A NON COMPACT VERSION OF BORSUK THEOREM FOR COMPONENTS

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N. Shekutkovski, T.A. Pachemska, G.Markoski, ***Maps of quasicomponents induced by a shape morphism***, Glasnik Matematicki 47 No.2 (2012)
http://web.math.pmf.unizg.hr/glasnik/forthcoming.html

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SOME HISTORICAL REMARKS OF THE SHAPE THEORY

The beginning of the shape theory is given in the paper of K. Borsuk (1968), *Concerning homotopy properties of compacta*, Fund. Math. 62, 223-254.

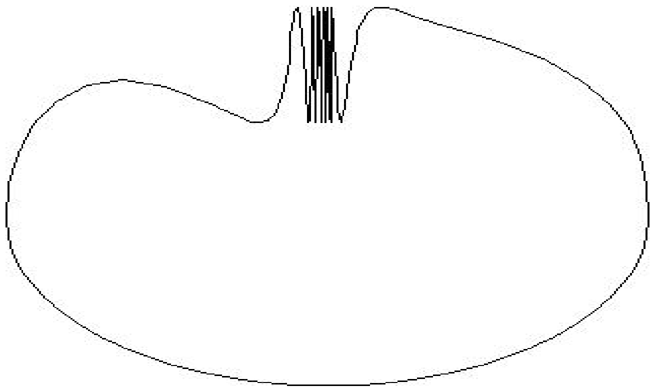
From that time thousend of papers are published and several books ([1],[2])

This improuvement of the theory of homotopy was constructed to solve the problem with compact metric spaces. The main idea is modification of theory of homotopy for spaces which have more general properties.

Borsuk is defined shape theory for compact metric spaces imbeded in pseudo – interior of Hilbert's cube, such as the homotopy clases was changed with the sequences of maps caled fundamental sequences.

Mardesic, Segal (1973) is constructed the approach with the inverse limit.

Homotopy clases of fundamental sequences was changed with morphism of shape. The new theory is used the simply idea that the main “shape” of compact metric space should it be (in some logical sence) limes of “shapes” of her aproximative sistem. For example, the Warshava’s circle has shape of circle.



They defined the functor $S : \mathrm{HTop} \to Sh$, caled **shape functor**. It is shown that the shape gives more brutal clasification of spaces than homotopy types.

The application of shape theory is expected in the situation which include global features of spaces with “irregular” local attitude. That spaces is appeared in many area of maths, such as fibers of mapings, the rest of compactification, sets of fix points, atractors in dinamical systems …

Many autors (Lisica, Schekutkovski, Sanhurjo, Porter …) give as an approach in shape theory using external spaces: neighbourhood of spaces located in Hilbert’s cube, polyhedra, ANR spaces,…

In the paper of N. Schekutkovski, *Intrinsic definition of strong shape for compact metric spaces,*Topology Proceedings 39 (2012), using higher level homotopies, the intrinsic definition of shape for compact metric spaces is given.

PRELIMINARIES

The main purpose is the investigation of non compact spaces and specially metric locally compact spaces. We assume all spaces to be metric and locally compact.

In the case of noncompact (and non locally connected) space the role of components of conectedness is replaced by quasicomponents.

There are not many books where the quasicomponents are treated. The only books where this topic is more thoroughly treated are the classical books of Kuratowski and Hocking -Young. Also, an important source is the paper of Ball ([3]),

The following is the usual definition of a quasicomponent of a space (Hocking – Young)

