Fast Multigrid Solution Method for Nested Edge-Based Finite Element Meshes

Vlatko Cingoski, Ryutaro Tokuda, So Noguchi, and Hideo Yamashita

***Abstract*—In this paper a fast multigrid solution method for edge-based finite element magnetostatic field computation with nested meshes is introduced and its efficiency is investigated. Special prolongation and restriction matrices were constructed according to the nature of the edge based field approximation. The comparison of the computation speed between the multigrid method and the ICCG method is also presented, showing that the multigrid method is very promising as a fast solution method for large system of equations.**

***Index Terms*—Edge finite elements, finite element methods, iterative methods, magnetostatics, multigrid method, relaxation methods.**

I. INTRODUCTION

RECENTLY, due to the tremendous developments in the computer software and hardware technology, large-scale simulations became possible. Solution of various electromagnetic field problems by means of the finite element method (FEM) results in a solution of a large system of simultaneous algebraic equations. To increase the solution speed of such problems, various methods have already been proposed, the multigrid method being one of the most promising [1], [2]. However, almost all of the present algorithms for multigrid solutions have been based on nodal approximations, either for the finite difference method or for the finite element method [1]–[4].

Recently, mainly as a result of its good computational properties, the edge-based finite element method has emerged widely for electromagnetic field computation in 2D and 3D [5]. While in ordinary nodal finite element analysis, the unknowns are sought at each node of a mesh, in case of edge-based finite element analysis, the unknowns are associated with the edges (branches) of the finite element mesh. Therefore, it is obvious that already developed schemes for multigrid solution on nodal finite elements can not be used directly for edge-based finite elements. Some efforts to implement multigrid solution schemes for edge based elements also appear recently [6], [7].

In this paper, we present an investigation of the efficiency of the multigrid solution method for edge-based nested finite element meshes. For that purpose, we also developed typical prolongation and restriction matrices suitable for edge-based nested meshes calculations. The efficiency of the proposed method is verified using two 2D models.

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V. Cingoski is with the Electric Power Company of Macedonia, Skopje, Macedonia (e-mail: vlatko@esmak.com.mk).

R. Tokuda, S. Noguchi, and H. Yamashita are with Hiroshima University, Japan (e-mail: yama@eml.hiroshima-u.ac.jp).

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II. FORMULATION

This paper deals with 2D magnetostatic problems. In the case of edge-based finite element analysis, the gauge condition $\phi = 0$ can be used and the electric scalar potential ϕ is eliminated from the $\boldsymbol{A}$–ϕ formulation. The governing equation is given by

$$\nabla \times \left(\frac{1}{\mu} \nabla \times \boldsymbol{A}\right) = \boldsymbol{J}_s \tag{1}$$

where $\boldsymbol{A}$ and $\boldsymbol{J}_s$ are magnetic vector potential and source current, respectively, and μ is permeability.

III. MULTIGRID METHOD

The multigrid method as its name suggests is a numerical method for the solution of partial differential equations discretized using several grids with different grid densities. In its core it uses the characteristic of some iterative method to rapidly smooth the high-frequency components of the iteration error on dense meshes [3]. The Gauss–Seidel or the Jacobi iterative methods possess this smoothing property, therefore they are usually used in order to rapidly eliminate the high-frequency components of the error on the fine mesh. Low-frequency components, on the other hand are transformed to the coarse mesh, where a few iteration steps of the above-mentioned iterative methods again eliminate them. This procedure is usually called *the coarse grid correction scheme* and is briefly described below:

- Step 1) Relax a few times on $\boldsymbol{K}_2\boldsymbol{x}_2 = \boldsymbol{f}_2$ to obtain an approximation solution $\tilde{\boldsymbol{x}}_2$. The matrix $\boldsymbol{K}_2$ is the system matrix, and vectors $\boldsymbol{x}_2$ and $\boldsymbol{f}_2$ are the unknown vector and source vector, respectively. This step is called *smoothing*.
- Step 2) Compute the residual vector $\boldsymbol{r}_2 = \boldsymbol{f}_2 - \boldsymbol{K}_2\tilde{\boldsymbol{x}}_2$.
- Step 3) Project the residual vector onto a coarser mesh using the restriction operator $\boldsymbol{r}_1 = \boldsymbol{R}\boldsymbol{r}_2$.
- Step 4) Exactly solve the residual equation $\boldsymbol{K}_1\boldsymbol{e}_1 = \boldsymbol{r}_1$ to obtain an error approximation $\boldsymbol{e}_1$.
- Step 5) Interpolate the approximation error to the finer mesh using the prolongation operator $\boldsymbol{e}_2 = \boldsymbol{P}\boldsymbol{e}_1$.
- Step 6) Compute the improved solution $\boldsymbol{x}_2 = \tilde{\boldsymbol{x}} + \boldsymbol{e}_2$.
- Step 7) Continue into new iterative cycle by doing again smoothing.

