C}^{T\times T}$.
The optimal input distribution is thus of the form

$$\mathbf{X} = \sqrt{\frac{PT}{M}}\mathbf{V}, \quad (4)$$

where $\mathbf{V} \in \mathbb{C}^{M\times T}$ is uniformly distributed in the *Stiefel manifold*, $\mathcal{V}^{\mathbb{C}}_{T,M}$ which is the collection of all $M \times T$ unitary matrices (which fulfill $\mathbf{VV}^{\mathrm{H}} = \mathbf{I}_M$).

B. Geometric interpretation

The fact that the optimal input has isotropic directions suggests the use of a different coordinate system [11], where the $M \times T$ transmit matrix X is represented as the linear subspace $\Omega_{\mathbf{X}}$ spanned by its row vectors, together with an $M \times M$ matrix $\mathbf{C}_{\mathbf{X}}$ which specifies the M row vectors of $\mathbf{X}$ with respect to a canonical basis in $\Omega_{\mathbf{X}}$

$$\begin{aligned} \mathbf{X} &\to (\mathbf{C}_{\mathbf{X}}, \Omega_{\mathbf{X}}) \\ \mathbb{C}^{M\times T} &\to \mathbb{C}^{M\times M} \times \mathcal{G}^{\mathbb{C}}_{T,M}, \end{aligned} \quad (5)$$

where $\mathcal{G}^{\mathbb{C}}_{T,M}$ denotes the collection (set) of all M-dimensional linear subspaces of $\mathbb{C}^T$ and is known as the (complex) Grassmann manifold, with (complex) dimension $\dim(\mathcal{G}^{\mathbb{C}}_{T,M}) = M(T-M)$.

For i. d. unitary input signal $\mathbf{X}$, the information-carrying object is the subspace $\Omega_{\mathbf{X}}$, i. e. $I(\mathbf{X};\mathbf{Y}) = I(\Omega_{\mathbf{X}};\mathbf{Y})$, which defines the Grassmann manifold $\mathcal{G}^{\mathbb{C}}_{T,M}$ as the relevant coding space. Additionally, $\dim\left(\mathcal{G}^{\mathbb{C}}_{T,M}\right)$ equals the pre-log term in the capacity expression (number of d. o. f.).

The instrumental in the derivation of (2) is the calculation of the entropy of an isotropically distributed matrix with the help of the decomposition (coordinate transformation) (5). Namely, for an i. d. random matrix $\mathbf{R} \in \mathbb{C}^{M\times T}$ admitting the decomposition (5), $\mathbf{R} \to (\mathbf{C}_R, \mathbf{\Omega}_R)$, the entropy $h(\mathbf{R})$ is calculated as

$$\begin{aligned} h(\mathbf{R}) \approx{}& h(\mathbf{C}_R) + \log_2|\mathcal{G}^{\mathbb{C}}_{T,M}| \\ &+ (T-M)\mathrm{E}\left[\log_2\det\left(\mathbf{RR}^{\mathrm{H}}\right)\right]. \end{aligned} \quad (6)$$

The term $|\mathcal{G}^{\mathbb{C}}_{T,M}|$ is the volume of the Grassmann manifold $\mathcal{G}^{\mathbb{C}}_{T,M}$ and appears in the capacity expression due to the coordinate transformation.

III. System Model and Problem Formulation

A. Two-way Relaying in the half-duplex Mode

We consider a wireless network with two users, A and B, one relay node R, and no direct link between the terminals. All the transceivers (terminals and relay) work in a half-duplex regime i. e. they can not transmit and receive simultaneously.

As in the point-to-point case, we assume block Rayleigh model where the channel is constant in a certain time block of length T, denoted as the *coherence time*. Although a block-fading structure represents a simplification of reality, it does capture the essential nature of fading and yields results that are very similar to those obtained with continuous fading models [13].

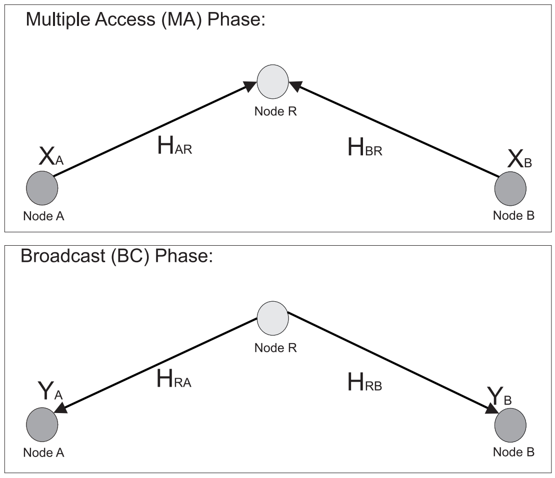

Fig. 1. Two-way relaying comprised of two phases, MA and BC.

