SIMPLE SUFFICIENT CONDITIONS FOR BOUNDED TURNING

N. TUNESKI, M. DARUS, AND E. GELOVA

ABSTRACT. Let f be an analytic function in the open unit disk normalized such that $f(0) = f'(0) - 1 = 0$. In this paper the modulus and the real part of the linear combination of $f'(z)$ and $f(z)/z$ is studied and conditions when f is with bounded turning are obtained.

1. Introduction and preliminaries

Let $\mathcal{A}$ denote the class of analytic functions in the unit disk $\mathbb{D} = \{z : |z| < 1\}$ that are normalized such that $f(0) = f'(0)-1 = 0$, i.e. $f(z) = z+a_2z^2+a_3z^3+\cdots$.

Function $f \in \mathcal{A}$ is in the class of *starlike functions*, S^*, if and only if

$$\mathrm{Re}\left[\frac{zf'(z)}{f(z)}\right] > 0, \quad z \in \mathbb{D}.$$

Such functions are univalent and their geometric characterization (which motivates the name of the class) is that they map the unit disk onto a starlike region, i.e. if $\omega \in f(\mathbb{D})$ then $t\omega \in f(\mathbb{D})$ for all $t \in [0, 1]$.

Another well known class of univalent functions is the *class of functions with bounded turning,*

$$R = \{f \in \mathcal{A} : \mathrm{Re}\, f'(z) > 0, z \in \mathbb{D}\}.$$

Here also, the name of class follows from its geometric characterization, i.e. from the fact that $\mathrm{Re}\, f'(z) > 0$ is equivalent with $|\arg f'(z)| < \pi/2$ and $\arg f'(z)$ is the angle of rotation of the image of a line segment from z under the mapping f.

More details on these classes can be found in [2]. One of the main results concerning them is due to Krzyz ([7]), claiming that S^* does not contain R and R does not contain S^*. This makes class R interesting and lots of research is dedicated to it. Some references in that direction are [6] – [9].

In this paper we will study the linear combination of two simple expressions, $f'(z)$ and $f(z)/z$, i.e. we will study the modulus and the real part of

$$\alpha \cdot f'(z) + \beta \cdot \frac{f(z)}{z} \tag{1}$$

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and receive criteria for a function $f \in \mathcal{A}$ to be of bounded turning. For that purpose we will use a method from the theory of differential subordinations. Valuable references on this topic are [1] and [3].

First we introduce subordination. Let $f, g \in \mathcal{A}$. Then we say that $f(z)$ is *subordinate* to $g(z)$, and write $f(z) \prec g(z)$, if there exists a function $\omega(z)$, analytic in the unit disc $\mathbb{D}$, such that $\omega(0) = 0$, $|\omega(z)| < 1$ and $f(z) = g(\omega(z))$ for all $z \in \mathbb{D}$. Specially, if $g(z)$ is univalent in $\mathbb{D}$ then $f(z) \prec g(z)$ if and only if $f(0) = g(0)$ and $f(\mathbb{D}) \subseteq g(\mathbb{D})$.

For obtaining the main result we will use the method of differential subordinations. The general theory of differential subordinations, as well as the theory of first-order differential subordinations, was introduced by Miller and Mocanu in [4] and [5]. Namely, if $\phi : \mathbb{C}^2 \to \mathbb{C}$ (where $\mathbb{C}$ is the complex plane) is analytic in a domain D, if $h(z)$ is univalent in $\mathbb{D}$, and if $p(z)$ is analytic in $\mathbb{D}$ with $(p(z), zp'(z)) \in D$ when $z \in \mathbb{D}$, then $p(z)$ is said to satisfy a first-order differential subordination if

$$\phi(p(z), zp'(z)) \prec h(z). \tag{2}$$

The univalent function $q(z)$ is said to be a *dominant* of the differential subordination (2) if $p(z) \prec q(z)$ for all $p(z)$ satisfying (2). If $\widetilde{q}(z)$ is a dominant of (2) and $\widetilde{q}(z) \prec q(z)$ for all dominants of (2), then we say that $\widetilde{q}(z)$ is the *best dominant* of the differential subordination (2).

From the theory of first-order differential subordinations we will make use of the following lemma.

