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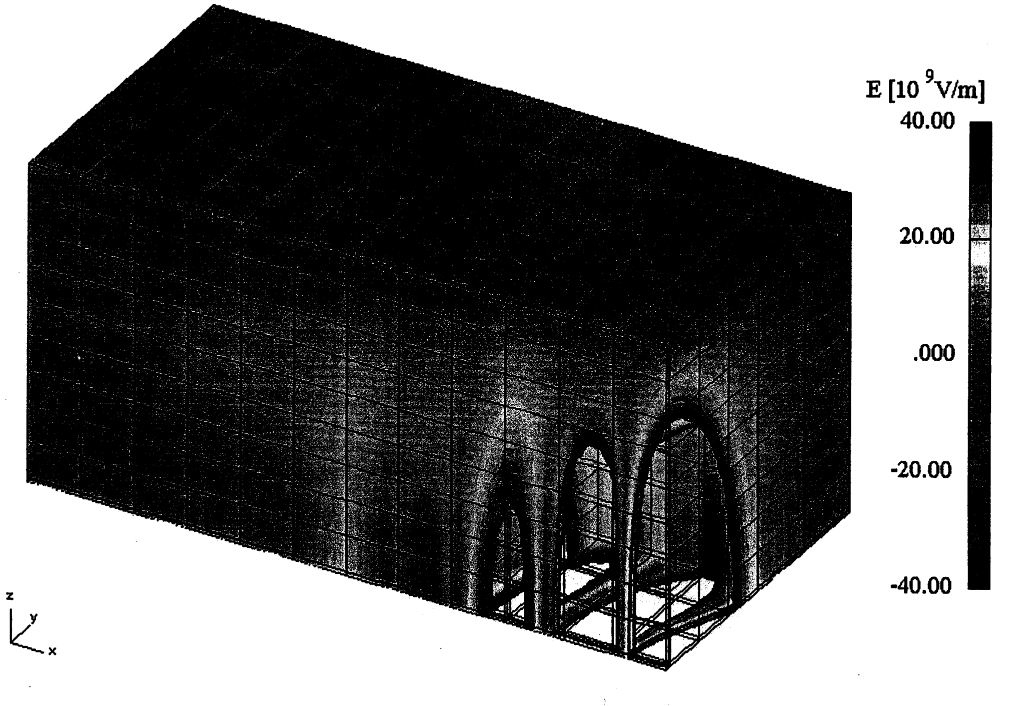
PROCEEDINGS

Part I



September 21-24, 1998, Graz, AUSTRIA

Department for Fundamentals and Theory in Electrical Engineering (IGTE)

Technical University of Graz

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On the Properties of Mixed Consistently and Non-Consistently First Order Edge Finite Elements

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Abstract—**In this paper, we discuss the properties of the mixed consistently and non-consistently linear (Whitney) edge finite elements for 3D eddy-current analysis. We report that by mixing these two types of finite elements the computational resources and efforts can be greatly reduced for the same accuracy of the results. Due to the nature of the edge-based shape functions which has the constant rotation and are divergence-free, we further show that the use of the mixed method is justified only for eddy-current analysis. The proposed mixed method was successfully applied for eddy-current non-destructive evaluation of thin cracks inside conductive materials.**

I. Introduction

The finite element analysis based on the first order non-consistently linear (Whitney) edge finite elements have been widely employed for various electromagnetic vector field problems, mainly as a result of several good properties of their shape functions, such as tangential continuity, constant rotation, divergence-free, etc. Additionally, these finite elements provides faster computation with smaller computer resources. In order to increase the accuracy and versatility of these finite elements, recently, several authors have made attempts to ease the construction and utilization of the higher order edge-based finite elements and their shape functions [2], [3]. However, the development of higher order edge-elements is difficult and rises a lot of mesh generation problems. Therefore, we believe that the Whitney linear edge finite elements with only one unknown variable per edge are still the most commonly employed vectorial edge-based finite elements.