The communication takes part in two phases, each of duration T. The first phase is the multiple access (MA) phase, where both users simultaneously transmit their information. The signals transmitted from the users are combined at the relay R, which performs a certain operation on the received signal, depending on the relaying strategy. In the next phase, denoted as broadcast phase (BC) the relay R broadcasts a signal to both users. Based on the received signal and the knowledge about its' own transmitted signal, each user decodes the information from the other user. We address the MIMO setup where user A and user B employ M_A and M_B transmit antennas respectively, and the relay has M_R antennas.

Within the MA phase of duration T, the channel between A and R is described by the matrix $\mathbf{H}_{AR}$, with elements which are i. i. d. circular complex Gaussian, $\mathcal{CN}(0,1)$. Similarly, the channel between B and R is described by the matrix $\mathbf{H}_{BR}$, with elements which are i. i. d. circular complex Gaussian, $\mathcal{CN}(0,1)$. The channels matrices in the BC phase are $\mathbf{H}_{RA}$ and $\mathbf{H}_{RB}$ respectively.

We well address two cases. In the first case we will assume that the channels $\mathbf{H}_{AR}$ and $\mathbf{H}_{RA}$, as well as $\mathbf{H}_{BR}$ and $\mathbf{H}_{RB}$ are reciprocal, i.e. $\mathbf{H}_{RA} = \mathbf{H}^{\mathrm{H}}_{AR}$ and $\mathbf{H}_{RB} = \mathbf{H}^{\mathrm{H}}_{BR}$. In the second case we will assume that the channels in the MA and the BC phase are independent, i. e the elements of $\mathbf{H}_{RA}$ and $\mathbf{H}_{RB}$ are i. i. d $\mathcal{CN}(0,1)$.

The signal transmitted from user A is a $M \times T$ matrix $\mathbf{X}_A$. We denote the codebook of user A as $\mathcal{X}_A$. Similarly, user B sends a $M \times T$ transmit matrix $\mathbf{X}_B$. The codebook of user B is denoted as $\mathcal{X}_B$. P is the average transmit power for one transmission of user A and user B. Further, we denote the average power for one transmission for the relay as P_R. Additionally, we have the constraint on the total network power, $2P + P_R = P_{tot}$ which serves for fair comparison, since it considers the transmit powers of all network nodes. Without making any assumptions about the network geometry (topology), results from the coherent setup [14] suggest that the power allocation $P = P_R/2 = P_{tot}/4$ maximizes the SNR per receive antenna.

B. Decode-and-Forward (DF) Two-way Relaying

The motivation to consider DF is that when the number of relay antennas is $M_R \geq M_A + M_B$, the relay can compensate for the fact that user A and user B do not cooperate. The performance limit of the DF scheme in the MA phase is then the achievable rate region for the multiple access channel with two users, employing respectively M_A and M_B transmit antennas, and a receiver employing M_R receive antennas. This system, on the other hand is upper-bounded by the MIMO point-to-point channel with $M_A + M_B$ transmit and M_R receive antennas [11].

1) MA phase: The signal received at the relay R in the MA phase is given as

$$\mathbf{Y}_R = \mathbf{H}_{AR}\mathbf{X}_A + \mathbf{H}_{BR}\mathbf{X}_B + \mathbf{Z}_R, \tag{7}$$

where $\mathbf{W}_R$ is the noise matrix at the relay R, with elements which are i. i. d. complex Gaussian, $\mathcal{CN}(0, \sigma^2)$.

We note that this is essentially the same setup as the non-coherent MAC. Now, having the received signal $\mathbf{Y}_R$, the relay performs decoding. We denote by $\hat{\mathbf{X}}_A$ and $\hat{\mathbf{X}}_B$ the decoded versions of the respective transmit signals, $\mathbf{X}_A$ and $\mathbf{X}_B$.

2) BC phase: In the BC phase the relay R broadcasts the signal $\mathbf{X}_R$ which, in the general case is a function of $\hat{\mathbf{X}}_A$ and $\hat{\mathbf{X}}_B$, $\mathbf{X}_R = f(\hat{\mathbf{X}}_A, \hat{\mathbf{X}}_B$, subject to the power constraint (11).

