# WAVELET-GALERKIN SOLUTION OF SOME ORDINARY DIFFERENTIAL EQUATIONS

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- **Spaces of functions**
- **Galerkin method for ODE**
- **Wavelets and MRA**
- **Wavelet-Galerkin method for ODE**
- **Application of W-G method**

## Spaces of functions

- $L^2(R)$

  $< f, g >= \int_R f(t)\overline{g(t)}dt$

- $L^2([0, 1])$
- $C^2([0, 1])$

## Galerkin metod for ODE

- *Sturm-Liouville equation*

$$Lu(t) \equiv -\frac{d}{dt}\left(a(t)\frac{du}{dt}\right) + b(t)u(t) = f(t),\ 0 \leq t \leq 1 \tag{1}$$

with DBC

$$u(0) = u(1) = 0. \tag{2}$$

- 1) $\{v_j\}$ - complete orthonormal system for $L^2([0,1])$
  2) every $v_j \in C^2([0,1])$
  3) $v_j(0) = v_j(1) = 0$.
- Approximation $u_s$ of the exact solution $u$

$$u_s = \sum_{k\in\Lambda} x_k v_k \tag{3}$$

- Criterion for coefficients $x_k$

$$< Lu_s, v_j >=< f, v_j >, \forall j \in \Lambda. \quad (4)$$

  - If we substitute the equation (3) in (4) we obtain

$$\sum_{k \in \Lambda} \langle Lv_k, v_j \rangle x_k = \langle f, v_j \rangle, \forall j \in \Lambda. \quad (5)$$

  - $A = [a_{j,k}]_{j,k \in \Lambda}$, $a_{j,k} = \langle Lv_k, v_j \rangle$ ;
    $X = (x_k)_{k \in \Lambda}$; $Y = (y_k)_{k \in \Lambda}$, $y_k = \langle f, v_k \rangle$

$$AX = Y. \quad (6)$$

  - Wavelet-Galerkin method: functions $v_j$ are wavelets

## Wavelets

- wavelet $\psi$: $L^2$ function which satisfy the admissibility condition

$$C_\psi = \int_{-\infty}^{\infty} \frac{|\hat{\psi}(\omega)|^2}{|\omega|} d\omega < \infty. \tag{7}$$

The condition (7) implies that

$$\hat{\psi}(0) = \int_{-\infty}^{\infty} \psi(t) dt = 0.$$

- wavelets

$$\psi_{a,b}(t) = \frac{1}{\sqrt{a}} \psi\left(\frac{t-b}{a}\right), a > 0, b \in R.$$

## Multiresolution analysis (MRA)

*Multiresolution analysis* of the space $L^2(R)$ consist of a sequence of closed subspace $\{V_j\}_{j=-\infty}^{\infty}$ with the following properties:

1. $V_j \subset V_{j+1}$
2. $\overline{\cup_{j\in Z} V_j} = L^2(R)$
3. $\cap_{j\in Z} V_j = \{0\}$
4. $f(t) \in V_j \Leftrightarrow f(2t) \in V_{j+1}$
5. $f(t) \in V_j \Leftrightarrow f(t-k) \in V_j, \forall k \in Z$
6. there exists a function $\phi$ (called *scaling function* or *father wavelet*) such that $\phi_{j,k}(t) = 2^{j/2}\phi(2^j t - k)$, $k \in Z$ constitute orthonormal basis for corresponding subspace $V_j$.

- Let $\phi \in L^2(R)$ be compactly supported scaling function of MRA. Then
  1)
  $$\int_{-\infty}^{\infty} \phi(t)dt \neq 0 \tag{8}$$
  2)
  $$\phi(t) = \sum_{k \in Z} a_k \phi(2t - k) \tag{9}$$
  where $a_k$ are real coefficients and $a_k \neq 0$ for only finitely many $k \in Z$ (the number of nonzero coefficients $a_k$ is denoted by $L$).
  3) $\phi_{j,k}(t) = 2^{j/2}\phi(2^j t - k)$, $j, k \in Z$ are orthonormal in $L^2(R)$ i.e.
  $$\int_{-\infty}^{\infty} \phi(t-n)\phi(t-k)dt = \delta_{k,n}, \tag{10}$$
  where
  $$\delta_{n,k} = \begin{cases} 0, & n \neq k \\ 1, & n = k \end{cases}. \tag{11}$$

- One can constuct wavelet $\psi$ such that

$$\psi_{j,k}(t) = 2^{j/2}\phi(2^j t - k),\ j,k \in Z$$

constitute an orthonormal basis for $L^2(R)$.