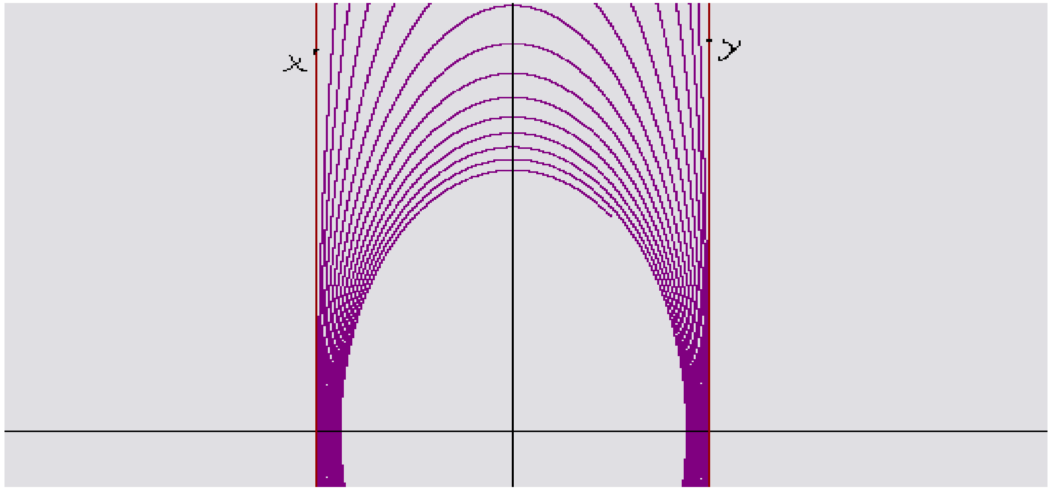
- Quasicomponent $Q(x)$ of a point x is the intersection of all clopen (= open and closed) subsets of X which containing the point x

We give an alternative description of quasicomponents based on the notion of functional separation.

Let A and B be subsets of a space X

- A and B are ***functionally separated*** in X if there exists a continuous function $f\colon X \to \{0,1\}$ such that $f(A) = 0$ and $f(B) = 1$.
- **Quasicomponent $Q(x)$ of a point x** consists of all points y which cannot be functionally separated from x.

Example 1. On the picture is presented the space X which consists of an upper part of the infinite spiral and of two parallel half lines. The spiral is approaching these two parallel lines. The points x and y cannot be functionally separated i.e. they are in the same quasicomponent.



Properties of the quasicomponents:

- $C(x) \subseteq Q(x)$, i.e. the component of x is contained in the quasicomponent of x for every $x \in X$.
- Let $x, y \in X$. If $y \in Q(x)$, then $Q(x) = Q(y)$.
- If $Q(x) \neq Q(y)$, then $Q(x) \cap Q(y) = \varnothing$.
- The quasicomponents of X are closed sets.
- Quasicomponents form a partition of X.
- Let X, Y be topological spaces and let $f\colon X \to Y$ be a continuous map. If $Q(x)$ is a quasicomponent of the point x then

$$f(Q(x)) \subseteq Q(f(x))$$

We can define an induced maping $f^*: Q(X) \to Q(Y)$ and we prove that this maping is continuous.

Also, the maping $p: X \to Q(X)$ defined by $p(x) = Q(x)$ is continuous.

Theorem: If f and g are homotopic and Q is a quasicomponent then for the induced maping f^*, $g^*: X \to Y$ we have $f^*(Q) = g^*(Q)$.

We can define a topology on the set $Q(X)$ of all clopen sets F. The base of the topology are the sets QF consisting of all quasicomponents contained in F.

The topological space $Q(X)$ is a compact space.

The following is well known theorem of Borsuk:

Theorem: (Borsuk) Let X and Y are compact metric spaces. Then for any shape morphism f : X ->Y, there exists an unique map $f\,\hat{}: C(X) \rightarrow C(Y)$ such that for any component C_0 of X, the restriction $f_|$ to C_0 is a shape morphism from C_0 to $f\,\hat{}\,(C_0)$.

The following question appears naturally:

Is this true for noncompact spaces and quasicomponents?

Question:(Borsuk Conference, 2005, Poland)

Is it possible to prove the same type of the Borsuk's theorem for some version of shape for noncompact spaces? (prof. N. Schekutkovski, Faculty of Natural Sciences and Mathematics, "St. Ciril and Methodius" University, Skopje, Macedonia)

Using the intrinsic definition of shape, we give a positive answer to the question (analogue of the Borsuk's theorem in the case of noncompact spaces)

INTRINSIC DEFINITION OF SHAPE

The most known approach for the intrinsic definition of shape of (metric) spaces is by use of functions $f : X \to Y$ which are near to continuous. The idea of ε - continuity (continuity up to $\varepsilon > 0$) leads to a continuity up to some cover $\mathcal{V}$ of Y i. e. $\mathcal{V}$ - continuity, and corresponding $\mathcal{V}$ - homotopy.