The signals received by user A and user B are given by

$$\begin{aligned} \mathbf{Y}_A &= \mathbf{H}_{RA}\mathbf{X}_R + \mathbf{W}_A \\ , \mathbf{Y}_B &= \mathbf{H}_{RB}\mathbf{X}_R + \mathbf{W}_B, \end{aligned} \tag{8}$$

where $\mathbf{W}_A$ and $\mathbf{W}_B$ are the corresponding noise matrices and have i.i.d. complex Gaussian, $\mathcal{CN}(0, \sigma^2)$ entries.

C. Problem Formulation

We are interested in the sum of the individual rates, achievable under a certain input distribution

$$R_{sum} = R_{A\to B} + R_{B\to A} \tag{9}$$

where $R_{A\to B}$ and $R_{B\to A}$ are the individual rates for the links $A \to B$ and $B \to A$ respectively, defined as

$$\begin{aligned} R_{A\to B} &\doteq \frac{1}{2} I\left(\mathbf{X}_A; \mathbf{Y}_B \mid \mathbf{X}_B\right); \\ R_{B\to A} &\doteq \frac{1}{2} I\left(\mathbf{X}_B; \mathbf{Y}_A \mid \mathbf{X}_A\right), \end{aligned} \tag{10}$$

subject to

$$\begin{aligned} \mathbb{E}\left[\operatorname{tr}\left(\mathbf{X}_A\mathbf{X}_A^{\mathrm{H}}\right)\right] &\leq PT; \\ \mathbb{E}\left[\operatorname{tr}\left(\mathbf{X}_B\mathbf{X}_B^{\mathrm{H}}\right)\right] &\leq PT; \\ \mathbb{E}\left[\operatorname{tr}\left(\mathbf{X}_R\mathbf{X}_R^{\mathrm{H}}\right)\right] &\leq P_R T. \end{aligned} \tag{11}$$

The pre-log factor $\frac{1}{2}$ in the individual rates is caused by the half-duplex constraint. We say that a rate pair (R_1, R_2) is *achievable* if there is a strategy which attains $R_A = R_1$ and $R_B = R_2$ simultaneously.

Since the two-way communication takes place in two phases, MA and BC, the achievable sum-rate is

$$R_{sum} = \min\left(R_{sum}^{(MA)}, R_{sum}^{(BC)}\right), \tag{12}$$

where $R_{sum}^{(MA)}$ and $R_{sum}^{(BC)}$ are the achievable sum-rates for the MA and the BC phase respectively.

IV. Performance of the DF Communication Strategy

A. Input distributions

We will assume independent, unitary, isotropically distributed input signals $\mathbf{X}_A$ and $\mathbf{X}_B$, of the form

$$\begin{aligned} \mathbf{X}_A &= \sqrt{\frac{PT}{M}}\mathbf{V}_A; \\ \mathbf{X}_B &= \sqrt{\frac{PT}{M}}\mathbf{V}_B, \end{aligned} \tag{13}$$

where $\mathbf{V}_A$ and $\mathbf{V}_B$ are uniformly distributed on the Stiefel manifold $\mathcal{V}_{T,M}^{\mathbb{C}}$. Although we do not know the optimal joint distribution $p(\mathbf{X}_A, \mathbf{X}_B)$ in general, this assumption is motivated by the results for the capacity achieving input distribution in the point-to-point case [11]. We note that by making this assumption, we actually derive a lower bound on the DF performance in the two-way relay channel.

B. Derivation of $R_{sum}^{(MA)}$

The channel in the MA phase corresponds to a non-coherent two-user MAC, given by

$$\mathbf{Y}_R = \mathbf{H}_{MA}\mathbf{X}, \tag{14}$$

where $\mathbf{X} = \begin{pmatrix} \mathbf{X}_A \\ \mathbf{X}_B \end{pmatrix}$ and $\mathbf{H}_{MA} = \begin{pmatrix} \mathbf{H}_{AR} & \mathbf{H}_{BR} \end{pmatrix}$.

We are interested in

$$\begin{aligned} R_{sum}^{(MA)} &= I\left(\mathbf{X}; \mathbf{Y}_R\right) \\ &= h(\mathbf{Y}_R) - h\left(\mathbf{Y}_R \mid \mathbf{X}\right), \end{aligned} \tag{15}$$

where $\mathbf{X}_A$ and $\mathbf{X}_B$ are drawn from the input distribution (13).

