CHARACTERISTICS OF PORTFOLIO UNDER RISK

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Abstract

The investor cannot associate a single number or payoff with investment in any asset because there is the existence of risk. The payoff should be described by a set of outcomes and each of their associated probability of occurence, which is called frequency function or return distribution. There is the examination of the two most frequently employed attributes of such a distribution: the expected return and the standard deviation. Because investors don't hold single assets, there is a concern with how one can compute the expected return and risk of a portfolio of assets given the attributes of the individual assets. One important aspect of this analysis is that the risk on a portfolio is more complex than a simple average of the risk on individual assets. So, this paper is showing how the risk of the portfolio of assets can be very different from the risk of the individual assets comprising the portfolio.

Key words: Average outcome, Dispersion, Variance, Portfoilo

JEL Classification: G11

Introduction

The nature of the opportunity set under risk is the basis of the paper. This paper is concerned with the expected return and risk of a portfolio of assets which can be computed given the attributes of the individual assets. The risk on a portfolio depends on whether the returns of individual assets tend to move together or weather some assets give good returns and other give bad returns. . It is more important to consider the combination of assets rather than just the assets themselves. Also, the distribution of outcomes on combination of assets can be different than the distributions on the individual assets. When two assets have their good and poor returns at opposite times, an investor can always find some combination of these assets that yields the same return under all market conditions.

Characteristics of a portfolio

The return on a portfolio of assets is weighted average of the return on the individual assets. The weight (applied to each return) is the fraction of the portfolio invested in that asset. If R_{Pj} is the jth return on the portfolio and X_i is the fraction of the investor's funds invested in the ith asset, and N is the number of assets, then

$$R_{Pj} = \sum_{i=1}^{N} (X_i R_{ij}) \tag{1}$$

The expected return is also a weighted average of the expected returns on the individual assets. Taking the expected value of the expression just given for the return on a portfolio yields

$$\overline{R}_P = E(R_P) = E\left(\sum_{i=1}^{N} X_i R_{ij}\right) \tag{2}$$

The expected value of the sum of various returns is the sum of the expected values. So,

$$\overline{R}_P = \sum_{i=1}^{N} E\left(X_i R_{ij}\right) \tag{3}$$

Finally, the expected value of a constant times a return is a constant times the expected return, or

$$\overline{R}_P = \sum_{i=1}^{N} \left(X_i \overline{R}_i\right) \tag{4}$$

This is general formula. To illustrate its use, we can consider the investment in assets 2 and 3 in Table 1. Assume that an investor has $1 to invest. If he select asset 2 and the market is good, he will have at the end of the period $1.16, if the market performance is average he will have $1.10, and if it is poor $1.04. But, if the investor invests $0.6 in asset 2 and $0.4 in asset 3, if the condition of the market is good, the investor will have $0.696 at the end of the period from asset 2 and $0.404 from asset 3, or $1.10. If the market conditions are average, he will receive $1.10. If the market conditions are poor, the investor will receive again $1.10. So, no matter what occurred, the investor would receive $1.10 on an investment of $1.00. This is a return of 0.10/1.00 = 10%.

Table 1 Returns on Various Investments

Market Condition	Return				
	Asset 1	Asset 2	Asset 3	Asset 4	Asset 5
Good	15	16	1	16	16
Average	9	10	10	10	10
Poor	3	4	19	4	4
Mean return	9	10	10	10	10
Variance	24	24	54	24	24
Standard deviation	4.9	4.9	7.35	4.90	4.9

If $0.60 is invetsted in asset 2 and $0.40 in asset 4, the fraction invested in asset 4 is 0.40/1.00. Furthermore, the expected return on asset 2 and asset 4 is 10%. Applying the formula for expected return on a portfolio yields.

$$\overline{R}_P = \left(\frac{0.60}{1.00}\right)(0.10) + \left(\frac{0.40}{1.00}\right)(0.10) = 0.10$$

