12185. *Proposed by George Stoica, Saint John, NB, Canada.* Let $n_1, \ldots, n_k$ be pairwise relatively prime integers greater than 1. For $i \in \{1, \ldots k\}$, let $f_i(x) = \sum_{m=1}^{n_i} x^{m-1}$. Let A be a $2n$-by-$2n$ matrix with real entries such that $\det f_j(A) = 0$ for all $j \in \{1, \ldots, k\}$. Prove $\det A = 1$.

12186. *Proposed by Anatoly Eydelzon, University of Texas at Dallas, Richardson, TX.* For $v = \langle x_1, \ldots, x_n \rangle$ in $\mathbb{R}^n$, let $\|v\|_p = \left(\sum_{i=1}^n |x_i|^p\right)^{1/p}$ and $\|v\|_\infty = \max_{1 \le i \le n} |x_i|$; these are the usual p-norm and ∞-norm on $\mathbb{R}^n$. For what v does the series

$$\sum_{p=1}^{\infty} \Big(\|v\|_p - \|v\|_\infty \Big)$$

converge?

12187. *Proposed by Khakimboy Egamberganov, Sorbonne University, Paris, France.* Given a scalene triangle ABC, let M be the midpoint of BC, and let m and s denote the median and symmedian lines, respectively, from A. (The *symmedian* line from A is the reflection of the median from A across the angle bisector from A.) Let K be the projection of C onto m, and let L be the projection of B onto s. Let P be the intersection of BL and CK, and let Q be the intersection of KL and BC. Prove that PM and AQ are perpendicular.

SOLUTIONS

Inequalities with Altitudes and Exradii

12068 [2018, 756]. *Proposed by D. M. Bătineţu-Giurgiu, "Matei Basarab" National College, Bucharest, Romania, and Neculai Stanciu, "George Emil Palade" School, Buzău, Romania.* Consider a triangle with altitudes h_a, h_b, and h_c and corresponding exradii r_a, r_b, and r_c. Let s, r, and R denote the triangle's semiperimeter, inradius, and circumradius, respectively.
(a) Prove

$$\frac{h_b + h_c}{h_a} r_a^2 + \frac{h_c + h_a}{h_b} r_b^2 + \frac{h_a + h_b}{h_c} r_c^2 \ge 2s^2.$$

(b) Prove

$$\frac{r_b + r_c}{r_a} h_a^2 + \frac{r_c + r_a}{r_b} h_b^2 + \frac{r_a + r_b}{r_c} h_c^2 \ge \frac{4s^2 r}{R}.$$

Solution by Said Amghibech, Quebec City, QC, Canada. By the AM–GM inequality, we have

$$\frac{h_a}{h_b} r_b^2 + \frac{h_b}{h_a} r_a^2 \ge 2 r_a r_b \quad \text{and} \quad \frac{r_a}{r_b} h_b^2 + \frac{r_b}{r_a} h_a^2 \ge 2 h_a h_b.$$

Summing the left sides of these inequalities cyclically gives the left sides of (a) and (b), respectively. Writing K for the area of the triangle, we have $K = r_a(s-a) = r_b(s-b) = r_c(s-c)$ and $K^2 = s(s-a)(s-b)(s-c)$. Hence

$$\begin{aligned} r_a r_b + r_a r_c + r_b r_c &= \frac{K^2}{(s-a)(s-b)} + \frac{K^2}{(s-a)(s-c)} + \frac{K^2}{(s-b)(s-c)} \\ &= s(s-c) + s(s-b) + s(s-c) = s^2. \end{aligned}$$

Part (a) now follows. Part (b) follows from

$$h_a h_b + h_a h_c + h_b h_c = \frac{2K}{a}\frac{2K}{b} + \frac{2K}{a}\frac{2K}{c} + \frac{2K}{b}\frac{2K}{c}$$
$$= \frac{4K^2(a+b+c)}{abc} = \frac{4K^2(a+b+c)}{4KR} = \frac{K}{R}(2s) = \frac{2s^2 r}{R}.$$

Editorial comment. A typographical error in the original printing has been corrected. Several solvers used Chebyshev's or Muirhead's inequality. Many solvers showed that equality holds if and only if the triangle is equilateral.

Also solved by M. Bataille (France), P. P. Dályay (Hungary), G. Fera (Italy), O. Geupel (Germany), L. Giugiuc (Romania), J. G. Heuver (Canada), W. Janous (Austria), K. T. L. Koo (China), S. S. Kumar, M. Lukarevski (Macedonia), D. Ş. Marinescu & M. Monea (Romania), P. Nüesch (Switzerland), A. Stadler (Switzerland), R. Stong, M. Vowe (Switzerland), T. Wiandt, M. R. Yegan (Iran), J. Zacharias, L. Zhou, GCHQ Problem Solving Group (UK), and the proposer.

A Trigonometric Double Integral

12070 [2018,851]. *Proposed by Cornel Ioan Vălean, Teremia Mare, Romania.* Prove

$$\int_0^{\pi/4}\int_0^{\pi/4} \frac{\cos x \cos y\,(y\,\sin y\,\cos x - x\,\sin x\,\cos y)}{\cos(2x)-\cos(2y)}\,dx\,dy = \frac{7\zeta(3)+4\pi\ln 2}{64},$$

where ζ is the Riemann zeta function.

Solution by Theo Koupelis, Broward College, Pembroke Pines, FL. Using the double angle and sum-to-product trigonometric identities, we can rewrite the integrand as

$$\frac{y\sin(2y)\big(1+\cos(2x)\big) - x\sin(2x)\big(1+\cos(2y)\big)}{-8\sin(x-y)\sin(x+y)},$$

a form that suggests the substitution $u = x + y$ and $v = x - y$. With this substitution, the integrand takes the form

$$\frac{1}{8}(\cos u + \cos v)\left(\frac{u}{\sin u} + \frac{v}{\sin v}\right)$$

which is

$$\frac{1}{8}\left(\frac{u}{\tan u} + \frac{v}{\tan v}\right) + \frac{1}{8}\left(\frac{u\,\cos v}{\sin u} + \frac{v\,\cos u}{\sin v}\right).$$

This change of variables for the integration has a corresponding Jacobian of $1/2$, and the region of integration changes from the square with vertices at $(0,0)$, $(\pi/4,0)$, $(\pi/4,\pi/4)$, and $(0,\pi/4)$ to the square S with vertices at $(0,0)$, $(\pi/4,\pi/4)$, $(\pi/2,0)$, and $(\pi/4,-\pi/4)$. As a result, the integral is equal to

$$\frac{1}{16}\iint_S\left(\frac{u}{\tan u} + \frac{v}{\tan v}\right)du\,dv + \frac{1}{16}\iint_S\left(\frac{u\,\cos v}{\sin u} + \frac{v\,\cos u}{\sin v}\right)du\,dv.$$

Denote the two summands in this expression by J_1 and J_2, respectively. The integrands for both J_1 and J_2 are even functions of v, and it therefore suffices to integrate over the triangle with vertices at $(0,0)$, $(\pi/4,\pi/4)$, $(\pi/2,0)$ and then double the result. Integration leads to the expressions

$$J_1 = \frac{\pi}{16}\int_0^{\pi/2}\frac{t}{\tan t}\,dt - \frac{1}{8}\int_0^{\pi/2}\frac{t^2}{\tan t}\,dt \quad\text{and}\quad J_2 = \frac{1}{8}\int_0^{\pi/2}\frac{t}{\tan t}\,dt.$$