The degrees of freedom of this channel are derived in [**?**], by applying a geometric approach. According to this, the pre-log factor in the sum-rate expression is given by

$$\Pi_{sum}^{(MA)} = (M_A + M_B)(T - M_A - M_B) \tag{16}$$

C. Derivation of $R_{sum}^{(BC)}$

In the following we assume that the relay has successfully decoded $\mathbf{X}_A$ and $\mathbf{X}_B$ in the MA phase, i.e $\hat{\mathbf{X}}_A = \mathbf{X}_A$ and $\hat{\mathbf{X}}_B = \mathbf{X}_B$. We note that this is assumption is reasonable, if in the MA phase we operate at a sum-rate smaller than the sum-rate $R_{sum}^{(MA)}$ achievable with unitary, isotropically distributed inputs $\mathbf{V}_A$ and $\mathbf{V}_B$.

We are interested in the mutual information between $\mathbf{X}_R$ and $\mathbf{Y} = \begin{pmatrix} \mathbf{Y}_A \\ \mathbf{Y}_B \end{pmatrix}$ given by

$$R_{sum}^{(BC)} = I\left(\mathbf{X}_R; \mathbf{Y}_A \mid \mathbf{X}_A\right) + I\left(\mathbf{X}_R; \mathbf{Y}_B \mid \mathbf{X}_B\right), \quad (17)$$

where

$$\begin{aligned} I\left(\mathbf{X}_R; \mathbf{Y}_A \mid \mathbf{X}_A\right) &= h\left(\mathbf{Y} \mid \mathbf{X}_A\right) - h\left(\mathbf{Y} \mid \mathbf{X}_A, \mathbf{X}_B\right), \\ I\left(\mathbf{X}_R; \mathbf{Y}_B \mid \mathbf{X}_B\right) &= h\left(\mathbf{Y} \mid \mathbf{X}_B\right) - h\left(\mathbf{Y} \mid \mathbf{X}_A, \mathbf{X}_B\right) \end{aligned} \quad (18)$$

Motivated from the results for the point-to-point MIMO channel, we choose the broadcast signal $\mathbf{X}_R$ in the following way

$$\begin{aligned} \mathbf{X}_R &= \begin{pmatrix} \tilde{\mathbf{X}}_A \\ \tilde{\mathbf{X}}_B \end{pmatrix} \\ &= \sqrt{\frac{PT}{M_A + M_B}} \tilde{\mathbf{V}} \end{aligned} \quad (19)$$

where

$$\tilde{\mathbf{V}} = \begin{pmatrix} \tilde{\mathbf{V}}_A \\ \tilde{\mathbf{V}}_B \end{pmatrix} \quad (20)$$

is uniformly distributed on the Stiefel manifold $\mathcal{V}_{T,K}^{\mathbb{C}}$ of unitary $(M_A + M_B) \times T$ matrices.

We note that it is also possible to transmit $\mathbf{V} = \begin{pmatrix} \mathbf{V}_A \\ \mathbf{V}_B \end{pmatrix}$, instead of $\tilde{\mathbf{V}} = \begin{pmatrix} \tilde{\mathbf{V}}_A \\ \tilde{\mathbf{V}}_B \end{pmatrix}$. However, this is suboptimal from an information-theoretic point of view, since $\mathbf{V}\mathbf{V}^{\mathrm{H}} \neq \mathbf{I}_{M_A+M_B}$ because $\mathbf{V}_A$ and $\mathbf{V}_B$ are independent (and thus not orthogonal in general).

We can see this relay function as a form of precoding,

$$\begin{pmatrix} \tilde{\mathbf{X}}_A \\ \tilde{\mathbf{X}}_B \end{pmatrix} = \begin{pmatrix} \mathbf{P}_{AA} & \mathbf{P}_{AB} \\ \mathbf{P}_{BA} & \mathbf{P}_{BB} \end{pmatrix} \begin{pmatrix} \mathbf{X}_A \\ \mathbf{X}_B \end{pmatrix}. \quad (21)$$