Let X, Y be a compact metric spaces and let $\mathcal{V}$ be a finite cover of Y.

Definition 1. The maping $f : X \to Y$ is ***$\mathcal{V}$ - continuous*** if for any $x \in X$, $\exists$ a neighborhood U of x, such that $f(U) \subseteq V$, for some member $V \in \mathcal{V}$

Equivalently, there exists a finite cover $\mathcal{U}$ of X, such that $f(\mathcal{U}) \subseteq \mathcal{V}$

Definition 2. The functions $f, g : X \to Y$ are $\mathcal{V}$ - ***homotopic*** if there is a function $F : I \times X \to Y$ (a $\mathcal{V}$ - ***homotopy***) such that:

1) $F : I \times X \to Y$ is $st(\mathcal{V})$ – ***continuous*** and

2) there exists a neighbourhood $N = [0, \varepsilon) \cup (1 - \varepsilon, 1]$ of $\{0,1\}$ such that $F|_{N \times X}$ is $\mathcal{V}$ - ***continuous***

3) $F(0, x) = f(x)$, $F(1, x) = g(x)$

Proposition: The relation of $\mathcal{V}$ - homotopy is an equivalence relation.

PROXIMATE NET

The main notion for the intrinsic definition of shape is the notion of proximate net from X onto Y.

A ***proximate net*** $(f_{\mathcal{V}}): X \to Y$ is a net of functions $f_{\mathcal{V}}: X \to Y$ indexed by all finite covers of Y, such that if $\mathcal{V} \succ \mathcal{W}$ then $f_{\mathcal{V}}$ and $f_{\mathcal{W}}$ are $\mathcal{V}$ - homotopic (i.e there exists a $\mathcal{V}$ - homotopy $(f_{\mathcal{V}\mathcal{W}}): I \times X \to Y$ connecting $f_{\mathcal{V}}$ and $f_{\mathcal{W}}$).

Two proximate nets $(f_{\mathcal{V}}):X\to Y$ and $(f'_{\mathcal{V}}):X\to Y$ are ***homotopic*** if there exists a proximate net $(F_{\mathcal{V}}):I\times X\to Y$ such that $F_{\mathcal{V}}$ connects $f_{\mathcal{V}}$ and $f'_{\mathcal{V}}$ for any $\mathcal{V}$.

$f_{\mathcal{V}} \approx f'_{\mathcal{V}}$ is an equivalence relation

We denote the class of homotopy with $[(f_{\mathcal{V}})]$.

Let $(f_{\mathcal{V}}):X\to Y$ and $(g_{\mathcal{W}}):Y\to Z$ are proximate nets. If for the cover $\mathcal{W}$ of Z, there exists a cover $\mathcal{V}$ of Y such that $g(\mathcal{V})\subseteq\mathcal{W}$, then the ***composition*** is a proximate net $(h_{\mathcal{W}})=(g_{\mathcal{W}}f_{\mathcal{V}}):X\to Z$ to.

Lemma. If $(f_v) \approx (f'_v)$ and $(g_w) \approx (g'_w)$ then $(g_w f_v) \approx (g'_w f'_v)$

Spaces and homotopy classes of proximate nets ***form the the category*** whose isomorphisms induces classifcation which coincide with shape classifcation, i.e., isomorphic spaces in this category have the same shape.

Definition: The proximate net is caled ***proximate sequence*** if the oriented set is the set of natural number ***N*.**

We mention that in the case of compact metric spaces we can work with proximate sequences instead of proximate nets ([4])

MAP OF QUASICOMPONENTS INDUCED BY A SHAPE MORPHISM

We are starting with the proof of the main result.

We will use the composition of proximate sequence, instead of the homotopy type.

Theorem 1. Let $\mathcal{W}$ be a cover consists of disjoint open sets in X and let $f : X \to Y$ be $\mathcal{W}$ **- continuous.** Then, for each component C of X, there exists $W_C \in \mathcal{W}$ such that $f_\omega(C) \subseteq W_C$.

