Investigation of the Efficiency of the Multigrid Method for Finite Element Electromagnetic Field Computations Using Nested Meshes

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Abstract— **Investigation of the efficiency of the multigrid solution methods for electromagnetic field computations using nested finite element meshes is presented. Two type of multigrid algorithms, the V-cycle and the W-cycle multigrids are investigated and the results for the convergence rate of the iterative process and the computation time are compared with those of the ICCG solution method which is commonly used solution method in finite element analysis. It is proven that the efficiency of the multigrid methods is better than that of the ICCG method especially for the solution of large systems of simultaneous algebraic equations.**

Index terms— **Finite element methods, multigrid methods, electrostatic analysis, magnetostatics, iterative methods, relaxation methods, linear algebra.**

I. Introduction

Tremendous developments in the computer software and hardware technology have enabled researchers to easier tackle higher dimensional and computationally expensive physical problems. Respectively, interests in various numerical methods which can be successfully utilized for the solution of a wide class of physical problems have increased rapidly. Therefore, the finite element method which is one of the most widely used numerical method for the solution of various physical problems that can be described using partial differential equations, became one of the methods where a vigorous research has been done recently. As a result, the solution of computationally demanding problems such as three-dimensional periodical or transient problems, coupled problems or inverse shape optimization problems become possible [1]. However, for such complex problems the computation time could be still extremely long mainly as a result of the necessity for solution of large systems of simultaneous linear or nonlinear algebraic equations. Consequently, if one can decrease the portion of the computation time needed for the solution of a system with large number of simultaneous algebraic equation, he/she could achieve decrease of the overall computation time.

In this paper, we present the investigation of the computational efficiency of the so called *multigrid methods* (MGMs) for the solution of large algebraic systems which appear in the electromagnetic field computations. Recently, several authors presented some initial research results in using the MGMs for the electromagnetic field computation, however, only as a preconditioner but not as a solution method [4], [5]. Here we make comparison between the efficiency of the commonly used ICCG (Incomplete Cholesky Conjugate Gradient) method and the two basic multigrid algorithms, V-cycle and W-cycle multigrids. First, we briefly address the mathematical basis and the reasons for improved efficiency of the MGMs. Next, we speak about the numerical implementation and the parameter selection for nested finite element meshes. Before we point some final remarks and conclude this paper, we also present comparison results obtained by applying the V-cycle and W-cycle MGMs and the ICCG method for two test models: one electrostatic and one magnetostatic model.

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II. Multigrid Solution Methods

For the electromagnetic finite element computations various linear system solvers are in common use, among them the ICCG method is probably the most popular. However, for nonlinear, time dependent or coupled problems even the ICCG method results in lengthly computation.

The MGMs are set of techniques for solving systems of algebraic equations using several finite element meshes with different mesh densities [2]. They can solve elliptic partial equations discretized on N grid points by finite element method in $\mathcal{O}$(N) operations, which is much faster than any other rapid iterative solution method which could go as far as $\mathcal{O}$(N logN) [3]. Moreover, as already mentioned in [5], the number of arithmetic operations for the ICCG method grows as much as $N^{3/2}$ for 2D and as much as $N^{4/3}$ for 3D problems, respectively, where N is the number of unknowns per finite element mesh. On the other side, the MGMs keep the number of arithmetic operations per number of unknowns almost independent of the size of the problem. This property together with their higher convergence rate make the MGMs very attractive possibility for the solution of large systems of algebraic equations.

A. Two-grid Method

Two-grid method is the simplest of all multigrid method and we will use it here to explain the main idea that lies behind the MGMs. It is based on the development of two meshes, a coarse one (later on *level #1*) and a dense mesh (later on *level #2*) as shown in Fig. 2. We call

these two meshes *nested meshes* because each node of a coarse mesh corresponds to a node of a dense mesh. i.e. these two meshes have common nodes. It is irrelevant how these two meshes are generated as long as they are nested, however, just for the information, nested meshes can be easily generated manually, automatically or even adaptively by simple subdivision of all or only some finite elements, e.g. those with large computation errors, of a coarse mesh. Later, we will show that the efficiency of the MGMs is independent of the method of meshing and that the computation speed depends only on the number of meshes and type of the algorithm used.