The precoding matrix P can be obtained from the LQ decomposition

With the above, the signal received by user A is

$$\begin{aligned} \mathbf{Y}_A &= \mathbf{H}_{RA}^{(A)} \tilde{\mathbf{X}}_A + \mathbf{H}_{RA}^{(B)} \tilde{\mathbf{X}}_B + \mathbf{W}_A \\ &= \mathbf{H}_{RA}^{(A)} \left(\mathbf{P}_{AA}\mathbf{X}_A + \mathbf{P}_{AB}\mathbf{X}_B\right) \\ &\quad + \mathbf{H}_{RA}^{(B)} \left(\mathbf{P}_{BA}\mathbf{X}_A + \mathbf{P}_{BB}\mathbf{X}_B\right) + \mathbf{W}_A \\ &= \left(\mathbf{H}_{RA}^{(A)}\mathbf{P}_{AA} + \mathbf{H}_{RA}^{(B)}\mathbf{P}_{BA}\right)\mathbf{X}_A \\ &\quad + \left(\mathbf{H}_{RA}^{(A)}\mathbf{P}_{AB} + \mathbf{H}_{RA}^{(B)}\mathbf{P}_{BB}\right)\mathbf{X}_B + \mathbf{W}_A, \end{aligned} \quad (22)$$

where

$$\mathbf{H}_{RA} = \begin{pmatrix} \mathbf{H}_{RA}^{(A)} & \mathbf{H}_{RA}^{(B)} \end{pmatrix}. \quad (23)$$

By analogy, the signal received by user B is

$$\begin{aligned} \mathbf{Y}_B = &\left(\mathbf{H}_{RB}^{(A)}\mathbf{P}_{AA} + \mathbf{H}_{RB}^{(B)}\mathbf{P}_{BA}\right)\mathbf{X}_A \\ &+ \left(\mathbf{H}_{RB}^{(A)}\mathbf{P}_{AB} + \mathbf{H}_{RB}^{(B)}\mathbf{P}_{BB}\right)\mathbf{X}_B + \mathbf{W}_B. \end{aligned} \quad (24)$$

Due to symmetry, it suffices to analyze the signal received by one of the users. let us, without loss of generality concentrate on user B.

We start by deriving $h(\mathbf{Y}_B \mid \mathbf{X}_B)$. Let us first denote $\mathbf{H}_A = \mathbf{H}_{RB}^{(A)}\mathbf{P}_{AA} + \mathbf{H}_{RB}^{(B)}\mathbf{P}_{BA}$ and $\mathbf{H}_B = \mathbf{H}_{RB}^{(A)}\mathbf{P}_{AB} + \mathbf{H}_{RB}^{(B)}\mathbf{P}_{BB}$.

Since conditioning does not increase entropy, we can write

$$\begin{aligned} h(\mathbf{Y}_B \mid \mathbf{X}_B) \geq &h(\mathbf{Y}_B \mid \mathbf{X}_B, \mathbf{H}_B = \mathbf{H}_{RB}\mathbf{H}_{BR}) \\ \approx &h(\sqrt{\gamma_R}\mathbf{H}_{RB}\mathbf{H}_{AR}\mathbf{X}_A \mid \mathbf{H}_{RB}) \\ = &MT\log_2 \gamma_R + h(\mathbf{H}_{AR}\mathbf{X}_A) \\ &+ M\mathbb{E}\left[\log_2 \det(\mathbf{H}_{RB}\mathbf{H}_{RB}^{\mathrm{H}})\right]. \end{aligned} \quad (25)$$

We note that $\mathbf{H}_{AR}\mathbf{X}_A$ is isotropically distributed. Hence, from [11] we have

$$\begin{aligned} h(\mathbf{H}_{AR}\mathbf{X}_A) = &MT\log_2 \frac{PT}{M} + h(\mathbf{C}_{\mathbf{H}_{AR}\mathbf{V}_A}) + \log_2 |\mathcal{G}_{T,M}^{\mathbb{C}}| \\ &+ (T-M)\mathrm{E}\left[\log_2 \det\left(\mathbf{H}_{AR}\mathbf{H}_{AR}^{\mathrm{H}}\right)\right] \\ = &MT\log_2 \frac{PT}{M} + h(\mathbf{H}_{AR}) + \log_2 |\mathcal{G}_{T,M}^{\mathbb{C}}| \\ &+ (T-M)\mathrm{E}\left[\log_2 \det\left(\mathbf{H}_{AR}\mathbf{H}_{AR}^{\mathrm{H}}\right)\right] \\ = &MT\log_2 \frac{PT}{M} + M^2\log_2 \pi e + \log_2 |\mathcal{G}_{T,M}^{\mathbb{C}}| \\ &+ (T-M)\mathrm{E}\left[\log_2 \det\left(\mathbf{H}_{AR}\mathbf{H}_{AR}^{\mathrm{H}}\right)\right]. \end{aligned} \quad (26)$$

