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ON THE GENERALIZED FRESNEL SINE INTEGRALS AND CONVOLUTION

LIMONKA LAZAROVA AND BILJANA JOLEVSKA-TUNESKA

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ABSTRACT. The generalized Fresnel sine integral $\mathrm{S_k}(x)$ and its associated functions $\mathrm{S_{k+}}(x)$, $\mathrm{S_{k-}}(x)$ are defined as locally summable functions on the real line. Some convolutions and neutrix convolutions of the generalized Fresnel sine integral and its associated functions are then found.

1. INTRODUCTION

The theory of distribution was first formulated by Sobolev and then developed systematically by L. Schwartz, [15]. It is a subject that is used not only in many mathematical disciplines such as functional analysis and applied mathematics but also in physics and engineering science, see [3, 4]. Together with Schwartz's theory, some singular functions such as the Dirac delta function δ and operations with them can be defined mathematically. But besides this, the theory has become insufficient to satisfy all the needs of the physicists, as for example evaluation of δ^2 or $\sqrt{\delta}$. Thus mathematicians started to develop new mathematical approaches, as the approach introduced by Fisher, [5, 6, 8], where he use the neutrix calculus to evaluate product and convolution products of distributions.

Besides the Dirac's delta function, the concept of finite parts of divergent integrals are at the origin of the theory. The Hadamard's finite part is a way of resulting finite values to the integral of functions with the technique of neglecting appropriately defined infinite quantities, see [10]. Since taking the neutrix limit of a function is to extract a finite part from a divergent quantity, this method can be regarded as an application of the neutrix calculus.

We also note that recently Ng and van Dam applied the neutrix calculus, in conjunction with the Hadamard integral, developed by van der Corput, to the quantum field theories, in particular to obtain finite results for the coefficients in the perturbation series. They also applied neutrix calculus to quantum field theory, and obtained finite renormalization in the loop calculations, see [12, 13].

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In this paper we will evaluate some product and neutrix product of generalized Fresnel integrals, see [1]. They are used in Fraunhfer diffraction and asymptotic of Weyl sums. The generalized Fresnel sine integral is defined by

$$\mathrm{S_k}(x) = \int_0^x \sin(u^k)du\,,$$

for $k = 1, 2, \ldots$ and the associated functions $S_{k+}(x)$ and $S_{k-}(x)$ are defined by

$$S_{k+}(x) = H(x)\, S_k(x)\,, \quad S_{k-}(x) = H(-x)\, S_k(x)\,,$$

for $k = 1, 2, \ldots$ where H denotes Heaviside's function.

In the following we define the function $L_{r,k}$ by

$$L_{r,k}(x) = \int_0^x u^r \sin\left(u^k\right) du\,,$$

for $r = 0, 1, 2, \ldots$ and $k = 1, 2, \ldots$.

In particular we have $L_{0,2}(x) = \int_0^x \sin\left(u^2\right) du = \frac{\pi}{2} S(x)$.

We define the functions $\sin_+ x^k$ and $\sin_- x^k$ by

$$\sin_+ x^k = H(x) \sin x^k \qquad \sin_- x^k = H(-x) \sin x^k\,.$$

2. Convolution product

The definition for the convolution product of two functions f and g is as follows:

Definition 2.1. *Let $f(x)$ and $g(x)$ be two countinous functions with bounded support. Their convolution produces a third function $h(x)$, which is denoted by $f * g$*

$$h(x) = (f * g)(x) = \int_{-\infty}^{+\infty} f(t)g(x-t)dt.$$

If convolution product of two functions f and g, $f * g$ exists then also $g * f$ exists and

$$f * g = g * f\,.$$

If $(f * g)'$ and $(f * g')$ (or $(f' * g)$) exists, then

$$(f * g)' = f * g'\,(f' * g)\,.$$

Such defined convolution product cannot be used when f and g are arbitrary distributions because, for one reason, two distributions cannot be multiplied in general. In order to extend the convolution process $f * g$ of two distributions f and g in $\mathcal{D}'$ there is the follow definition, see [16].

Definition 2.2. *Let $f(x)$ and $g(x)$ be two distributions in $\mathcal{D}'$ and f and g has a bounded support, moreover the both supports are bounded on the left (or right), then the convolution products $f * g$ of distributions is defined by expression*

$$\langle (f * g)(x), \varphi(x) \rangle = \langle f(y), \langle g(x), \varphi(x+y) \rangle \rangle\,.$$

The convolution of two distributions is commutative, see [16] and [9].