Theorem 2. *If W is a covering of Y consisting of disjoint open sets (shortly disjoint covering) and $f : X \to Y$ is a W – continuous function, then for each quasicomponent Q of X, there exists $W_Q \in W$ such that $f_Q(Q) \subseteq W_Q$*

Proof. Let $x \in X$, Q is quasicomponent of X and $W_Q \in W$ such that $f(x) \in W_Q$. We define a continuous map $h : Y \to \{0, 1\}$ by $h(W_Q) = \{0\}$ and $h(Y \setminus W_Q) = \{1\}$.

We will prove that $hf : X \to \{0, 1\}$ is continuous.

For an arbitrary point $z \in X$ there exists a neighborhood Uz and there exists $W \in W$ such that $f(Uz) \subseteq W$.

Now, if $W = W_Q$, then $hf(Uz) \subseteq h(W_Q) \subseteq \{0\}$, while if $W = WQ$, $hf(Uz) \subseteq h(W) \subseteq \{1\}$. It follows that the composition $hf : X \to \{0, 1\}$ is continuous.

If we suppose that there is a point $y \in Q$ such that $f(y) \notin W_Q$, then x and y will be functionally separated by the map $hf : X \to \{0, 1\}$. We conclude that $f(Q) \subseteq W_Q$.

Corollary. If W is a covering consisting of disjoint open sets, and $H : X \times I \to Y$ is a W-homotopy connecting W – continuous functions $f : X \to Y$ and $g : X \to Y$, then for each component $Q \in \mathrm{C}(X)$, there exists $W_Q \in W$ such that $f(Q) \subseteq W_Q$ and $g(Q) \subseteq W_Q$.

Theorem 3. *Let $(f_V) : X \to Y$ be a proximate net, and W be a covering consisting of disjoint open sets. Then, for each quasicomponent Q of X, there exists $W_Q \in W$ such that for every $V \prec W$, $f_V(Q) \subseteq W_Q$ (and specially $fW(Q) \subseteq W_Q$).*

Corollary. Let $(f_V) : X \to Y$ be a proximate net, and W be a covering consisting of disjoint open sets. Then, for each component $Q \in \mathrm{C}(X)$, there exists $W_Q \in W$ such that for every $V \prec W$, $fV(Q) \subseteq W_Q$ (and specially $fW(Q) \subseteq W_Q$).

Compact case (Borsuk):

Theorem 1: Let X and Y be compact metric spaces and let $(f_n): X \to Y$ be a proximate sequence. Then there exists an induced mapping $(f_n)_{\#}: C(X) \to C(Y)$

Sketch of the proof:

Let $(fn): X \to Y$ be a proximate sequences over covering (Vn) and let C be a component of connectedness of the point $x \in X$. There exists a point of accumulation $y \in Y$, of the sequence $(fn(x))$, i.e. there exists a subsequence (fn_i) such that $fn_i(x) \to y$. Suppose $y \in D$, where D is the component of connectedness of Y.

We will show that D does not depend on the choice of the accumulation point.

Suppose the contrary, there exists another point of accumulation $z \in E$ different from D, E a component of Y, such that $fn_j(x) \to z$.

Then there exist an open disjoint sets V and W such that $D \subseteq V$, $E \subseteq W$, and $W = \{V, W\}$ is a covering of Y.

There exists n_0 such that fn_0 is W-continuous functions and for $n \geq n_0$, fn and fn_0 are homotopic as Vn_0-continuous functions, and it follows as W-continuous functions. Then if $fn_0(x) \in V$ it follows that $fn(x) \in V$ for all $n \geq n_0$. And this is a contradiction. (thesame contradiction is obtained if we suppose $fn_0(x) \in W$).

We will show that D does not depend on the choice of $x \in C$.

If $x' \in C$ and the component E of Y is chosen, in the same way like the component D is chosen for x, then there exist two open disjoint sets V and W such that $V \cup W = Y$, and $D \subseteq V, E \subseteq W$. Put $\mathcal{W} = \{V, W\}$. By theorem 2.1, there exists n such that $f_n(C) \subseteq V$ and $f_n(C) \subseteq W$ which is a contradiction.