What remains is to evaluate $h(\mathbf{Y}_B \mid \mathbf{X}_A, \mathbf{X}_B)$. We start by observing that given $\mathbf{X}_A$ and $\mathbf{X}_B$, $\mathbf{Y}_B$ is not Gaussian, since $\mathbf{H}_A$, $\mathbf{H}_B$ and $\mathbf{W}_B$ are not Gaussian. Nevertheless, the following holds

$$h(\mathbf{Y}_B \mid \mathbf{X}_A, \mathbf{X}_B) \leq h(\mathbf{N}_B), \quad (27)$$

where $\mathbf{N}_B$ is Gaussian with the same covariance matrix as the one of $\mathbf{Y}_B \mid \mathbf{X}_A, \mathbf{X}_B$,

$$\begin{aligned} \mathbb{E}\left[\mathbf{N}^{\mathrm{H}}\mathbf{N}\right] &= \mathbb{E}\left[\mathbf{Y}_B^{\mathrm{H}}\mathbf{Y}_B \mid \mathbf{X}_A, \mathbf{X}_B\right] \\ &= \frac{M\gamma_R PT}{M}\mathbf{V}_A^{\mathrm{H}}\mathbf{V}_A + \frac{M\gamma_R PT}{M}\mathbf{V}_B^{\mathrm{H}}\mathbf{V}_B + \nu^2\mathbf{I}_T. \end{aligned} \quad (28)$$

Hence, we can write

$$\begin{aligned} h(\mathbf{Y}_B \mid \mathbf{X}_A, \mathbf{X}_B) \leq &M\mathbb{E}[\log_2 \det(\nu^2\mathbf{I}_T + \frac{M\gamma_R PT}{M}\mathbf{V}_A^{\mathrm{H}}\mathbf{V}_A \\ &+ \frac{M\gamma_R PT}{M}\mathbf{V}_B^{\mathrm{H}}\mathbf{V}_B)] + \log_2\left(\pi e\right)^{TM} \\ = &M\mathbb{E}[\log_2 \det(\mathbf{I}_{2M} + \frac{M\gamma_R PT}{M\nu^2}\mathbf{V}_A^{\mathrm{H}}\mathbf{V}_A \\ &+ \frac{M\gamma_R PT}{M\nu^2}\mathbf{V}_B^{\mathrm{H}}\mathbf{V}_B)] + MT\log_2\left(\pi e\nu^2\right) \\ \approx &M\mathbb{E}[\log_2 \det(\mathbf{V}_A^{\mathrm{H}}\mathbf{V}_A + \mathbf{V}_B^{\mathrm{H}}\mathbf{V}_B)] \\ &+ 2M^2\log_2 \frac{M\gamma_R PT}{M\nu^2} + MT\log_2 \pi e\nu^2. \end{aligned} \quad (29)$$

From (25), (26) and (29), for $I(\mathbf{X}_A, \mathbf{X}_B; \mathbf{Y}_B)$ we obtain

$$\begin{aligned} I(\mathbf{X}_A; \mathbf{Y}_B \mid \mathbf{Y}_B) \geq & M(T-2M)\log_2 \frac{\gamma_R PT}{\nu^2} \\ & + \log_2 |\mathcal{G}^{\mathbb{C}}_{T,M}| - MT\log_2 M \\ & + (T-M)\mathrm{E}\left[\log_2 \det\left(\mathbf{H}_{AR}\mathbf{H}_{AR}^{\mathrm{H}}\right)\right] \\ & + M\mathbb{E}\left[\log_2 \det(\mathbf{H}_{RB}\mathbf{H}_{RB}^{\mathrm{H}})\right] \\ & - M\mathbb{E}[\log_2 \det(\mathbf{V}_A^{\mathrm{H}}\mathbf{V}_A + \mathbf{V}_B^{\mathrm{H}}\mathbf{V}_B)] \\ & - M(T-M)\log_2 \pi e, \\ = & M(T-2M)\log_2 \frac{\gamma_R PT}{\nu^2} \\ & + \log_2 |\mathcal{G}^{\mathbb{C}}_{T,M}| - MT\log_2 M \\ & + T\mathrm{E}\left[\log_2 \det\left(\mathbf{H}_{AR}\mathbf{H}_{AR}^{\mathrm{H}}\right)\right] \\ & - M\mathbb{E}[\log_2 \det(\mathbf{V}_A^{\mathrm{H}}\mathbf{V}_A + \mathbf{V}_B^{\mathrm{H}}\mathbf{V}_B)] \\ & - M(T-M)\log_2 \pi e, \end{aligned} \quad (30)$$

where the last equation follows from the fact that

$$\mathrm{E}\left[\log_2 \det\left(\mathbf{H}_{AR}\mathbf{H}_{AR}^{\mathrm{H}}\right)\right] = \mathbb{E}\left[\log_2 \det(\mathbf{H}_{RB}\mathbf{H}_{RB}^{\mathrm{H}})\right]. \quad (31)$$