In the following we prove two theorems that are generalization of some results obtained in [14].

Theorem 2.1. *The convolution product* $(\sin_+ x^k) * x_+^r$ *exists and*

$$(\sin_+ x^k) * x_+^r = \sum_{i=0}^{r} \binom{r}{i} (-1)^{r-i} L_{r-i,k}(x) x_+^i \tag{2.1}$$

for $r = 0, 1, 2, \dots$ *and* $k = 1, 2, \dots$

Proof. If $x < 0$ then $\left(\sin_+ x^k\right) * x_+^r = 0$.

If $x > 0$, then

$$\begin{aligned}\left(\sin_+ x^k\right) * x_+^r &= \int_0^x \sin t^k (x-t)^r \, dt = \\ &= \int_0^x \sin t^k \sum_{i=0}^{r} \tbinom{r}{i} x^i (-t)^{r-i} dt = \\ &= \textstyle\sum_{i=0}^r \binom{r}{i} (-1)^{r-i} \int_0^x t^{r-i} \sin t^k x^i \, dt = \\ &= \textstyle\sum_{i=0}^r \binom{r}{i} (-1)^{r-i} L_{r-i,k}\left(x\right) x_+^i\end{aligned}$$

□

Corollary 2.1. *The convolution product* $(\sin_- x^k) * x_-^r$ *exists and*

$$\left(\sin_- x^k\right) * x_-^r = \sum_{i=0}^{r} \binom{r}{i} L_{r-i,k}\left(x\right) x_-^i . \tag{2.2}$$

for $r = 0, 1, 2, \dots$ *and* $k = 1, 2, \dots$

Proof. The equation (2.2) follows on replacing x by $-x$ in (2.1). □

Theorem 2.2. *The convolution product* $\mathrm{S}_{k+}(x) * x_+^r$ *exists and*

$$\mathrm{S}_{k+}(x) * x_+^r = \frac{1}{r+1} \sum_{i=0}^{r+1} \binom{r+1}{i} (-1)^{r-i+1} L_{r-i+1,k}(x) x_+^i \tag{2.3}$$

for $r = 0, 1, 2, \dots$ *and* $k = 1, 2, \dots$

Proof. If $x < 0$, then $\mathrm{S}_{k+}(x) * x_+^r = 0$.

If $x > 0$, then we have,

$$\begin{aligned}
S_{k+}(x) * x_+^r &= \int_0^x (x-t)^r S_k(t)\,dt = \\
&= \int_0^x (x-t)^r \int_0^t \sin\left(u^k\right) du dt = \\
&= \int_0^x \sin\left(u^k\right) \int_u^x (x-t)^r dt du = \\
&= -\tfrac{1}{r+1} \int_0^x \sin\left(u^k\right)(x-u)^{r+1}(-1) = \\
&= \tfrac{1}{r+1} \sum_{i=0}^{r+i} \tbinom{r+1}{i} (-1)^{r+1-i} \int_0^x u^{r+1-i} x^i \sin\left(u^k\right) x^i = \\
&= \tfrac{1}{r+1} \sum_{i=0}^{r+i} \tbinom{r+1}{i} (-1)^{r+1-i} L_{r-i+1,k}(x)\, x_+^i
\end{aligned}$$

□

Corollary 2.2. *The convolution product $S_{k-}(x) * x_-^r$ exists and*

$$S_{k-}(x) * x_-^r = \frac{1}{r+1} \sum_{i=0}^{r+i} \binom{r+1}{i} L_{r-i+1,k}(x)\, x_-^i \tag{2.4}$$

for $r = 0, 1, 2, \ldots$ and $k = 1, 2, \ldots$.

Proof. The equation (2.4) comes from equation (2.3), on replacing x by $-x$. □

3. Neutrix convolution product

In order to extend the convolution product to a larger class of distributions, the neutrix convolution product was introduced by Fisher, see [5, 6, 7, 8]. For the further extension, first of all, we let τ in $\mathcal{D}'$ be a function with the following properties:

$\tau(x) = \tau(-x)$,
$0 \le \tau(x) \le 1$,
$\tau(x) = 1$ for $|x| \le \frac{1}{2}$,
$\tau(x) = 0$ for $|x| \ge 1$.