Using a dense finite element mesh we could generate a system of equations of shape $\mathbf{K}\,\mathbf{x} = \mathbf{f}$, where $\mathbf{K}$ is the matrix of the system, $\mathbf{x}$ is the unknown vector, and $\mathbf{f}$ is the source vector. If we try to solve the above system for several iterations using some kind of iterative method (e.g. the Gauss-Seidel or the Jacobi iterative method) we can obtain the approximated solution $\tilde{\mathbf{x}}$ for the problem $\mathbf{K}\,\mathbf{x} = \mathbf{f}$ with the algebraic error $\mathbf{e}$ between the exact solution $\mathbf{x}$ and the approximated solution $\tilde{\mathbf{x}}$, $\mathbf{e} = \mathbf{x} - \tilde{\mathbf{x}}$. Since the exact solution $\mathbf{x}$ is unknown, respectively, the error $\mathbf{e}$ can not be computed explicitly. However, we can compute the residual vector $\mathbf{r}$ according to

$$\mathbf{r} = \mathbf{f} - \mathbf{K}\,\tilde{\mathbf{x}} \quad , \tag{1}$$

using the approximated solution vector $\tilde{\mathbf{x}}$. It can be shown that by simple rearrangement of (1) and its subtraction from the governing equation $\mathbf{K}\,\mathbf{x} = \mathbf{f}$, one can get the very important relationship between the unknown error vector $\mathbf{e}$ and the known residual vector $\mathbf{r}$

$$\mathbf{K}\,\mathbf{e} = \mathbf{r} \quad . \tag{2}$$

Finally, the exact solution $\mathbf{x}$ can be easily computed knowing the error vector $\mathbf{e}$ and the approximated solution $\tilde{\mathbf{x}}$, i.e. $\mathbf{x} = \tilde{\mathbf{x}} + \mathbf{e}$.

The main reason why the MGMs exhibits fast convergence rates can be simply explained using expansion of the error $\mathbf{e}$ into discrete Fourier series. In this case, the error components can be divided into two sets: *smooth components* which are those components on the lower half of the frequency spectrum, and *nonsmooth components* which lie on the higher frequency range [6]. It is well known that for dense meshes which have large smooth error components the relaxation methods converge slowly. However, in the same time many relaxation methods reduce the nonsmooth components fast - they are good smoothers. Therefore, it is apparent that if we use relaxation method only few times, i.e. smooth the nonsmooth components on a dense mesh and then transform somehow the obtained residual to a coarser mesh the convergence could be improved. The MGMs use this property intensively to increase the computation speed of the iterative procedure. For transformation of the residual and error vectors between two adjoint meshes a set of specific transformation matrices called *restriction* and *prolongation* matrices must be developed. In our research we use nested meshes for which simple interpolation and extrapolation can be defined such as method of injection or the five-point or the nine-point interpolation [2].

B. Types of Multigrid Methods

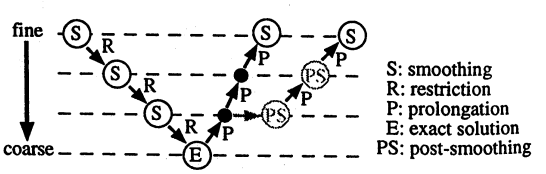
There is not a single MGM which can be applied with the same success for all problems. The user must accommodate his/her problem to the main frame of MGMs and find the multigrid algorithm which is most suitable. Several MG algorithms have already been established and they are intensively described in MG references [2]. In this paper we investigated two of them: the *V-cycle* and the *W-cycle* MGMs. As can be seen from Fig. 1, both algorithms get their names according to the shape of the cycle that they perform. The existence of pre-smoothing and post-smoothing, as well as the number of smoothing steps per each cycle can be freely defined by the user and opens a wide area of investigation on which algorithms better suits to a particular problem.

III. Applications

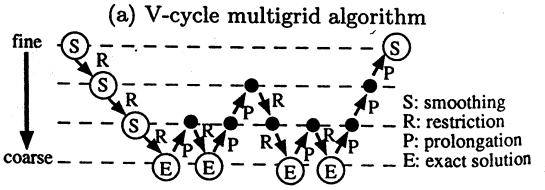
A. Electrostatic Test Problem

A simple 2D electrostatic model of a cylindrical condenser shown in Fig. 2a was used for efficiency investigation. Six nested meshes were generated consequently starting from initial mesh with 21 nodes and 24 elements till the finest mesh with 12,545 nodes and 24,576 elements. As can be seen from Fig. 2, additionally to the increasing of the mesh density, the curvilinear boundary of the model was constantly improved after each subdivision step in order to improve the accuracy of the results. Fig. 3 shows the convergence properties for all three solutions methods. It is readily apparent that MGMs show better convergence rates than the ICCG method. They converge after only 4 (W-cycle) and 5 cycles (V-cycle), respectively, while the ICCG method needed 246 iterations. Regarding the computation time, using 6 nested finite element meshes, both the V-cycle and the W-cycle MGMs were almost 5.5 times faster than the ICCG method which was utilized only on the densest finite element mesh.

The correlations between the computation time and the size of the problem for V-cycle, W-cycle and the ICCG iterative processes, respectively, are given in Fig. 4. One can easily see that the efficiency of the MGMs increases strongly with the increase of the size of the system matrix of the problem that has to be iteratively solved.



