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A Mixed Solving Procedure for Ungauged 3D Edge Finite Element Analysis

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ABSTRACT – **A new mixed procedure for solving the system of algebraic equations generated by the edge finite element method (EFEM) for ungauged vector potential formulation is proposed. The singularity of the matrix of the system obtained by ungauged vector formulation results in unstable and nonuniform convergence rates of the Preconditioned Conjugate Gradient (PCG) iteration procedure. The proposed method uses the excellent initial convergence rate of the PCG procedure, and enables switching from the PCG procedure to the Gauss-Seidel procedure when numerical instability of the former method occurs, continuing with a slower but more stable convergence rate. The proposed mixed PCG Gauss-Seidel procedure requires monitoring of the PCG iteration process, however, it does not require generation of tree graphs, rearranging of unknowns inside the matrix of the system nor any additional memory.**

I. INTRODUCTION

The recently developed edge finite element method (EFEM) exhibits many advantages over traditional nodal finite element method such as short computational time, less required memory and satisfaction of proper boundary conditions only across inter-material boundaries. Using the EFEM, where unknowns are actually line integrals of magnetic vector potential over a set of edges of regularly constructed *3D* mesh, the unknown values of magnetic vector potential are determined up to a gradient function [1]. To obtain a unique solution for the magnetic vector potential, knowing that the line integral of a gradient function over a closed loop must vanish, we must perform magnetic potential gauging with an arbitrary vector field $\mathbf{u}$ which does not possess closed field lines [2]. It is the same as assigning arbitrary values (e.g. zero) to all edges that belong to a tree graph. The results show, however, that although the matrix is nonsingular, the PCG iteration process is poorly conditioned, and a large number of iterations is required.

Efforts to obtain an optimal tree graph have been investigated [3], but only poor convergence rates have been achieved. Some authors have proposed, however, another self-gauging method [4], which solves the problem of poor convergence rates with computational time expenses for renumbering and rearranging the unknowns inside the matrix — nearly the same as the generation of a tree graph.

In this paper, the authors propose a new mixed algorithm for solving the singular system of algebraic equations generated by ungauged vector potential EFEM. This method is free of the aforementioned drawbacks, such as the generation of tree graphs or the rearranging of unknowns. The proposed method requires only the monitoring of the PCG iteration process with no additional memory and is easily applicable especially for solving nonlinear electromagnetic and eddy-current problems and problems with arbitrary shape of coils where the solenoidal character of source current $div\ \mathbf{J}_0 = 0$ can not be exactly satisfied in the whole analysis region. The electric vector potential $\mathbf{T}_0$ defined as

$$\mathbf{J}_0 = \nabla \times \mathbf{T}_0 \quad , \tag{1}$$

exactly satisfies the solenoidal character of the source current $\mathbf{J}_0$ but only for each finite element separately, and for previously known direction of the source current. However, if the direction of the source current $\mathbf{J}_0$ is unknown or it is highly changeable as in the case of circular or other arbitrary shape of coils, then the solenoidal character of the source current could be exactly satisfied only by explicit solution of equation

$$\nabla \times \left(\frac{1}{\sigma} \nabla \times \mathbf{T}_0\right) = 0 \quad , \tag{2}$$

which obviously is further computation time and memory consuming.

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II. SOLUTION OF A SYSTEM OF LINEAR ALGEBRAIC EQUATIONS

Whenever we treat any physical problem described in its finite element approximation, the solution is always obtained by solving a system of linear algebraic equations. The method for solving such a system, therefore, is of great significance. The practical methods for solving a system of linear algebraic equations from a numerical point of view can be divided into two large categories: direct methods and methods of successive approximation, or iterative methods. The direct methods provide a solution of the system of equations, usually by employing two operations: forward elimination and backward substitution. They are easy applicable, however, they require considerable computation time and memory expense. Starting from a certain approximated value of the unknowns, the iterative methods obtain new and "improved" approximated values at each iteration step. This procedure is repeated until the desired error is reached. As the number of equations in the system of algebraic equations increases, the methods of successive approximation become favorable over direct methods.