Lemma 1 ([5])**.** *Let $q(z)$ be univalent in the unit disk $\mathbb{D}$, and let $\theta(\omega)$ and $\phi(\omega)$ be analytic in a domain D containing $q(\mathbb{D})$, with $\phi(\omega) \neq 0$ when $\omega \in q(\mathbb{D})$. Set $Q(z) = zq'(z)\phi(q(z))$, $h(z) = \theta(q(z)) + Q(z)$, and suppose that*

i) $Q(z)$ is starlike in the unit disk $\mathbb{D}$; and
ii) $\operatorname{Re} \frac{zh'(z)}{Q(z)} = \operatorname{Re} \left\{ \frac{\theta'(q(z))}{\phi(q(z))} + \frac{zQ'(z)}{Q(z)} \right\} > 0$, $z \in \mathbb{D}$.

If $p(z)$ is analytic in $\mathbb{D}$, with $p(0) = q(0)$, $p(\mathbb{D}) \subseteq D$ and

$$\theta(p(z)) + zp'(z)\phi(p(z)) \prec \theta(q(z)) + zq'(z)\phi(q(z)) = h(z) \tag{3}$$

then $p(z) \prec q(z)$, and $q(z)$ is the best dominant of (3).

Now, using Lemma 1 we will prove the following result.

Lemma 2. *Let $f \in \mathcal{A}$ and $\alpha, \beta \in \mathbb{C} \setminus \{0\}$ be such that $\alpha + \beta = 0$ or $\alpha + \beta = 1$. Also, let $q(z)$ be univalent in the unit disk $\mathbb{D}$ satisfying $q(0) = 0$ and*

$$\operatorname{Re} \left[1 + \frac{zq''(z)}{q'(z)} \right] > 0, \quad z \in \mathbb{D}. \tag{4}$$

Additionally, $\operatorname{Re}\frac{1}{\alpha} > -1$ *and*

$$\operatorname{Re}\left[1+\frac{zq''(z)}{q'(z)}\right] > -\operatorname{Re}\frac{1}{\alpha}, \quad z\in\mathbb{D}, \tag{5}$$

in the case when $\alpha+\beta=1$*. If*

$$\alpha\cdot f'(z)+\beta\cdot\frac{f(z)}{z} \prec (\alpha+\beta)\cdot[q(z)+1]+\alpha zq'(z)\equiv h(z) \tag{6}$$

then $\frac{f(z)}{z}-1\prec q(z)$*, and* $q(z)$ *is the best dominant of (6).*

Proof. Functions $\theta(\omega)=(\alpha+\beta)\cdot(\omega+1)$ and $\phi(\omega)=\alpha$ are analytic in a domain $D=\mathbb{C}$ which contains $q(\mathbb{D})$ and $\phi(\omega)\neq 0$ when $\omega\in q(\mathbb{D})$. Further, $Q(z)=zq'(z)\phi(q(z))=\alpha zq'(z)$ is starlike since

$$\operatorname{Re}\frac{zQ'(z)}{Q(z)} = \operatorname{Re}\left[1+\frac{zq''(z)}{q'(z)}\right]>0, \quad z\in\mathbb{D},$$

and for the function $h(z)=\theta(q(z))+Q(z)=Q(z)$ we have

$$\operatorname{Re}\frac{zh'(z)}{Q(z)} = \operatorname{Re}\left[1+\frac{\alpha+\beta}{\alpha}+\frac{zq''(z)}{q'(z)}\right]>0, \quad z\in\mathbb{D},$$

for $\alpha+\beta=0$ due to (4) and for $\alpha+\beta=1$ due to (5).

Now, let choose $p(z)=\frac{f(z)}{z}-1$ which is analytic in $\mathbb{D}$, $p(0)=q(0)=0$ and $p(\mathbb{D})\subseteq D=\mathbb{C}$. Finally, having in mind that subordinations (3) and (6) are equivalent, from Lemma 1 we receive the conclusions of Lemma 2. □

2. Results over the modulus of (1)

In this section we will study the modulus of (1) and receive conclusions that will lead to criteria for a function f to be in the class R.