(a) V-cycle multigrid algorithm



(b) W-cycle multigrid algorithm

Fig. 1. Typical multigrid algorithms.

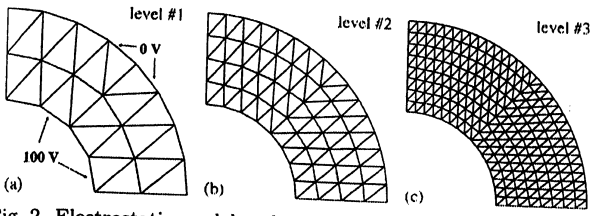


Fig. 2. Electrostatic model and mesh samples for three mesh levels.

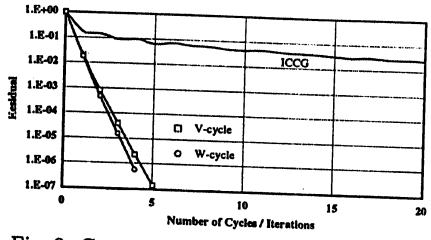


Fig. 3. Comparison of the convergence rates.

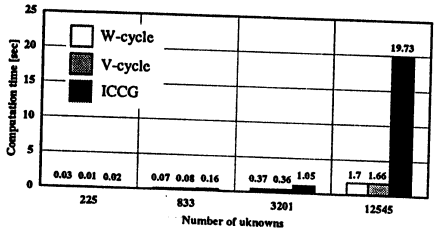


Fig. 4. Computation time vs. number of unknowns.

B. Magnetostatic Test Model

A simple bus-bar model was used to investigate the effectiveness of the MGMs computation scheme for 2D magnetostatic problems. Five finite element meshes from 55 nodes and 83 elements up to 10,825 nodes and 21,248 elements were generated in order to investigate the properties of MGMs to handle non uniformly generated meshes (see Fig. 5). In order to preserved nested character of the meshes, for the generation of a denser mesh each finite element of a coarse mesh was subdivided into four finite elements using mid-edge points. The prolongation and the restriction matrices were derived using linear interpolation algorithms [6].

The obtained results presented in Figs. 6 and 7 show that both, the V-cycle and the W-cycle MGMs exhibits very fast convergence rates. The iteration process converges after 12 cycle for V-cycle and 10 cycles for W-cycle MGMs, respectively, in comparison with 187 iterations needed by the ICCG method for solution on the densest mesh. The computation speed-up was again about 5 times. Finally, similarly with the electrostatic model, the efficiency of the MGMs solver increases with the increase of the model size as shown in Fig. 7.

IV. Conclusions

In this paper, we presented an investigation of the efficiency of two types of MGMs, V-cycle MGM and W-cycle MGM for solution of electromagnetic field problems using finite element analysis based on nested meshes. The aim of this paper was to investigate the convergence rates of MGMs and to make comparison with those of the widely used ICCG method. Additionally, the computation time was investigated and compared with that of the ICCG method. The results show that the MGMs exhibit higher convergence rates than that those of the ICCG method, and that the MGMs provide fast and accurate computation for a shorter computation time. Finally, it was shown that the efficiency of MGMs increases with the increase of the size and complexity of the system of equations that has to be solved.There are few differences in V-cycle MGM and W-cycle MGM.

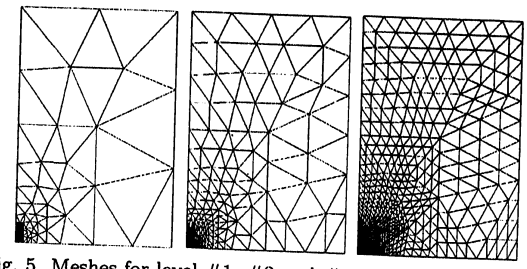


Fig. 5. Meshes for level #1, #2 and #3 of the bus-bar model.

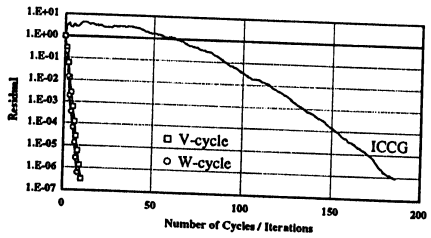


Fig. 6. Comparison of the convergence rates.

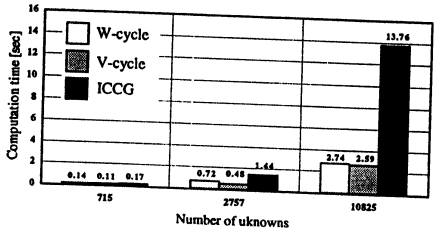


Fig. 7. Computation time vs. number of unknowns.

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