At step 4, when the residual equation is solved, the coarse grid correction scheme can be used recursively. Then, the number of meshes becomes unrestricted and several iteration schemes are possible. Fig. 2(a)–(c) show the schedule of grids for each cycle on four grid levels. The schedules shown in Fig. 2(a) and (b) are

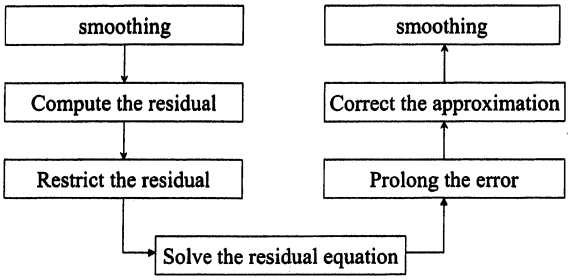

Fig. 1. The coarse grid correction scheme.

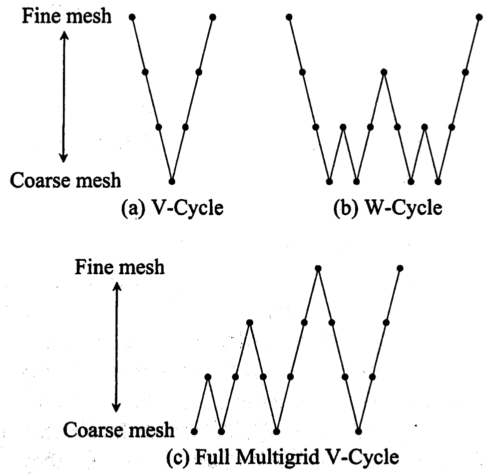

Fig. 2. Typical multigrid cycles.

called the V-cycle and the W-cycle, respectively, because of the pattern of this diagram. At each level, the coarse grid correction scheme is performed recursively one time for the V-cycle and two times for the W-cycle. For the full multigrid V-cycle, coarse grids are used to obtain improved initial guess for the V-cycles.

IV. Prolongation and Restriction Operators

For the interpolation of the edge values from coarser to finer mesh we need to define a *prolongation operator.* Additionally, for the projection of the edge values from finer to coarse grid we need to define an opposite operator than the prolongation one, what we call the *restriction operator.*

First, we explain the construction of the prolongation operator $\boldsymbol{P}$. Let $P_i(x_i, y_i)$ be the middle point of an edge i for which edge we want to get the prolongation value, while $A_{x,i}$ and $A_{y,i}$ are the values of the unknown magnetic vector potential at point P_i calculated using the coarser mesh:

$$\begin{bmatrix} A_{x,i} \\ A_{y,i} \end{bmatrix} = N \begin{bmatrix} A_I \\ A_J \\ A_k \end{bmatrix}, \qquad (1)$$

where N is the edge-based shape functions of the first order and A_I, A_J, and A_K are the unknowns along edges I, J, and K on the coarse mesh. Next, the value of the unknown component A'_i along the direction of the edge i is given by

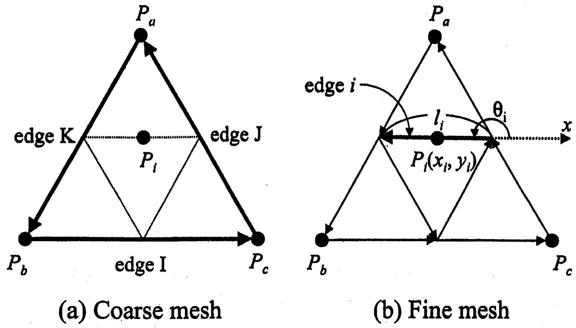

Fig. 3. Definition of the prolongation operator.