Although, the first order Whitney edge elements provide fast computation on modest computer platforms (especially for 3D problems), unfortunately, in the same time, they are not linearly consistent (hence their name non-consistently linear.) They provide linear approximation only inside the elements and have the constant value of the unknown variable along element's edges [4]. To develop consistently linear edge-based finite elements we have to construct finite elements which have two unknowns per each edge [5]. These finite elements improve the accuracy of the approximation from order $O(h)$, which is the error order for the non-consistently (Whitney) linear edge finite elements, up to order $O(h^2)$. However, the price that one must pay for the increase of the accuracy is:

- doubling the total number of unknowns ,
- aggravating the sparsity of the global matrix of the system,
- increasing the required computer memory almost two times, and
- prolongating the computation time.

In this paper, we show that this price is worth to be paid only for eddy-current analysis. For magnetostatic analysis due to the nature of the shape functions, we will show that the results obtained by Whitney (non-consistently linear) elements and the consistently linear first order finite elements are exactly the same. Moreover, because in eddy-current analysis usually only a small portion of the analysis domain is the conductive region, we propose usage of mixed edge finite elements: consistently linear edge elements inside the conducting region, and non-consistently linear edge elements elsewhere. To develop such a mixed computation method, we have to construct a new mixed edge finite elements placed on the border between non-consistently and consistently linear edge elements. The proposed computational approach provides results with the same accuracy as if we use consistently linear edge elements in the entire analysis region, however, with less computer resources and faster. Additionally, this method exhibits:

- very small influence of the element shapes on the convergence of the iterative solution method and the accuracy of the results, which are the properties of the consistently linear elements, and
- fast convergence rate which is a characteristic of the non-consistently linear (Whitney) edge finite elements.

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The proposed mixed method is successfully applied for 3D eddy-current non-destructive evaluation of the shape, position and parameters of thin cracks inside conductive materials.

II. Governing Equation for Eddy-Current Analysis

The governing equation for eddy-current field analysis using magnetic vector potential formulation is given with this equation

$$rot\ \nu\ rot\mathbf{A} + \sigma\frac{\partial \mathbf{A}}{\partial t} = \mathbf{J}_0 \quad , \qquad (1)$$

where $\mathbf{A}$ is the magnetic vector potential, ν and σ are the magnetic permeability and the electric conductivity coefficients, respectively, and $\mathbf{J}_0$ is the source current density vector. Using the following approximation procedure

$$\mathbf{A} = \sum_i \mathbf{N}_i A_i \quad , \qquad (2)$$

where A_i is the unknown vector potential values along edge i, and $\mathbf{N}_i$ is the edge-based shape function for edge i, the governing equation (1), according to the Galerkin procedure, can be transformed into the following functional

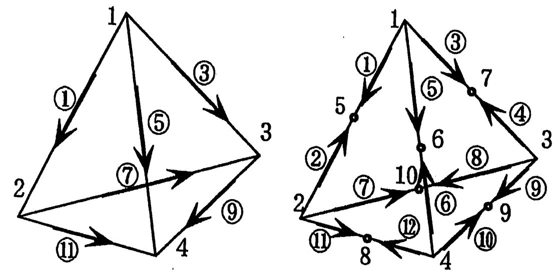
$$\iiint_{\Omega} \nu\ rot\mathbf{N}_i \cdot rot\mathbf{A}\ dV - \iiint_{\Omega_c} \mathbf{N}_i \cdot \mathbf{J}_0\ dV + \iiint_{\Omega_e} \mathrm{j}\,\omega\sigma\mathbf{N}_i \cdot \mathbf{A}\ dV = 0 \quad . \qquad (3)$$

In the above equation, the integration domains Ω, Ω_c and Ω_e stands for the entire analysis area, the coil and eddy-current conductive area, respectively. Equation (3) is general and it is applicable for different types of edge elements. We would like to point out that, as can be seen from (2), the edge-based shape functions $\mathbf{N}_i$ are not any more scalar functions like in ordinary nodal finite element analysis, but vector functions. Respectively, the unknown variables A_i are scalars. In what follows, we will shortly describe the differences between the shape functions and their properties for non-consistently and consistently linear edge finite elements.

III. First Order Edge Finite Elements

A. Non-consistently Linear (Whitney) Edge Elements

Fig. 1a shows the ordinary non-consistently linear (Whitney) tetrahedral edge finite element with one unknown per each of its six edges. This element is called non-consistently linear because, it provides partial linear approximation; linear approximation only inside the element, and constant values of the approximated function along its edges. The shape function for arbitrary edge



(a) Non-consistently (Whitney) element

(b) Consistently linear element

Fig. 1. First order edge-based finite elements.