A. Preconditioned Conjugate Gradient (PCG) Method

The PCG method is widely used for solving systems of algebraic equations generated by the finite element method. Here, we will briefly point out some characteristics important to our further discussion.

To achieve a quick solution with low memory requirements in finite element analysis, two procedures are significant: storage and solution. The PCG procedure is an

iterative procedure, or more accurately a semi-iterative procedure, because usually it enables reaching the solution of the system of algebraic equations after fewer iteration steps than number of equations. In this procedure, the solution of the algebraic system of n equations

$$\mathbf{S}\,\mathbf{x} = \mathbf{b} \quad , \tag{3}$$

where $\mathbf{S}$ is $n \times n$ matrix of the system, $\mathbf{x}$ is the vector of unknown variable, and $\mathbf{b}$ is the vector of the right-hand side or vector of the source, is modified by its preconditioning with the preconditioning matrix $\mathbf{B}$ into the following system

$$\mathbf{BSB}^t[\mathbf{B}^{-t}\mathbf{x}] = \mathbf{Bb} \quad . \tag{4}$$

As a preconditioning matrix, it is possible to use the inverse of an approximated Cholesky factor (ICCG). In our case, however, we used the Evans's preconditioning [5], where the matrix of the system is factorized as

$$\mathbf{S} = \mathbf{DAD} \quad , \tag{5}$$

where the terms of the diagonal matrix $\mathbf{D}$ and matrix $\mathbf{A}$ are

$$D_{ij} = \begin{cases} \sqrt{S_{ij}}, & i = j \\ 0, & i \neq j \end{cases} \tag{6}$$

$$A_{ij} = \frac{S_{ij}}{D_{ii}D_{jj}} \quad . \tag{7}$$

This factorization is important for two main reasons:

1. Only the diagonal terms of matrix $\mathbf{D}$ need to be stored, and
2. Since all diagonal terms of matrix $\mathbf{A}$ are 1, we may write the rest of matrix $\mathbf{A}$ over matrix $\mathbf{S}$. This procedure saves a significant amount of memory and enables, whenever necessary, to recover matrix $\mathbf{S}$ from matrix $\mathbf{A}$ directly, with no additional memory requirements.

These two benefits, as clearly seen, are crucial in keeping memory requirements to a minimum and allowing the easy reconstruction of the matrix of the system $\mathbf{S}$ at any moment during the analysis.

B. Gauss-Seidel Iteration Procedure

The Gauss-Seidel iteration procedure is an improved algorithm of the simple iteration procedure [6], where after finding an approximation for a component, we immediately use this approximation to find the next component and so on. This iteration procedure has the following characteristics:

1. A slower convergence rate than the PCG procedure, as here any successive approximation is not always in the correct direction (not orthogonal) toward the exact solution, and
2. The convergence rate is strongly influenced by the initial approximated solution: If the initial solution is close enough to the exact solution, the iteration process converges much faster.

These two characteristics also have significance to our further discussion.

III. Ungauged Vector Potential Formulation in EFEM

Problems that arise in the solution of a set of algebraic equations generated by EFEM using ungauged magnetic vector potential formulation have already been discussed [1] - [4]. For purpose of continuity in our discussion, these problems are briefly presented here.

Using magnetic vector potential formulation, the governing equation for magnetostatic problems becomes

$$\nabla \times \left(\frac{1}{\mu}\nabla \times \mathbf{A}\right) = \mathbf{J}_0 \quad , \tag{8}$$

where $\mathbf{A}$ is magnetic vector potential, $\mathbf{J}_0$ is source current density vector and μ is magnetic permeability. The uniqueness of the solution of magnetic vector potential $\mathbf{A}$ can be ensured by setting its *curl* and *divergence*:

$$\nabla \times \mathbf{A} = \mathbf{B} \quad , \tag{9}$$

$$\nabla \cdot \mathbf{A} = 0 \quad . \tag{10}$$

Eq. (10) is usually employed in nodal finite element analysis as a Coulomb gauge condition, but in the case of edge finite elements, it is not applicable. The reason is the nature of the first order edge finite element shape functions, which posses no *divergence* and have a constant *curl*. Therefore, (10) is automatically ensured, and the uniqueness of the solution is not provided since any function $\mathbf{A}' = \mathbf{A} \pm \nabla\phi$, also exactly satisfies (8); that is, the values of magnetic vector potential $\mathbf{A}$ are determined up to a gradient function [1]