- Daubechies scaling function

$$\phi(t) = \sum_{k=0}^{L-1} a_k \phi(2t - k) \tag{12}$$

- Daubechies wavelet function

$$\psi(t) = \sum_{k=2-L}^{1} (-1)^k a_{1-k} \phi(2t - k) \tag{13}$$

where $L$ *is a positive even integer and denotes the genus of the Daubechies wavelet.*

## Wavelet-Galerkin method for ODE

First type of Sturm-Liouville differential equation

$$u''(t) + \alpha u(t) = f(t), \quad t \in [0, 1], \tag{14}$$

with DBC

$$u(0) = u(1) = 0 \tag{15}$$

- Application: electromagnetic radiation, seismology and acoustics.
- Approximate solution

$$u_j(t) = \sum_{k=1-L}^{2^j} c_k \phi_{j,k}(t), k \in Z, \tag{16}$$

where $\phi$ is the scaling function of MRA.

### Remark

- There are no closed-form formulas for the Daubechies wavelets and scaling functions.
- W-G method with Daubechies scaling functions: homogeneous differential equations.
- Application of this method to nonhomogeneous differential equations

$$\phi(t) = \begin{cases} \frac{1}{6}(2+t)^3, & t \in (-2,-1) \\ \frac{1}{6}(4-6t^2-3t^3), & t \in (-1,0) \\ \frac{1}{6}(4-6t^2+3t^3), & t \in (0,1) \\ \frac{1}{6}(2-t)^3, & t \in (1,2) \\ 0, & t \notin (-2,2) \end{cases} \tag{17}$$

satisfies

$$\phi(t) = \frac{1}{8}\phi(2t+2) + \frac{1}{2}\phi(2t+1) + \frac{3}{4}\phi(2t) + \frac{1}{2}\phi(2t-1) + \frac{1}{8}\phi(2t-2),$$

so $L = 5$.

- For $j = 0$

$$u_0(t) = \sum_{k=-4}^{1} c_k \phi(t-k), t \in [0,1]. \tag{18}$$

$$\frac{d^2}{dt^2} \sum_{k=-4}^{1} c_k \phi(t-k) + \alpha \sum_{k=-4}^{1} c_k \phi(t-k) = f(t). \tag{19}$$

- $\alpha = -1$

Taking inner product with $\phi(t-n)$, $n \in \{-4, -3, -2, -1, 0, 1\}$, we obtain

$$\sum_{k=-4}^{1} c_k \Omega_{n-k} - \sum_{k=-4}^{1} c_k a_{n,k} = b_n, \tag{20}$$

where

$$\Omega_{n-k} = \int_{-4}^{5} \phi^{''}(t-k)\phi(t-n)dt, \tag{21}$$

$$a_{n,k} = \int_{-4}^{5} \phi(t-k)\phi(t-n)dt, \tag{22}$$

$$b_n = \int_{-4}^{5} \phi(t-n)f(t)dt. \tag{23}$$

- By using DBC (15) we obtain

$$u_0(0) = \sum_{k=-4}^{1} c_k \phi(-k) = 0 \tag{24}$$

and

$$u_0(1) = \sum_{k=-4}^{1} c_k \phi(1-k) = 0 \tag{25}$$

- We replace the first and the last equation of system (20) by (24) and (25) respectively and obtain the matrix equation

$$TC = B \tag{26}$$

$$T = \begin{bmatrix} \phi(4) & \phi(3) & \phi(2) & \phi(1) & \phi(0) & \phi(-1) \\ \Omega_0 - a_{-4,-4} & \Omega_{-1} - a_{-4,-3} & \Omega_{-2} - a_{-4,-2} & \Omega_{-3} - a_{-4,-1} & \Omega_{-4} - a_{-4,0} & \Omega_{-5} - a_{-4,1} \\ \Omega_1 - a_{-3,-4} & \Omega_0 - a_{-3,-3} & \Omega_{-1} - a_{-3,-2} & \Omega_{-2} - a_{-3,-1} & \Omega_{-3} - a_{-3,0} & \Omega_{-4} - a_{-3,1} \\ \Omega_2 - a_{-2,-4} & \Omega_1 - a_{-2,-3} & \Omega_0 - a_{-2,-2} & \Omega_{-1} - a_{-2,-1} & \Omega_{-2} - a_{-2,0} & \Omega_{-3} - a_{-2,1} \\ \Omega_3 - a_{-1,-4} & \Omega_2 - a_{-1,-3} & \Omega_1 - a_{-1,-2} & \Omega_0 - a_{-1,-1} & \Omega_{-1} - a_{-1,0} & \Omega_{-2} - a_{-1,1} \\ \Omega_4 - a_{0,-4} & \Omega_3 - a_{0,-3} & \Omega_2 - a_{0,-2} & \Omega_1 - a_{0,-1} & \Omega_0 - a_{0,0} & \Omega_{-1} - a_{0,1} \\ \phi(5) & \phi(4) & \phi(3) & \phi(2) & \phi(1) & \phi(0) \end{bmatrix}$$

$$C = \begin{bmatrix} c_{-4} \\ c_{-3} \\ c_{-2} \\ c_{-1} \\ c_0 \\ c_1 \end{bmatrix}, B = \begin{bmatrix} 0 \\ b_{-3} \\ b_{-2} \\ b_{-1} \\ b_0 \\ 0 \end{bmatrix}.$$

**Example.**

$$u^{''}(t) - u(t) = t - 1, \qquad 0 \leq t \leq 1, \tag{27}$$

with DBC $u(0) = u(1) = 0$.