Theorem 1. *Let* $f\in\mathcal{A}$*,* $\mu>0$ *and* $\alpha,\beta\in\mathbb{C}\setminus\{0\}$ *be such that* $\alpha+\beta=0$ *or* $\alpha+\beta=1$*. Also, let* $\operatorname{Re}\frac{1}{\alpha}>-1$ *in the case when* $\alpha+\beta=1$*. If*

$$\left|\alpha\cdot f'(z)+\beta\cdot\frac{f(z)}{z}-(\alpha+\beta)\right|<\delta\equiv\left\{\begin{array}{cc}\mu\cdot|\alpha|, & \alpha+\beta=0\\ \mu\cdot|1+\alpha|, & \alpha+\beta=1\end{array}\right., \tag{7}$$

for all $z\in\mathbb{D}$*, then*

$$\left|\frac{f(z)}{z}-1\right|<\mu, \quad z\in\mathbb{D}. \tag{8}$$

This implication is sharp, i.e., in the inequality (8), μ *can not be replaced by a smaller number so that the implication holds. Also,*

$$|f'(z)-1|<\lambda\equiv\left\{\begin{array}{cc}2\mu, & \alpha+\beta=0\\ \mu\cdot\left(\left|1+\frac{1}{\alpha}\right|+\left|1-\frac{1}{\alpha}\right|\right), & \alpha+\beta=1\end{array}\right\}, \quad z\in\mathbb{D}.$$

This implication is also sharp, i.e., λ *can not be replaced by a smaller number so that the implication holds, if*

(i) $\alpha+\beta=0$; *or*

(ii) $\alpha+\beta=1$ *and* $\left|1+\frac{1}{\alpha}\right|+\left|1-\frac{1}{\alpha}\right|=2$.

Additionally, if $\mu\le\frac{1}{2}$ *for* $\alpha+\beta=0$ *or* $\left|1+\frac{1}{\alpha}\right|+\left|1-\frac{1}{\alpha}\right|\le\frac{1}{\mu}$ *for* $\alpha+\beta=1$ *then* $f\in R$.

Proof. Choosing $q(z)=\mu z$ we have $1+\frac{zq''(z)}{q'(z)}=1$, meaning that (4) and (5) form Lemma 2 hold. Further, for the function $h(z)$ defined in (6) we have

$$h(z)=\alpha+\beta+\mu z(2\alpha+\beta),$$

meaning that subordination (6) is equivalent to

$$\left|\alpha\cdot f'(z)+\beta\cdot\frac{f(z)}{z}-(\alpha+\beta)\right|<\mu\cdot|2\alpha+\beta|=\delta,\quad z\in\mathbb{D},$$

i.e. to (7). Therefore, (8) follows directly from Lemma 2 and the definition of subordination.

Further, for all $z\in\mathbb{D}$,

$$\left|\alpha\cdot f'(z)+\beta\cdot\frac{f(z)}{z}-(\alpha+\beta)\right|=\left|\alpha\cdot[f'(z)-1]+\beta\cdot\left[\frac{f(z)}{z}-1\right]\right|$$

and

$$\begin{aligned}|\alpha|\cdot|f'(z)-1|&\le\left|\alpha\cdot[f'(z)-1]+\beta\cdot\left[\frac{f(z)}{z}-1\right]\right|+\left|\beta\cdot\left[\frac{f(z)}{z}-1\right]\right|\\&<\delta+|\beta|\cdot\mu=|\alpha|\cdot\lambda,\end{aligned}$$

since $|w_1|\le|w_1+w_2|+|w_2|$. Therefore, the implication of this corollary holds.

Both implication are sharp as the function $f_*(z)=z+\mu z^2$ shows, since

$$\left|\alpha\cdot f_*'(z)+\beta\cdot\frac{f_*(z)}{z}-(\alpha+\beta)\right|=\mu\cdot|2\alpha+\beta|\cdot|z|=\delta\cdot|z|,\quad z\in\mathbb{D},$$

$$\left|\frac{f_*(z)}{z}-1\right|=\mu\cdot|z|,\quad z\in\mathbb{D},$$

$$|f_*'(z)-1|=2\cdot\mu\cdot|z|,\quad z\in\mathbb{D},$$

and $2\mu=\lambda$ if (i) or (ii) hold. □

3. Results over the real part of (1)

In this section we will study the real part of the expression (1) and receive criteria over it that will embed a function $f\in\mathcal{A}$ in the class R.

Theorem 2. *Let* $f\in\mathcal{A}$, $\mu>0$ *and* $\alpha,\beta\in\mathbb{C}\setminus\{0\}$ *be such that* $\alpha+\beta=0$ *or* $\alpha+\beta=1$. *Also, let* $\operatorname{Re}\frac{1}{\alpha}>0$ *in the case when* $\alpha+\beta=1$. *If*

$$\alpha\cdot f'(z)+\beta\cdot\frac{f(z)}{z}\prec(\alpha+\beta)\left(1+\frac{2\mu z}{1-z}\right)+\frac{2\alpha\mu z}{(1-z)^2}\equiv h_2(z)\tag{9}$$

then

$$\operatorname{Re}\left[\frac{f(z)}{z}\right] > 1-\mu, \quad z\in\mathbb{D}. \tag{10}$$

This implication is sharp, i.e., in the inequality (10), μ can not be replaced by a bigger number so that the implication holds.