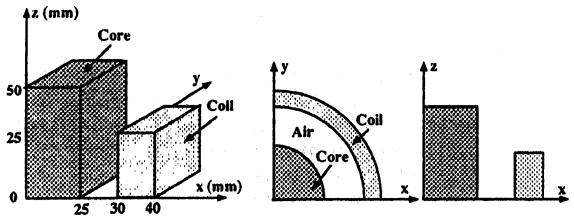
$$\mathbf{A} = -\nabla\phi + \mathbf{A}_c \quad , \tag{11}$$

where the unknown function $\mathbf{A}$ is separated into two parts, a scalar field ϕ, the gradient of which is associated with all tree graph edges in the *3D* mesh, and a co-tree field $\mathbf{A}_c$. By assigning arbitrary values to the scalar function ϕ (usually zero), a unique solution for the magnetic vector potential $\mathbf{A}_c$ can be achieved. This is the same procedure as gauging the magnetic vector potential $\mathbf{A}$ with an arbitrary vector field $\mathbf{u}$, which does not possess closed field lines [2]:

$$\mathbf{A} \cdot \mathbf{u} = 0 \quad . \tag{12}$$

Applying this gauging procedure to a system of algebraic equations generated by EFEM leads to a nonsingular matrix of the system. Although the matrix becomes nonsingular, the PCG iteration process is poorly conditioned, and a large number of iterations is required [3]. Conversely, abandoning this gauging procedure leads to a singular matrix of the system, which results in an initially fast convergence rate of the PCG method. Unfortunately, this iteration procedure fails to be uniform and stable if the iteration process is necessary for a very low residual. Experience shows that when obtained with a high residual rate, the results are acceptable from the point of view of accuracy. To verify this conclusion, two simple test models, presented in Fig. 1, were developed.

The first model in which the divergence of the source current density vector $\mathbf{J}_0$ is satisfied exactly (Fig. 1(a)), shows a good convergence rate and was established to determine the required residual error that results with an acceptably accurate solution. Separate test runs were performed using both the PCG and Gauss-Seidel iteration processes. To analyze the influence of the initial approximated solution $\mathbf{x}_i^{(0)}$ in the Gauss-Seidel procedure,



a) b)

Fig. 1. Test models.

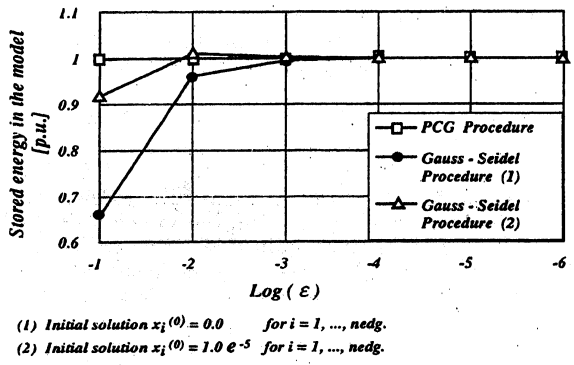


Fig. 2. Stored energy in the model vs. residual value.

two test runs were performed: with zero initial solution $\mathbf{x}_i^{(0)} = 0.0$, and with initial solution $\mathbf{x}_i^{(0)} = 1.0^{-5}$, a value which is much closer to the expected value for magnetic vector potential $\mathbf{A}$ in the analyzed model around the area of interest (between the core and the coil). The monitoring parameter was the stored energy in the system, which, with a decrease of the residual, should converge to its stationary value. The obtained results are presented in Fig. 2. We used the following stopping criterion

$$\varepsilon = \frac{\|\mathbf{b} - \mathbf{S}\mathbf{x}\|}{\|\mathbf{b}\|} \quad . \tag{13}$$