Now, if we assume the power allocation $P = P_R/2$, in the high SNR regime (when $\sigma^2 \to 0$), we have that $\gamma_R \approx 1$ and $\nu^2 \approx M\sigma^2 + \sigma^2$. Hence, (32) becomes

$$\begin{aligned} I(\mathbf{X}_A; \mathbf{Y}_B \mid \mathbf{Y}_B) \geq & M(T-2M)\log_2 \frac{PT}{(\sigma^2 + \frac{\sigma^2}{M})M} \\ & + \log_2 |\mathcal{G}^{\mathbb{C}}_{T,M}| - MT\log_2 M \\ & + T\mathrm{E}\left[\log_2 \det\left(\mathbf{H}_{AR}\mathbf{H}_{AR}^{\mathrm{H}}\right)\right] \\ & - M\mathbb{E}[\log_2 \det(\mathbf{V}_A^{\mathrm{H}}\mathbf{V}_A + \mathbf{V}_B^{\mathrm{H}}\mathbf{V}_B)] \\ & - M(T-M)\log_2 \pi e, \end{aligned} \quad (32)$$

V. Performance of a random coding scheme

In order to assess the performance of non-coherent coding in the DF setup, we look at a random code construction which is inspired by the insights obtained from the analysis performed in the previous Section.

A. MA Phase

In the MA phase, we choose the codewords of $\mathcal{X}_A$ and $\mathcal{X}_B$ to be independent and uniformly distributed in $\mathcal{G}^{\mathbb{C}}_{T,M_A}$ and $\mathcal{G}^{\mathbb{C}}_{T,M_A}$ respectively. The motivation behind is that in the non-coherent setup the information-carrying objects are random subspaces, i.e

$$I(\mathbf{X}, \mathbf{Y}_R) = I(\mathbf{\Omega}_\mathbf{X}, \mathbf{Y}_R), \quad (33)$$

where $\mathbf{\Omega}_\mathbf{X}$ is the subspace spanned by the rows of $\mathbf{X} = \begin{pmatrix} \mathbf{X}_A \\ \mathbf{X}_B \end{pmatrix}$. An unique representation of the subspace $\mathbf{\Omega}_\mathbf{X}$ can be obtained by performing the LQ decomposition

$$\mathbf{X} = \mathbf{LQ}, \quad (34)$$

where $\mathbf{L}$ is lower-triangular and $\mathbf{Q}$ is $(M_A + M_B) \times t$ unitary.

For two codewords $\mathbf{X}_{A_i} \in \mathcal{X}_A$ and $\mathbf{X}_{B_j} \in \mathcal{X}_B$, we can think of $\mathbf{Q}_{i,j}$ as being a *joint* codeword. We denote by $\mathcal{Q}$ the joint codebook which consists of all $\mathbf{Q}_{i,j}$ where $i = 1, \ldots, |\mathcal{X}_A|$ and $j = 1, \ldots, |\mathcal{X}_B|$. Let us denote by $R_A^* = \frac{1}{T}\log_2 |\mathcal{X}_A|$ and $R_B^* = \frac{1}{T}\log_2 |\mathcal{X}_B$ the rates of user A and user B respectively. For the sum-rate we have $R_{sum}^* = R_A^* + R_B^* = \frac{1}{T}\log_2 |\mathcal{Q}|$.

We perform the joint decoding at the relay R by looking at the most likely transmitted subspace $\mathbf{\Omega}_\mathbf{X}$, which is done by projecting the received matrix $\mathbf{Y}_R$ on all possible codewords $\mathbf{Q}_{i,j} \in \mathcal{Q}$

$$\hat{\mathbf{Q}} = \arg \max_{\mathbf{Q}_{i,j} \in \mathcal{Q}} \|\mathbf{Y}_R \mathbf{Q}_{i,j}^{\mathrm{H}}\|_F^2 \quad (35)$$

If the rate pair (R_A^*, R_B^*) is within the achievable region of the MAC, in the high SNR regime we can assume that $\hat{\mathbf{Q}} = \mathbf{Q}$.