Proof. The implication of this theorem follows directly from Lemma 2 for $q(z) = \frac{2\mu z}{1-z}$. Condition $\operatorname{Re}\frac{1}{\alpha} > 0$ stands in stead of $\operatorname{Re}\frac{1}{\alpha} > -1$ in order (5) to hold. The result is sharp due to the function $f_*(z) = z + z\cdot q(z)$ such that

$$\alpha\cdot f_*'(z) + \beta\cdot\frac{f_*(z)}{z} = (\alpha+\beta)\left(1+\frac{2\mu z}{1-z}\right) + \frac{2\alpha\mu z}{(1-z)^2}$$

and $\operatorname{Re}\frac{f(z)}{z} = 1-\mu$ for $z=-1$. □

In the case when $\alpha+\beta = 1$ we receive the following corollary.

Corollary 1. *Let $f\in\mathcal{A}$, $\alpha>0$ and $\mu>0$. If*

$$\operatorname{Re}\left[\alpha\cdot f'(z) + (1-\alpha)\cdot\frac{f(z)}{z}\right] > 1-\mu\cdot\left(1+\frac{\alpha}{2}\right), \quad z\in\mathbb{D}, \tag{11}$$

then

$$\operatorname{Re}\left[\frac{f(z)}{z}\right] > 1-\mu, \quad z\in\mathbb{D}.$$

If, additionally,

(i) $\alpha > 1$ *and* $\mu\le 1$; *or*

(ii) $\alpha < 1$ *and* $\mu \ge 1$;

then

$$\operatorname{Re} f'(z) > 1-\frac{3}{2}\cdot\mu, \quad z\in\mathbb{D}. \tag{12}$$

These results are sharp.

Proof. Let $\alpha+\beta=1$. So, for the function h_2 defined in (9) we have

$$h_2(z) = 1+\frac{2\mu z}{1-z}+\frac{2\alpha\mu z}{(1-z)^2},$$

$h_2(0) = 1$ and

$$h_2(e^{i\theta}) = 1-\frac{\mu\alpha}{2}(1+t^2)-\mu+\mu t i,$$

where $t = \operatorname{ctg}(\theta/2)$. Therefore,

$$X = \operatorname{Re} h(e^{i\theta}) = 1-\mu\left(\frac{\alpha}{2}+1\right)-\frac{\alpha}{2\mu}\cdot Y^2,$$

where

$$Y = \operatorname{Im} h(e^{i\theta}) = \mu t$$

attains all real numbers. This leads to

$$h_2(e^{i\theta}) = \left\{ x + iy : x = 1 - \mu \left(1 + \frac{\alpha}{2}\right) - \frac{\alpha}{2\mu} \cdot y^2, y \in \mathbb{R} \right\}.$$

From here, having in mind the definition of subordination, the inequality (11) and the fact that

$$\left\{ x + iy : x > 1 - \mu \left(1 + \frac{\alpha}{2}\right), y \in \mathbb{R} \right\} \subseteq h_2(\mathbb{D}),$$

we receive subordination (9). Therefore, from Theorem 2 follows

$$\mathrm{Re} \left[\frac{f(z)}{z}\right] > 1 - \mu, \quad z \in \mathbb{D}.$$

Further, in the case when (i) or (ii) holds we have

$$\begin{aligned} \mathrm{Re}\, f'(z) &= \frac{1}{\alpha} \cdot \left\{ \mathrm{Re} \left[\alpha \cdot f'(z) + (1-\alpha) \cdot \frac{f(z)}{z}\right] - (1-\alpha) \cdot \mathrm{Re} \left[\frac{f(z)}{z}\right] \right\} \\ &> \frac{1}{\alpha} \cdot \left[1 - \mu \left(1 + \frac{\alpha}{2}\right) - (1-\alpha)(1-\mu)\right] = 1 - \frac{3}{2} \cdot \mu, \end{aligned}$$

for all $z \in \mathbb{D}$.