- Exact solution

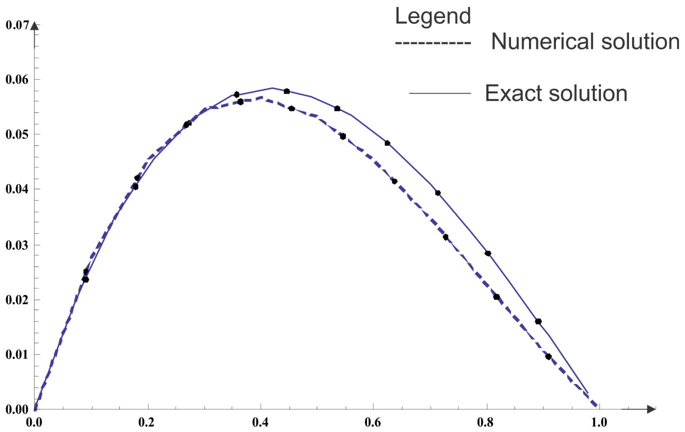
$$u(t) = -\frac{1}{1-e^2}e^t + \frac{e^2}{1-e^2}e^{-t} - t + 1.$$

- Approximate solution obtained by W-G method is

$$u_0(t) = c_{-1}\phi(t+1) + c_0\phi(t) + c_1\phi(t-1), \quad t \in [0,1].$$

## Comparison of results

| Case | numerical solution $u_0$ | exact solution $u$ | absolute error |
|---|---|---|---|
| 0 | 0 | 0 | 0 |
| 0.1 | 0.0278843 | 0.0265183 | 0.00136595 |
| 0.2 | 0.0457588 | 0.0442945 | 0.00146426 |
| 0.3 | 0.0550535 | 0.0545074 | 0.000546154 |
| 0.4 | 0.0571985 | 0.0582599 | 0.00106145 |
| 0.5 | 0.0536236 | 0.0565906 | 0.00296699 |
| 0.6 | 0.0457588 | 0.0504834 | 0.00472462 |
| 0.7 | 0.0350341 | 0.0408782 | 0.0058441 |
| 0.8 | 0.0228794 | 0.0286795 | 0.00580016 |
| 0.9 | 0.0107247 | 0.0147663 | 0.00404158 |
| 1 | 0 | 0 | 0 |


Legend
---------- Numerical solution
Exact solution
0.07
0.06
0.05
0.04
0.03
0.02
0.01
0.00
0.0
0.2
0.4
0.6
0.8
1.0

Second type of Sturm-Liouville differential equation

$$u''(t) + g(t)u'(t) = f(t), \quad t \in [0, 1], \tag{28}$$

with DBC

$$u(0) = u(1) = 0, \tag{29}$$

$$\sum_{k=-4}^{1} c_k \Omega_{n-k} + \sum_{k=-4}^{1} c_k d_{n,k} = b_n, \tag{30}$$

$n \in \{-4, -3, -2, -1, 0, 1\}$, where

$$\Omega_{n-k} = \int_{-4}^{5} \phi^{''}(t-k)\phi(t-n)dt, \tag{31}$$

$$d_{n,k} = \int_{-4}^{5} g(t)\phi'(t-k)\phi(t-n)dt, \tag{32}$$

$$b_n = \int_{-4}^{5} \phi(t-n)f(t)dt. \tag{33}$$

**Example.** Van Der Pol equation

$$u^{''}(t) + \mu(t^2 - 1)u^{'}(t) = -t, \qquad 0 \leq t \leq 1, \tag{34}$$

with DBC

$$u(0) = u(1) = 0. \tag{35}$$

- Stable oscillations(relaxation-oscillations/ limit cycles) in electrical circuits employing vacuum tubes
- Physical and biological sciences, seismology
- No exact analytic solution

Numerical solution $u_0$ of the Van Der Pol equation for different values of parametar $\mu$

| Case | $\mu$=0.05 | $\mu$=0.1 | $\mu$=1 |
|---|---|---|---|
| 0 | 0 | 0 | 0 |
| 0.1 | 0.0838955 | 0.218264 | 0.240269 |
| 0.2 | 0.137675 | 0.358176 | 0.394288 |
| 0.3 | 0.16564 | 0.430931 | 0.474378 |
| 0.4 | 0.172093 | 0.44772 | 0.49286 |
| 0.5 | 0.161338 | 0.419738 | 0.462057 |
| 0.6 | 0.137675 | 0.358176 | 0.394288 |
| 0.7 | 0.105407 | 0.274229 | 0.301877 |
| 0.8 | 0,0688374 | 0.179088 | 0.197144 |
| 0.9 | 0.0322675 | 0.0839475 | 0.0924113 |
| 1 | 0 | 0 | 0 |

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