From Fig. 2, it is apparent that in case of the PCG iteration procedure, the stored energy in the model quickly converges to its stationary value. The value of the residual error for which the stored energy reached its stationary value depends on the mesh density. Experience shows that results obtained with residual error of order 10^{-4} in the PCG iteration procedure almost always can be accepted as close enough to the exact results.

As for the Gauss-Seidel iteration process, however, the stored energy slowly approaches its stationary value reached for residual error smaller than 10^{-5}. In case of initial solution $\mathbf{x}_i^{(0)} = 1.0^{-5}$, the stationary value of the stored energy was reached faster and for larger residual error than that of the zero initial solution. The following conclusions are therefore obvious:

- The results obtained by the PCG iteration procedure converge very quickly to the exact results. The error criterion (13) of order smaller than 10^{-4} almost always can be accepted as low enough which results in an acceptably accurate solution.
- The Gauss-Seidel iteration procedure enables a uniform but slower approach towards the exact solution. For the same error criterion, the results obtained for an initial solution closer enough to the exact solution are more acceptable than those obtained for the zero initial solution.

The second test model (Fig. 1(b)) is axi-symmetrical and shows a poor PCG convergence rate mainly due to the fact that it does not satisfy exactly the solenoidal character of the source current vector $\mathbf{J}_0$. As a result, the smallest residual error achieved by the PCG iteration procedure for extremely dense division mesh using the error criterion (13) was of order 10^{-3}. Afterwards, the PCG iteration process starts to diverge (see Fig. 3). To improve the convergence rate, the following procedures are applicable: developing a more dense division mesh in the region of the source to improve the solenoidal character of the source current, explicitly solving (2) in order to input source current values $\mathbf{J}_0$ or developing the tree graph and proceeding with gauged magnetic vector potential formulation. Each procedure results in an increase of the memory and computation time requirements, especially the first procedure, with its unnecessarily extremely dense division mesh in the area of the slowly changing field, in other words, the source area. To solve this problem, we propose a new approach based on mixed solution techniques between the PCG and the Gauss-Seidel iteration procedures, free from the above drawbacks.

IV. Mixed Solving Procedure

As already pointed out, the PCG method is able to solve the singular system of equations for ungauged EFEM, but with residuals that can not be improved beyond a certain limit. At times, however, this residual is not small enough, resulting in an unacceptable solution of the problem (as for the model in Fig. 1(b)). This situation almost always occurs in problems where the solenoidal character of the source current $div\ \mathbf{J}_0 = 0$ is not exactly satisfied, such as in the case of circular or other complicated coil shapes and where (2) has to be solved explicitly. This problem also usually occurs in the solution of eddy-current and non-linear magnetostatic problems. In these cases, we want to "forcibly" decrease the residual until the desired error is attained avoiding solution of (2). In the proposed method, this "forcing" is done by employing the Gauss-Seidel solving procedure.

The simplified algorithm of the proposed procedure is as follows:

- Generating the matrix of the system using ungauged magnetic potential and EFEM.
- Solving the system by the PCG iteration procedure until numerical instability occurs.
- Switching from the PCG procedure to Gauss-Seidel procedure using as an initial approximated solution the values of unknowns obtained from the last PCG iteration.
- Proceeding with the Gauss-Seidel procedure until the desired error criterion is attained.