The results are sharp due to the function $f_*(z) = z + \frac{2\mu z^2}{1-z}$ such that $f_*(z)/z = 1 + \frac{2\mu z}{1-z} \equiv g(z)$, $g(\mathbb{D}) = \{x + iy : x > 1 - \mu, y \in \mathbb{R}\}$,

$$\alpha \cdot f_*'(z) + (1-\alpha) \cdot \frac{f_*(z)}{z} = h_2(z)$$

and

$$\mathrm{Re}\, f_*'(z) = \mathrm{Re}\, h_2(z) = 1 - \frac{3}{2} \cdot \mu \quad \text{for} \quad z = -1.$$

□

In a similar way as in Corollary 1, for the case $\alpha = -\beta = 1$ we receive

Corollary 2. *Let $f \in \mathcal{A}$ and $\mu > 0$. If*

$$\mathrm{Re} \left[f'(z) - \frac{f(z)}{z}\right] > -\frac{\mu}{2}, \quad z \in \mathbb{D}, \tag{13}$$

then $\mathrm{Re} \left[\frac{f(z)}{z}\right] > 1 - \mu$, $z \in \mathbb{D}$, *and* $\mathrm{Re}\, f'(z) > 1 - \frac{3}{2} \cdot \mu$, $z \in \mathbb{D}$. *If, additionally,* $\mu \le \frac{2}{3}$, *then* $\mathrm{Re}\, f'(z) > 0$, $z \in \mathbb{D}$, *i.e.* $f \in R$. *Both implications are sharp.*

4. Examples

The following example exhibits some concrete conclusions that can be obtained from the results of the previous sections by specifying the values α, β and μ.

Example 1. *Let $f \in \mathcal{A}$.*

(i) *If* $\left|f'(z) - \frac{f(z)}{z}\right| < \frac{1}{2}$ $(z \in \mathbb{D})$ *then* $|f'(z) - 1| < 1$ $(z \in \mathbb{D})$ *and* $f \in R$. *($\alpha = -\beta = 1$ and $\mu = \frac{1}{2}$ in Theorem 1);*

(ii) *If* $\left|f'(z)+\frac{f(z)}{z}-2\right|<1$ $(z\in\mathbb{D})$ *then* $|f'(z)-1|<1$ $(z\in\mathbb{D})$ *and* $f\in R$. *(*$\alpha=\beta=\frac{1}{2}$ *and* $\mu=\frac{1}{4}$ *in Theorem 1);*

(iii) *If* $\alpha>0$ *and* $\mathrm{Re}\left[\alpha\cdot f'(z)+(1-\alpha)\cdot\frac{f(z)}{z}\right]>-\frac{\alpha}{2}$ $(z\in\mathbb{D})$ *then* $\mathrm{Re}\left[\frac{f(z)}{z}\right]>0$ $(z\in\mathbb{D})$ *and* $\mathrm{Re}\,f'(z)>-1/2$ $(z\in\mathbb{D})$. *(*$\mu=1$ *in Corollary 1);*

(iv) *If* $\mathrm{Re}\left[f'(z)+\frac{f(z)}{z}\right]>-\frac{1}{2}$ $(z\in\mathbb{D})$ *then* $\mathrm{Re}\left[\frac{f(z)}{z}\right]>0$ $(z\in\mathbb{D})$ *and* $\mathrm{Re}\,f'(z)>-1/2$ $(z\in\mathbb{D})$. *(*$\alpha=1/2$ *and* $\mu=1$ *in Corollary 1);*

(v) *If* $\mathrm{Re}\left[f'(z)-\frac{f(z)}{z}\right]>-\frac{1}{3}$ $(z\in\mathbb{D})$ *then* $\mathrm{Re}\,f'(z)>0$ $(z\in\mathbb{D})$ *and* $f\in R$. *(*$\mu=\frac{2}{3}$ *in Corollary 2);*

Remark 1. *It is worth noting that in part (iii) of the previous example, the conclusion does not depend on* α.

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Faculty of Mechanical Engineering, Ss. Cyril and Methodius University in Skopje, Karpoš II b.b., 1000 Skopje, Republic of Macedonia.

E-mail address: nikola.tuneski@mf.edu.mk

School of Mathematical Sciences,Faculty of science and Technology Universiti Kebangsaan Malaysia, Bangi 43600, Selangor Darul Ehsan, Malaysia

E-mail address: maslina@ukm.my

University Goce Delcev, Štip, Republic of Macedonia

E-mail address: gelovae@yahoo.com