$$A'_i = [\cos\theta_i \quad \sin\theta_i] \begin{bmatrix} A_{x,i} \\ A_{y,i} \end{bmatrix}, \qquad (2)$$

where $A_{x,i}$ and $A_{y,i}$ are the unknown values at point P_i, and θ_i is the angle between x-axis and the direction of edge i [as shown in Fig. 3(a) and (b)]. Because A'_i is a value per unit length, the value that is assigned to edge i on the fine mesh can be given by multiplication of A'_i and the length l_i of edge i,

$$A_i = A'_i l_i. \qquad (3)$$

In general, from (1)–(3), the prolongation operator can be mathematically expressed as

$$A_i = [p_{iI} \quad p_{iJ} \quad p_{iK}] \begin{bmatrix} A_I \\ A_J \\ A_K \end{bmatrix}, \qquad (4)$$

where the coefficients p_{iI}, p_{iJ}, and p_{iK} are computed according to the geometry of the edge finite element as shown in Fig. 3. Finally, the global prolongation matrix $\boldsymbol{P}$ for the entire system can be constructed by summation of each contribution $[p_1 \quad p_2 \quad p_3]$ from all edges in the model. The size of the prolongation matrix $\boldsymbol{P}$ is $n_{fine} \times n_{coarse}$, where n_{fine} and n_{coarse} are the number of edges on the fine and the coarse finite element mesh, respectively.

The restriction matrix $\boldsymbol{R}$ is constructed as a transpose matrix of the above defined prolongation matrix $\boldsymbol{P}$. Prolongation and restriction matrices $\boldsymbol{P}$ and $\boldsymbol{R}$ are calculated according to the geometry of the meshes at pre-processing phase. During multigrid cycles, prolongation is performed by multiplication of the prolongation matrix $\boldsymbol{P}$ and the error vector $\boldsymbol{e}$. Similarly, restriction is performed by multiplication of the restriction matrix $\boldsymbol{R}$ and the residual vector $\boldsymbol{r}$.

V. Investigation of the Efficiency of the Multigrid Solution Method Using Edge FEM

The investigation of the efficiency of the proposed multigrid solution scheme was carried out for 2D magnetostatic models shown in Figs. 4 and 7. Test model #1, shown in Fig. 4 has a

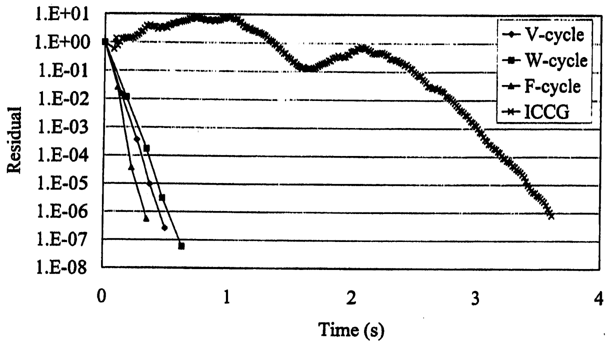

Fig. 8. Convergence characteristics.

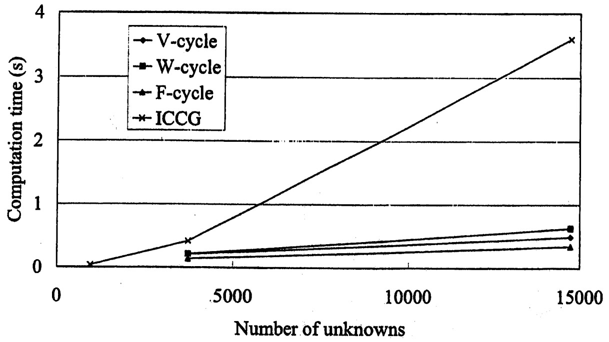

Fig. 9. Computation time versus number of unknowns.

computation time versus number of unknowns. Similarly, computation time of one multigrid method is almost independent of the number of unknowns. The memory requirements were 5.7 MB for the multigrid method, and 4.6 MB for the ICCG method.

VI. Conclusions

We presented an investigation of the efficiency of the multigrid method for solution of a system of linear algebraic equations obtained using edge finite element analysis based on nested meshes. Multigrid methods improve the convergence rate and decrease the computation time in comparison with the ICCG method with a modest increase of the used memory. The efficiency of the multigrid method increases with the increase of the size of the system matrix of the problem.

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