It follows that the function $(f_n)\#: CX \to CY$ defined by $(f_n)\#(C) = D$ is well defined.

This maping is caled **induced maping** and we are proved that this is well defined and $\mathcal{W}$-continuous.

Theorem 2: Let $(f_n),(g_n):X\to Y$ are two proximate sequences. If $(f_n)\sim(g_n)$ (i.e. they are homotopic as a proximate sequences), then the induced functions are equal i.e. $(f_n)_{\#}=(g_n)_{\#}:C(X)\to C(Y)$

Noncompact case (Generalization of Borsuk's theorem):

Let X and Y are noncompact topological spaces with a compact space of quasicomponent QY.

Theorem 3. *If X and Y are topological spaces with compact space of quasicomponents Q, then any proximate net* $(f_v):X\to Y$ *induces a continuous function* $(f_V)\#:QX\to QY$.

Sketch of the proof:

Let $(f_\nu): X \to Y$ be an proximative sequence.

Let Q be the quasicomponent of the point $x \in X$.

Then, for the cover of disjoint open sets $\mathcal{W}$ of *Y*, there exists $W_Q \in \mathcal{W}$, such that $f_\nu(Q) \subseteq W_Q$. Moreover, if $\mathcal{W}'$ is another covering of *Y*, such that $\mathcal{W} \subseteq \mathcal{W}'$, and if $f_{W'}(Q) \subseteq W'_Q$, then $W'_Q \subseteq W_Q$. **(N. Shekutkovski, G. Markoski, Proper shape over finite coveringss, Topology and its Applictions 158 (2011), 2016-2021)**

The set $QW_Q = \{A \mid A \in QX, A \subseteq W_Q\}$ is closed in QY wich is compact, so, the intersection of all QW_Q, where W_Q is taken over $\mathcal{W}$ is not empty, and consists one quasicomponent T. We define $(f_\nu)_\#(Q) = T$.

Theorem 4: Let $(f_\nu), (g_\nu) : X \to Y$ be a homotopic proximate nets $(f_\nu) \sim (g_\nu)$. Then, for the induced maps holds $(f_\nu)_\# = (g_\nu)_\#$.

Proof. Let $(H_V) : X \times I \to Y$ be a proximate net, i.e. $H_V : X \times I \to Y$ is a homotopy connecting V - continuous functions f_ν, $g_\nu : X \to Y$. Let Q be a quasicomponent of X such that $(f_V)\#\ (Q) = T$, $(g_\nu)\#\ (Q) = R$ and $T = R$. Then there exist an open disjoint sets V and W such that $T \subseteq V$, $R \subseteq W$, and $W = \{V, W\}$ is a covering of Y . $H_w : X \times I \to Y$ is a stW-continuous, and since $stW = W$, $(H_W) : X \times I \to Y$ is W-continuous.
Let x be an arbitrary point from the quasicomponent Q. Then $H_w((x, 0)) = f_w(x) \in V$, while $H_w((x, 1)) = g_w(x) \in W$. This is a contradiction, since by Theorem 1, $H_w(Q \times I) \subseteq V$ or $H_w(Q \times I) \subseteq W$. So, $T = R$, i.e. for the induced maps holds: $(f_V)\# = (g_V)\#$.

The last theorem is the main result, an analogue of Borsuk's theorem, in the case of non compact spaces. It is a consequence of theorems 4.1,4.2 and 2.3.

Main theorem: Let X and Y are locally compact metric spaces with compact spases of quasicomponents QX and QY. If $f : X \to Y$ is shape morphism represented by proximate net $(f_\nu) : X \to Y$, then there exists a uniqe maping $(f_\nu)_\# : Q(X) \to Q(Y)$ such that the restriction of f to each quasicomponent represented by with the restriction of the proximate net $(f_\nu) : Q \to (f_\nu)_\#(Q)$ is the shape morphism to.

Moreover, if the proximative nets are homotopic, then they induced the same shape morphism.

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Thank you for your attention