The function τ_ν for $\nu > 0$ is now defined by:

$$\tau_\nu = \begin{cases} 1, & |x| \le \nu, \\ \tau\left(\nu^\nu x - \nu^{\nu+1}\right), & x > \nu, \\ \tau\left(\nu^\nu x + \nu^{\nu+1}\right), & x < -\nu. \end{cases}$$

The following definition of the non-commutative neutrix convolution was given in [5].

Definition 3.1. *Let f and g be distributions in $\mathcal{D}'$ and let $f_\nu = f\tau_\nu$ for $\nu > 0$. Then the non-commutative neutrix convolution $f \circledast g$ is defined as the neutrix limit of the sequence $\{f_\nu * g\}$, provided the limit h exists in the sense that*

$$\underset{\nu\to\infty}{\mathrm{N-lim}} \langle f_\nu * g, \varphi\rangle = \langle h, \varphi\rangle$$

for all φ in $\mathcal{D}$, where N is the neutrix, see van der Corput [2], having domain N' the positive reals and range N'' the real numbers, with negligible functions finite linear *sums of the functions*

$$\nu^{\lambda}\ln^{r-1}\nu, \quad \ln^{r}\nu, \quad \nu^{r}\sin\nu^{k}, \quad \nu^{r}\cos\nu^{k}: \ \lambda>0, \ r=1,2,\ldots \quad k=1,2,\ldots$$

and all functions which converge to zero in the normal sense as n tends to infinity.

It is easily seen that any results proved with the original definition of the convolution hold with the new definition of the neutrix convolution. The following results proved in [5] hold, first showing that the neutrix convolution is a generalization of the convolution.

Theorem 3.1. *Let f and g be two distributions and suppose that the convolution $f \circledast g$ exists, then the convolution $f \circledast g$ exists and*

$$(f \circledast g)' = f \circledast g\,.$$

Now let $L_{r,k} = N - \lim\limits_{\nu\to\infty} L_{r,k}(\nu)$. Then we have the following theorem:

Theorem 3.2. *The neutrix convolution $\left(\sin_{+}x^{k}\right) \circledast x^{r}$ exists and*

$$\left(\sin_{+}x^{k}\right) \circledast x^{r} = \sum_{i=0}^{r}\binom{r}{i}(-1)^{r-i}L_{r-i,k}x^{i} \tag{3.1}$$

for $r=0,1,2,\ldots$ and $k=1,2,\ldots$.

Proof. Let

$$\left(\sin_{+}x^{k}\right)_{\nu} = \left(\sin_{+}x^{k}\right)_{\nu}\tau_{\nu}(x)\,.$$

Then the convolution product $\left(\sin_{+}x^{k}\right)_{\nu} * x^{r}$ exists by definition 2.2 and we have

$$\left(\sin_{+}x^{k}\right)_{\nu} * x^{r} = \int_{0}^{\nu}\sin t^{k}(x-t)^{r}\,dt + \int_{\nu}^{\nu+\nu^{-\nu}}\tau_{\nu}(t)\sin t^{k}(x-t)^{r}\,dt. \tag{3.2}$$

For the first integral, from (2.1) we have:

$$\int_{0}^{\nu}\sin t^{k}(x-t)^{r}\,dt = \sum_{i=0}^{r}\binom{r}{i}(-1)^{r-i}L_{r-i,k}(\nu)\,x^{i}$$

and it follows that

$$\mathrm{N}\!-\!\lim_{\nu\to\infty}\int_{0}^{\nu}\sin t^{k}(x-t)^{r}\,dt = \sum_{i=0}^{r}\binom{r}{i}(-1)^{r-i}L_{r-i,k}x^{i} \tag{3.3}$$

Further, it can easily be seen that for fixed x we have:

$$\lim_{\nu\to\infty}\int_{\nu}^{\nu+\nu^{-\nu}}\tau_{\nu}(t)\sin t^{k}(x-t)^{r}\,dt = 0\,, \tag{3.4}$$

so (3.1) follows from (3.2), (3.3) and (3.4), proving the theorem. $\square$

Corollary 3.1. *The neutrix convolution* $\left(\sin_{-}x^k\right) \circledast x^r$ *exists and*

$$\left(\sin_{-}x^k\right) \circledast x^r = \sum_{i=0}^{r} \binom{r}{i} (-1)^{r-i+1} L_{r-i,k} x^i \tag{3.5}$$

for $r = 0, 1, 2, \ldots$ *and* $k = 1, 2, \ldots$.