$\{i, j\}$ can be easily derived using the following expression [1]

$$\mathbf{N}_{i,j} = L_i \nabla L_j - L_j \nabla L_i \quad , \qquad (4)$$

where L_i and L_j are the volume coordinates of the tetrahedron. The volume coordinates L_i $i = 1 \sim 4$ can be computed using the following matrix equation

$$\begin{bmatrix} L_1 \\ L_2 \\ L_3 \\ L_4 \end{bmatrix} = \begin{bmatrix} p_1 & q_1 & r_1 & s_1 \\ p_2 & q_2 & r_2 & s_2 \\ p_3 & q_3 & r_3 & s_3 \\ p_4 & q_4 & r_4 & s_4 \end{bmatrix} \begin{bmatrix} 1 \\ x \\ y \\ z \end{bmatrix} \quad , \qquad (5)$$

Matrix coefficients p_i, q_i, r_i and s_i can be computed from the coordinates values at all four tetrahedron vertices, while vector $\mathbf{x} = [1, x, y, z]^T$ is the coordinate vector of an arbitrary point inside the tetrahedron where the volume coordinates have to be computed.

The global matrix of the system is a complex matrix with real and imaginary coefficients computed according to the first and third parts of (3). The second part of (3) results in the source current vector on the right hand side. In order to construct the global matrix of the system, initially, the rotation of the shape functions must be computed, which according to (4) and (5) yields

$$rot\ \mathbf{N}_{i,j} = 2\left\{(r_i s_j - r_j s_i)\,\mathbf{i} + (q_j s_i - q_i s_j)\,\mathbf{j} + (q_i r_j - q_j r_i)\,\mathbf{k}\right\} \quad , \qquad (6)$$

where $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ are the unit vectors in x, y and z direction, respectively. From (6), it is readily apparent that the rotation of the shape function yields a constant function. Therefore, the value of the computed magnetic flux density vector $\mathbf{B} = rot\mathbf{A}$ is a constant value everywhere inside the finite element, which agrees well with the definition of the non-consistently (Whitney) edge finite element as a linear finite element.

B. Consistently Linear Edge Finite Element

Let us now make the same investigation done in the previous paragraph for the consistently linear edge finite element shown in Fig. 1b which has two unknowns per each edge. This element has one additional node per each edge placed at the center of the edge, for example, at the center of the edge $\{1,2\}$ node 5. The shape functions for both edges that belong to the same tetrahedron edge can be represented as follows

$$\begin{aligned} \mathbf{N}_{i,k} &= L_i \nabla L_j \quad , \\ \mathbf{N}_{j,k} &= L_j \nabla L_i \quad , \end{aligned} \qquad (7)$$

where i and j are the terminal edge nodes, and k is the central node of the edge $\{i,j\}$. The resemblance between both shape functions given in (4) and (7) is obvious. However, the later ones provide linear distribution of the unknown variable along the edge, while the former one has the constant value. Next, let us compute the rotation of these linear shape functions given in (7)

$$\begin{aligned} rot\ \mathbf{N}_{i,k} &= (r_i s_j - r_j s_i)\,\mathbf{i} + (q_j s_i - q_i s_j)\,\mathbf{j} \\ &+ (q_i r_j - q_j r_i)\,\mathbf{k} \quad , \\ rot\ \mathbf{N}_{j,k} &= (r_j s_i - r_i s_j)\,\mathbf{i} + (q_i s_j - q_j s_i)\,\mathbf{j} \\ &+ (q_j r_i - q_i r_j)\,\mathbf{k} \quad . \end{aligned} \qquad (8)$$

Two conclusions are easily derived: (1) rotation of the shape functions (7) again yields constant functions, therefore, the computed magnetic flux density vector will have the constant value everywhere inside the finite element, i.e. this finite element is also linear, and (2) the sum of both rotations given in (7) is exactly the same as the rotation function computed according to the non-consistently linear (Whitney) edge element given in (4). Therefore, the contribution to the global matrix of the system from both types of edge elements is the same. The only difference appears from the imaginary part of the global matrix where we use directly the shape functions, not their rotations (see (3)).