B. BC Phase

In the BC phase the relay R forwards the information about the codewords $\mathbf{X}_A$ and $\mathbf{X}_B$. Since this phase, similar as the MA phase, requires no channel knowledge (neither transmit nor receive), the communication is also performed based on subspaces. The relay R sends information about the subspace $\Omega_\mathbf{X}$

C. Discussion

The term $\mathrm{E}\left[\log_2 \det\left(\mathbf{H}_{AR}\mathbf{H}_{AR}^{\mathrm{H}}\right)\right]$ in the expression (32) can be further written as

$$\mathrm{E}\left[\log_2 \det\left(\mathbf{H}_{AR}\mathbf{H}_{AR}^{\mathrm{H}}\right)\right] = \sum_{i=1}^{M} \mathbb{E}\left[\log_2 \chi_{2i}^2\right], \quad (36)$$

where χ_{2i}^2 is Chi-square distributed of dimension $2i$ [11]. The term $\mathbb{E}[\log_2 \det(\mathbf{V}_A^{\mathrm{H}}\mathbf{V}_A + \mathbf{V}_B^{\mathrm{H}}\mathbf{V}_B)]$, on the other hand, is a measure for the "orthogonality defect" of the matrix $\mathbf{V} = \left(\mathbf{V}_A // \mathbf{V}_B \right)$ and appears in the expression since user A and user B do not cooperate, i. e they send independent messages.

The exact characterization of this term is of interest when we are interested not only in the pre-log factors, but also in the constant terms which appear in the capacity expressions.

VI. Examples and Practical Considerations

An achievable pre-log region for the two-way relay channel in the non-coherent setup, with $M = 2$ and $T = 12$ is shown in Fig. 1. We note that we use the fact that any point (pre-log pair) which lies on the line between two corner points is also achievable (by time sharing).

The region is compared to the TDMA case, both coherent and non-coherent. For the particular choice of the parameters, the joint scheme outperforms TDMA, both coherent and non-coherent. Actually, it can be shown that, given that T is sufficiently large, the two-way relaying AF scheme always outperforms TDMA. It follows directly from (32) that when $T \geq 3M$ two-way relaying with AF outperforms non-coherent TDMA. When $T \geq 4M$, two-way relaying with AF outperforms coherent TDMA as well.

In the context of emerging systems such as 3GPP LTE or IEEE 802.16 WiMAX, symbol periods of around $10-20$ ms still exhibit flat-fading and the block fading model applies. For pedestrian velocities, T is in the range of several hundreds, for vehicular velocities up to $v = 120Km/h$, T is around 10, and for high-speed trains with velocities $v \geq 300km/h$,

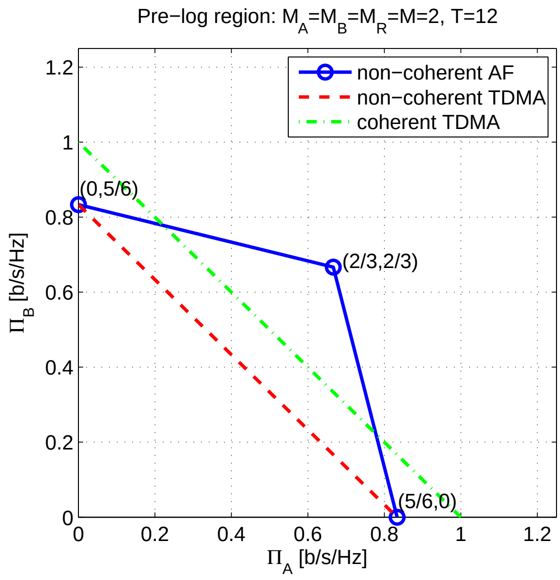

Fig. 2. An achievable pre-log region for the block two-way relay channel. The coherence time is $T = 12$, user A and B have $M_A = M_B = 2$ antennas.

$T \leq 5$. Hence, in the first example, two-way relaying would be preferable over TDMA for practical numbers of transmit antennas. In the second case this would still hold for $M \leq 2$. In the last case this would only hold for $M = 1$ and already for $M > 1$, TDMA is the preferred strategy.

VII. Conclusions

We performed an analysis on the achievable rate region of the two-way relaying channel with amplify-and-forward (AF) at the relay node. We concentrated on the non-coherent setup where neither the terminals nor the relay have knowledge of the channel realizations. As a byproduct we presented an achievable pre-log region of the AF scheme. The analysis was supported by a geometric interpretation, based on the paradigm of subspace-based communication.

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