Proof. The equation (3.5) follows from the equation (3.1) by replacing x by $-x$. □

Corollary 3.2. *The neutrix convolution* $\sin(x^k) \circledast x^r$ *exists and*

$$\sin(x^k) \circledast x^r = 0\,, \tag{3.6}$$

for $r = 0, 1, 2, \ldots$ *and* $k = 1, 2, \ldots$.

Proof. Equation (3.6) follows from equations (3.5) and (3.1) . □

Theorem 3.3. *The neutrix convolution* $S_{k+}(x) \circledast x^r$ *exists and*

$$S_{k+}(x) \circledast x^r = \frac{1}{r+1} \sum_{i=0}^{r+1} \left(\begin{array}{c} r+1 \\ i \end{array} \right) (-1)^{r-i+1} L_{r-i+1,k} x^i \tag{3.7}$$

for $r = 0, 1, 2, \ldots$ *and* $k = 1, 2, \ldots$.

Proof. Let $\left(S_{k+}(x)\right)_\nu = S_{k+}(x)\,\tau_\nu(x)$. Then the convolution product $\left(S_{k+}(x)\right)_\nu * x^r$ exists by definition 2.2 and we have:

$$\left(S_{k+}(x)\right)_\nu * x^r = \int_0^\nu S_k(t)\,(x-t)^r\,dt + \int_\nu^{\nu+\nu^{-\nu}} \tau_\nu(t)\,S_k(t)\,(x-t)^r\,dt \tag{3.8}$$

Next

$$\begin{aligned}
\int_0^\nu S_k(t)\,(x-t)^r\,dt &= \int_0^\nu (x-t)^r \int_0^t \sin u^k\,du\,dt = \\
&= \int_0^\nu \sin u^k \int_u^\nu (x-t)^r\,dt\,du = \\
&= \int_0^\nu \sin u^k\,du \left(-\tfrac{1}{r+1}\right) \left((x-\nu)^{r+1} - (x-u)^{r+1}\right) = \\
&= -\tfrac{1}{r+1} \int_0^\nu \sin u^k\,du \left(\sum_{i=0}^{r+1} \left(\begin{array}{c} r+1 \\ i \end{array} \right) x^i \left((-\nu)^{r+1-i} - (-u)^{r+1-i}\right) \right) = \\
&= -\tfrac{1}{r+1} \int_0^\nu \sum_{i=0}^{r+1} x^i \left((-\nu)^{r+1-i} - (-u)^{r+1-i}\right) \sin u^k\,du
\end{aligned}$$

and

$$\underset{\nu\to\infty}{\mathrm{N-lim}} \int_0^\nu S_k(t)\,(x-t)^r\,dt = -\frac{1}{r+1} \sum_{i=0}^{r+1} (-1)^{r+1-i} L_{r+1-i,k} x^i\,. \tag{3.9}$$

For each fixed x we have that

$$\lim_{\nu\to\infty} \int_\nu^{\nu+\nu^{-\nu}} \tau_\nu(t)\,S_k(t)\,(x-t)^r\,dt = 0. \tag{3.10}$$

So the equation (3.7) comes from equations (3.8), (3.9) and (3.10). □

Corollary 3.3. *The neutrix convolution* $S_{k-}(x) \circledast x^r$ *exists and*

$$S_{k-}(x) \circledast x^r = \frac{1}{r+1}\sum_{i=0}^{r+1}\binom{r+1}{i}(-1)^{r-i}L_{r-i+1,k}x^i \tag{3.11}$$

for $r = 0, 1, 2, \ldots$ *and* $k = 1, 2, \ldots$.

Proof. The equation (3.11) follows from the equation (3.7) by replacing x by $-x$. □

Corollary 3.4. *The neutrix convolution* $S_k(x) \circledast x^r$ *exists and*

$$S_k(x) \circledast x^r = 0 \tag{3.12}$$

for $r = 0, 1, 2, \ldots.$ *and* $k = 1, 2, \ldots$.

Proof. The equation (3.12) follows from the equation (3.7) and (3.11). □

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University of Goce Delčev
Faculty of Computer Science, Department of Mathematics
Štip, Republic of Macedonia
E-mail address: limonka.lazarova@ugd.edu.mk

Ss. Cyril and Methodious University in Skopje,
Faculty of Electrical Engineering and Informational Technologies
Karpos II bb, Skopje, Republic of Macedonia
E-mail address: biljanaj@feit.ukim.edu.mk