C. *Magnetostatics vs. Eddy-current Analysis*

As already mentioned in the above paragraph, in case of the real matrix of the system the generated system of linear algebraic equation is the same for both, non-consistently linear (Whitney) and consistently linear edge finite elements. Therefore, in case of magnetostatic field analysis where the global system matrix is real, the computed results using Whitney elements and the consistently linear edge elements are exactly the same. To be more precise, the element matrix constructed using consistently linear edge finite elements has the following shape

$$\begin{bmatrix} b_{1,1} & b_{1,2} & \dots & b_{1,12} \\ b_{2,1} & b_{2,2} & \dots & b_{2,12} \\ \dots & \dots & \dots & \dots \\ b_{12,1} & b_{12,2} & \dots & b_{12,12} \end{bmatrix} \begin{bmatrix} A_1 \\ A_2 \\ \dots \\ A_{12} \end{bmatrix} = \begin{bmatrix} J_1 \\ J_2 \\ \dots \\ J_{12} \end{bmatrix} , \qquad (9)$$

where coefficients $b_{i,j}$ can be computed using the following expression

$$b_{i,j} = (\nu_x a_{i,x} a_{j,x} + \nu_y a_{i,y} a_{j,y} + \nu_z a_{i,z} a_{j,z})\, V_e \quad . \qquad (10)$$

The contributions $a_{i,x}$, $a_{i,y}$ and $a_{i,z}$ are the x, y and z components of the shape function rotation for edge i (see (8))

$$\mathbf{a}_i = \begin{Bmatrix} a_{i,x} \\ a_{i,y} \\ a_{i,z} \end{Bmatrix} = rot\ \mathbf{N}_i \quad , \qquad (11)$$

while V_e is the volume of the finite element. As one could expected, the element matrix becomes a full 12×12 real matrix.

On the other side, the element matrix constructed using non-consistently linear (Whitney) edge finite elements is a full 6×6 real matrix of shape

$$4 \begin{bmatrix} b_{1,1} & b_{1,3} & \dots & b_{1,11} \\ b_{1,3} & b_{3,3} & \dots & b_{3,11} \\ \dots & \dots & \dots & \dots \\ b_{1,11} & b_{3,11} & \dots & b_{11,11} \end{bmatrix} \begin{bmatrix} A_1 \\ A_3 \\ \dots \\ A_{11} \end{bmatrix} = 2 \begin{bmatrix} J_1 \\ J_3 \\ \dots \\ J_{11} \end{bmatrix} . \qquad (12)$$

By careful observation of the (6), (8) and (9), one can easily see that the obtained results after solution for $A_1, A_2, \dots, A_{12}$ for the consistently element, and $A_1, A_3, \dots, A_{11}$ for the non-consistently element (see Fig. 1) will be the same, or to be more precise, the following is valid:

Consistently Linear Element		Whitney Element
$A_1 = A_2$	$\equiv$	$\frac{A_1}{2}$,
$A_3 = A_4$	$\equiv$	$\frac{A_3}{2}$,
	$\dots$	
$A_{11} = A_{12}$	$\equiv$	$\frac{A_{11}}{2}$, (13)

The same is valid for the source current vectors where in case of consistently linear elements the values of the source currents along two edges that lie on the same tetrahedron edge, e.g edge 1 and 2 in Fig. 1b, are only one half of the source current value assigned to edge 1 (see Fig. 1a) in case of non-consistently linear (Whitney) element.

In conclusion, we would like to point out that utilizing consistently linear edge finite elements is justified only for the eddy-current analysis where the linear system of algebraic equations with complex coefficients has to be solved. In case of magnetostatic field analysis where the coefficients are real, the accuracy of the results is the same as that obtained by means of non-consistently liner Whitney edge finite elements.