The second characteristic is the variance. The variance on a portfolio is a little more difficult to determine than the expected return. In the case of two – asset example the variance of a portfolio *P*, designated by σ_P^2, is simply the expected value of the squared deviations of the return on the portfolio from the mean return on the portfolio, or $\sigma_P^2 = E\left(R_P - \overline{R}_P\right)^2$.Substituting in this expression the formulas for return on the portfolio and mean return yields in the two – security case

$$\sigma_P^2 = E\left(R_p - \overline{R}_P\right)^2 = E\left[X_1 R_{1j} + X_2 R_{2j} - \left(X_1 \overline{R}_1 + X_2 \overline{R_2}\right)\right]^2 = E\left[X_1\left(R_{1j} - \overline{R}_1\right) + X_2\left(R_{2j} - \overline{R}_2\right)\right]^2 \tag{5}$$

where $\overline{R}_i$ stands for the expected value of security *i* with respect to all possible outcomes. Because

$$\left(X + Y\right)^2 = X^2 + XY + XY + Y^2 = X^2 + 2XY + Y^2$$

Appling this to the previous expression derive that

$$\sigma_P^2 = E\left[X_1^2\left(R_{1j} - \overline{R}_1\right)^2 + 2X_1X_2\left(R_{1j} - \overline{R}_1\right)\left(R_{2j} - \overline{R}_2\right) + X_2^2\left(R_{2j} - \overline{R}_2\right)^2\right] \tag{6}$$

Applying the two rules that the expected value of the sum of a series of returns is equal to the sum of the expected value of each return, and that expected value of a constant times a return is equal to the constant times the expected return, we have

$$\begin{aligned}\sigma_P^2 &= X_1^2 E\left[\left(R_{1j} - \overline{R}_1\right)^2\right] + 2X_1X_2E\left[\left(R_{1j} - \overline{R}_1\right)\left(R_{2j} - \overline{R}_2\right)\right] + X_2^2E\left[\left(R_{2j} - \overline{R}_2\right)^2\right] \\ &= X_1^2\sigma_1^2 + 2X_1X_2E\left[\left(R_{1j} - \overline{R}_1\right)\left(R_{2j} - \overline{R}_2\right)\right] + X_2^2\sigma_2^2\end{aligned} \tag{7}$$

$E\left[\left(R_{1j} - \overline{R}_1\right)\left(R_{2j} - \overline{R}_2\right)\right]$ is the covariance and will be designated as σ_{12}. Substituting the symbol σ_{12} for $E\left[\left(R_{1j} - \overline{R}_1\right)\left(R_{2j} - \overline{R}_2\right)\right]$ yields

$$\sigma_P^2 = X_1^2\sigma_1^2 + X_2^2\sigma_2^2 + 2X_1X_2\sigma_{12} \tag{8}$$

Here covariance is the expected value of the product of two deviations: the deviations of the returns on security 1 from its mean $\left(R_{1j} - \overline{R}_1\right)$ and the deviations of security 2 from its mean $\left(R_{2j} - \overline{R}_2\right)$. In this sense it is very much like the variance. However, it is the product of two different deviations. As such it can be positive or negative. It will be large when the good outcomes for each stock occur together and when the bad outcomes for each stock occur together. In this case, for good outcomes the covariance will be the product of two large positive numbers, which is positive. When the bad outcomes occur, the covariance will be the product of two large negative numbers, which is positive. This will result in a large value for the covariance and a large variance for the portfolio. In contrast, if good outcomes for one asset are associated with bad outcomes of the other, the covariance is negative. It is negative because a plus deviation for one asset is associated with a minus deviation for the second and the product of a plus and a minus is negative.

The covariance is a measure of how returns on assets move together. Insofar as they have positive and negative deviations at similar times, the covariance is a large positive number. If they have the positive and negative deviations at dissimilar times, then the covariance is negative. If the positive and negative deviations are unrelated, it tends to be zero.