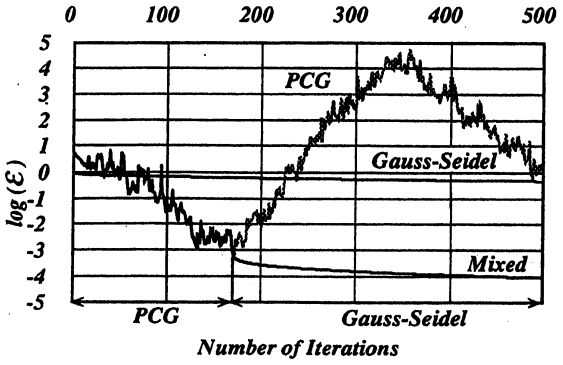


Fig. 3 Convergence rate for PCG, Gauss-Seidel and mixed solving procedures.

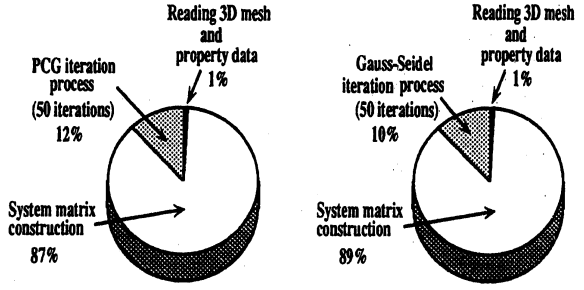


Fig. 4 A CPU usage for edge finite element analysis using the PCG and the Gauss-Seidel solving procedures, respectively.

The convergence rates for ungauged PCG, Gauss–Seidel and mixed procedures are presented in Fig. 3 for an axisymmetrical model with a large minimum PCG iteration rate (Fig. 1(b)). After the smallest residual that we were able to achieve by PCG iteration procedure was reached (for example 10^{-3}), we switched to the Gauss-Seidel iteration procedure using the results obtained from the last iteration of the PCG procedure as an initial approximated solution and the desired error criterion of 10^{-4} was relatively quickly attained. From Fig. 3 it is obvious that the mixed procedure enables a better convergence rate than either procedure alone, especially the Gauss–Seidel. The reasons, we believe, are twofold:

- We are still able to maintain an the excellent convergence rate of the PCG procedure at the beginning of the iteration process,
- We enter into the Gauss-Seidel procedure with an initial solution close enough to the exact solution to obtain a fast and uniform convergence rate.

We want to point out that due to fewer computational operations, the time per iteration is about 15 percent less for the Gauss-Seidel procedure than for the PCG procedure.

We also want to emphasize the main problem in the implementation of the proposed mixed solving procedure and its solution, i.e. the problem of identification when to stop the PCG iteration process and switch to the Gauss-Seidel iteration procedure. This problem occurs due to a nonuniform convergence rate of the PCG method. In other words, the exact determination of the smallest residual possible achieved by the PCG method is not easily accessible. We propose two algorithms for the solution of this problem:

1. After the generation of a system of equations, we store the matrix **S** and right-hand side **b** on the external memory device. Afterwards, monitoring the PCG iteration process, we are able to find the smallest residual value and the number of iterations when it occurs and restart the iteration process with the mixed solving procedure.
2. We monitor the PCG iteration process, and when numerical instability occurs, we again restart the iteration process with the mixed process. In this case, we use the factorization of the matrix **S** described in the second paragraph of this paper. This factorization easily enables reconstruction of the matrix **S**.

Both procedures required no additional operating memory, but additional computational work in the first procedure for input-output activities and in the second procedure for reconstruction of the matrix **S** was necessary. Both procedures, however, require monitoring of the PCG iteration procedure because of its high irregularity. The restarting procedure does not affect the total computational time because the largest portion of the total computational time is consumed by the construction of the matrix **S** which fortunately has to be performed only once (Fig. 4).

V. Conclusions

A new mixed PCG–Gauss-Seidel procedure for solving ungauged magnetic potential formulation in EFEM is proposed. This method requires the monitoring of the PCG iteration process, but not the construction of arbitrary or optimal tree graphs, rearrangement of the number of unknowns, nor additional memory. It is easily applicable to any model, especially to non-linear magnetostatic and eddy-current problems and for models where the solenoidal character of source current is hard to be satisfied exactly.

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