IV. Mixed Linear Edge Finite Elements

In the previous section we saw that utilizing consistently linear edge finite elements is justified only for eddy-current field analysis. Another important point is that these linear elements provide better accuracy then the

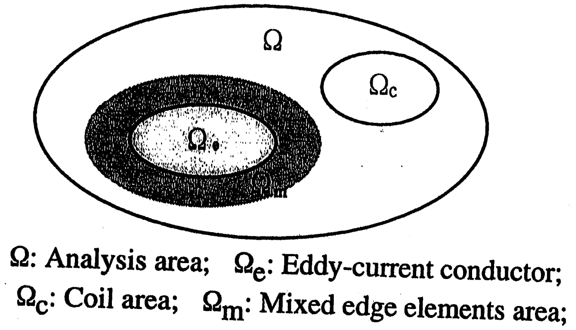
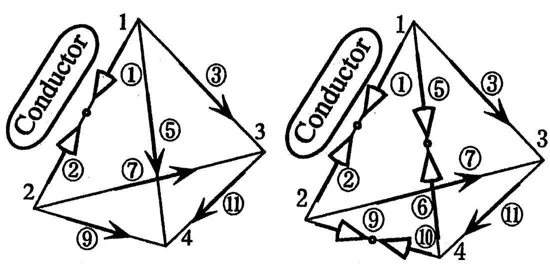


Fig. 2. Computation space and subspaces.



(a) Mixed edge element with 7 unknowns

(b) Mixed edge element with 9 unknowns

Fig. 3. Mixed linear tetrahedron edge elements.

ordinary Whitney non-consistently linear edge elements. However, as can be seen from (9) the element matrix for consistently linear edge element is four times larger than that of the Whitney element. Consequently, the sparsity of the global matrix is aggravated, larger computer resources are needed and the computation time is increased. Therefore, development of a mixed consistently and non-consistently linear edge elements (hereinafter - mixed edge elements) appears as a natural solution. Figure 2 shows the general computation space for eddy-current field analysis Ω divided into three computational subspaces: the eddy-current area Ω_e, the coil area Ω_c, and the mixed edge elements area Ω_m. In each of these areas we use different type of edge elements: inside the eddy-current area, consistently linear elements with two unknowns per edge, in the mixed edge element area, mixed edge elements which as we can see later have some edges with only one unknown and some edges with two unknown, and for the rest of the analysis domain, we use only non-consistently linear (Whitney) edge elements.

Two types of mixed linear tetrahedron edge elements with 7 and with 9 unknowns are shown in Fig. 3. As can be seen, these elements are placed on the boundary between eddy-current conductive area and the rest of the analysis domain. Since, two tetrahedrons can have only one common edge or only one common face, these mixed finite edge elements can be easily constructed by setting two unknowns per common edge, or two unknowns for all edges that lie on common facet. Therefore, only these two mixed elements are enough to descretized the entire mixed edge element area. Development of the shape functions for these mixed edge elements is straightforward and can be easily developed according to (4) and (7).

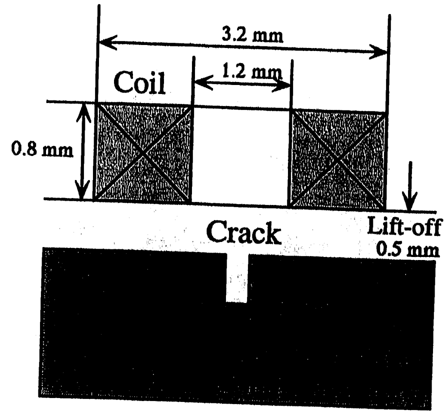


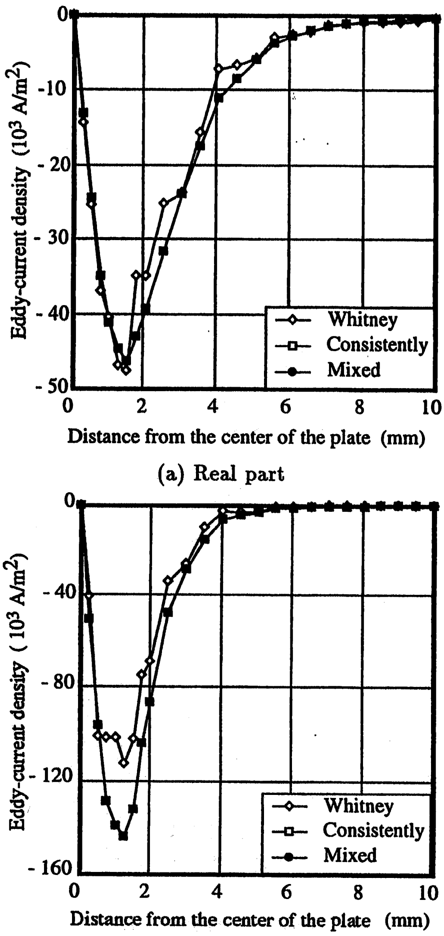
Fig. 4. Model of pancake probe for non-destructive eddy-current evaluation.

V. Obtained Results and Comparisons

We analyzed eddy-current distributions inside a thin conductive plate using a pancake type probe for non-destructive eddy-current evaluation. Although geometrically simple, these problem is not that easy due to the small size of the crack, shallow thickness of the plate and high frequency values of the source currents. Additionally, the signal which must be detected as a result of the eddy-current flow inside the plate, and its changes due to the crack existence is very small. Therefore, we must consider a fast numerical analysis method which provides highly accurate results.

A. Comparison Among Different Edge Finite Elements

A simplified model of pancake type probe used for evaluations is shown in Fig. 4. The model consists of a square shaped coil with 140 turns, the source current value of 1 At and frequency of 300 kHz, and a conductive plate with a crack with length of 5 mm and width of only 0.2 mm. Figs. 5a and 5b show the comparison between computed real and imaginary components of the eddy-current density distributions along line placed on the top face of the conductive plate. The center of the coil corresponds with the 0 coordinate of the x-axis in both figures. The results obtained using three types of edge finite elements are presented: Whitney edge elements, consistently linear edge



(a) Real part

(b) Imaginary part

Fig. 5. Comparison of the obtained eddy-current density distributions for a ppancake ECT probe using different type linear edge elements.

elements, and mixed edge elements. It can be observed that the obtained results for consistently linear and for mixed edge elements are exactly the same, and they overlap for the entire observation area. The Whitney edge elements provide poor accuracy results with rough eddy-current distribution. We also investigated the computer resources and the required CPU time for each of these three type of finite elements. Two iterative methods were used for iterative solution of the generated system of linear equation: Conjugate Gradient method (CG), and the Incomplete Cholesky Conjugate Gradient method (ICCG). Several models were investigated: (1) models with the same number but different type of tetrahedron edge finite elements, and (2) models with approximately same number of unknowns for different type tetrahedron edge elements. The obtained results using the following iteration

TABLE I
RESULTS FOR THE SAME NUMBER OF ELEMENTS

		Whitney	Consistently	Mixed
Elements		42,960	42,960	42,960
Unknowns		53,166	106,332	65,203
Nonzero entries		399,058	1,537,078	67,2574
Iterations	CG	10,638	19,015	24,194
	ICCG	2,750	30,000[a]	3,845
RAM (Mbytes)	CG	22.3	51.4	30.3
	ICCG	29.3	77.1	39.9
CPU time (s)	CG	3,708	18,783	12,563
	ICCG	2,172	56,340	4,323

CPU: Ultra SPARC / 143MHz
[a]Reached total number of iterations without satisfying $\varepsilon < 10^{-6}$

stopping criterion $\varepsilon < 10^{-6}$ are summarized in Table I and Table II, respectively. We would like to point out that in these Tables, 'Whitney' stands for model constructed entirely of Whitney edge elements in the entire analysis domain, 'Consistently' stands for model constructed of only consistently linear edge elements in the entire analysis domain, and 'Mixed' stands for model where inside the eddy-current conducting area consistently linear edge elements were used, inside the mixed edge element area mixed linear edge elements (see Fig. 3), and in the rest of the domain Whitney edge elements. According to the results presented in Tables I and II, the following conclusions can be drawn:

- For the same number of elements, mixed approach provides more accurate results and smoother eddy-current distribution than the Whitney edge finite elements;
- Mixed approach optimizes the computer resources and CPU time requirements;
- Using only consistently linear edge elements results in twice more number of unknowns, about four times increase of the number of nonzero entries, large RAM requirements and poor convergence rates, therefore, these elements should be avoided, especially for magnetostatic analysis;
- It is better to use consistently linear edge elements inside the eddy-current conductive area instead of doubling the number of Whitney elements, because the former ones provide consistently linear distribution, while the later ones provide linear distribution only inside the elements but not also along its edges.

B. Comparison Between Measured and Computed Results

To further verify our computational approach based on mixed edge finite elements, we made comparison between computed signal of the pancake type eddy-current testing (ECT) probe and the measured results for the same

TABLE II
RESULTS FOR APPROXIMATELY SAME NUMBER OF UNKNOWNS

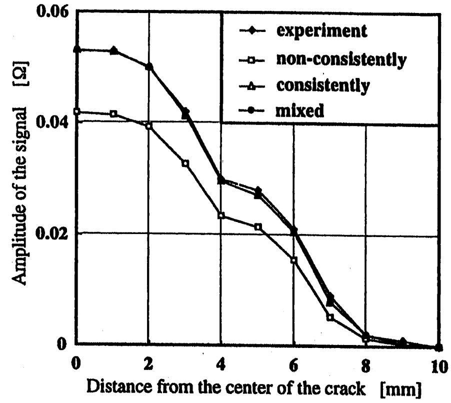
		Whitney	Consistently	Mixed
Elements		85,920	42,960	66,588
Unknowns		105,226	106,332	104,931
Nonzero entries		811,118	1,537,078	1,155,152
Iterations	CG	7,462	19,015	31,129
	ICCG	2,029	30,000[a]	3,354
RAM (MBytes)	CG	41.2	51.4	45.6
	ICCG	55.9	77.1	65.3
CPU time (s)	CG	6,688	18,783	27,292
	ICCG	4,576	56,340	7,265

CPU: Ultra SPARC / 143MHz
[a] Reached total number of iterations without satisfying $\varepsilon < 10^{-6}$

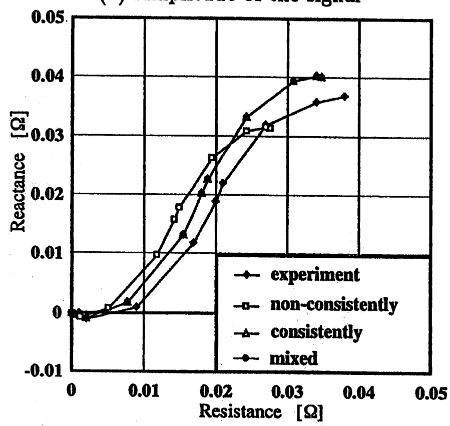
model. The impedance of the signal was computed according to the following equation:

$$\mathbf{Z} = \frac{-j\omega N_t \int_c \mathbf{A}_c \cdot d\ell}{\mathbf{I}} , \qquad (14)$$

where $\mathbf{A}_c$ is the magnetic vector potential inside the pick-up coil area, N_t is the number of turns in the coil, I is the source current per one turn, and l is the perimeter length of the coil. The comparison between measured and computed results for the outer defect with depths of 40% (OD40%)of the plate thickness (outer defect means that the defect and the probe are on the opposite sides of the plate), is presented in Figs. 6a and 6b. As can be seen from Figs. 6a, the consistently edge elements and the mixed elements approaches provide exactly the same results which agree very well with the experiment, while the agreement between measured and computed results using Whitney non-consistently linear edge elements is poor. Good agreement can also be observed for the trace of the signal shown in Fig. 6b.



(a) Amplitude of the signal



(b) Phase of the signal

Fig. 6. Amplitude and phase trace for pancake ECT probe – Outer defect 40% (OD 40%).

VI. CONCLUSIONS

We discussed the properties and the differences between two types first order edge finite elements, non-consistently linear (Whitney) edge elements and consistently linear edge elements for 3D eddy-current analysis. We showed that the usage of consistently linear edge elements is justified only for eddy-current analysis, while for magnetostatic field analysis the obtained results have the same order of accuracy as those obtained using non-consistently linear elements. The accuracy of the results could be improved from order $O(h)$ up to order $O(h^2)$ if the non-consistently linear elements are replaced with the consistently linear ones. This, however, leads to increase of the computer memory requirements and the computation time, therefore it is not computationally effective solution. To solve this problem, in this paper, we proposed a new mixed finite element approach for which a new family of mixed linear edge finite elements were proposed. This approach provides numerical results with the same accuracy as those obtained by the consistently linear elements with short computation time on the modest computer resources. The proposed approach was successfully applied for 3D eddy-current analysis of a pancake type non-destructive evaluation probe. The computed results showed a very good agreement with the measurements verifying our proposed computational method.

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