Dividing the covariance between two assets by the product of the standard deviation of each asset produces a variable with the same properties as the covariance but with a range of –1 to +1, which is the correlation coefficient. Letting ρ_{ik} stand for the correlation between securities *i* and *k* the correlation coefficient is defined as

$$\rho_{ik} = \frac{\sigma_{ik}}{\sigma_i\sigma_k} \tag{9}$$

Dividing by the product of the standard deviations does not change the properties of the covariance. It simply scales it to have values between -1 to +1. Let apply these formulas.

Table 2 Calculating Covariance

Condition of Market	Deviations Security 1	Deviations Security 2	Products of Deviations	Deviations Security 1	Deviations Security 3	Products of Deviations
Good	(15 - 9)	(16 - 10)	36	(15 - 9)	(1 - 10)	-54
Average	(9 - 9)	(10 - 10)	0	(9 - 9)	(10 - 10)	0
Poor	(3 - 9)	(4 - 10)	36	(3 - 9)	(19 - 10)	-54
			72			-108

Table 2 shows the intermediate calculations necessary to determine the covariance between securities 1 and 2 and securities 1 and 3. The sum of the deviations between securities 1 and 2 is 72. Therefore, the covariance is 72/3 = 24 and the correlation coefficient is $24/\sqrt{24}\sqrt{24}$. For assets 1 and 3 the sum of the deviations is -108. The covariance is -108/3 = -36 and the correlation coefficient is $-36/\sqrt{24}\sqrt{54}$. Similar calculations can be made for all other pairs of assets, and the results are contained in Table 3.

Table 3 Covariance and Correlation Coefficients (in Brackets) Between Assets

1	2	3	4	5
1	24 (+ 1)	-36 (- 1)	0 0	24 (+ 1)
2		-36 (- 1)	0 0	24 (+ 1)
3			0 0	-36 (- 1)
4				0 0
5				

We examined the results obtained by an investor with $1.00 to spend who put $0.60 in asset 2 and $0.40 in asset 3. Applying the expression for variance of the portfolio we have

$$\sigma_P^2 = \left(\frac{0.60}{1.00}\right)^2 24 + \left(\frac{0.40}{1.00}\right)^2 54 + 2\left(\frac{0.60}{1.00}\right)\left(\frac{0.40}{1.00}\right)(-36) = 0$$

The correlation coefficient between securities 2 and 3 is -1. This meant that good and bad returns of assets 2 and 3 tended to occur at opposite times. When this situation occurs, a portfolio can always be constructed with zero risk.

Without the formula, we calculated the variance directly. There are nine possible returns when we combined assets 2 and 4. They are $1.16, $1.13, $1.13, $1.10, $1.10, $1.10, $1.07, $1.07and $1.04. Since start of the investment is $1.00, the returns are easy to determine. The return is 16%, 13%, 13%, 10%, 10%,10%, 7%, 7%, and 4%. By examination it is easy to see that the mean return is 10%. The deviations are 6, 3, 3, 0, 0, 0, -3, -3, -6. The squared deviations are 36, 9, 9, 0, 0, 0, 9, 9, 36, and the average squared deviation or variance is 108/9 = 12.

The formula for variance of a portfolio can be generalized to more than two assets. We first started with a three- asset case. Substituting the expression for return on a portfolio and expected return of a portfolio in the general formula for variance yields

$$\sigma_P^2 = E(R_P - \overline{R}_P)^2$$
$$= E\left[X_1 R_{1j} + X_2 R_{2j} + X_3 R_{3j} - \left(X_1 \overline{R}_1 + X_2 \overline{R}_2 + X_3 \overline{R}_3\right)\right]^2$$

Rearranging,

$$\sigma_P^2 = E\left[X_1 (R_{1j} - \overline{R}_1) + X_2 (R_{2j} - \overline{R}_2) + X_3 (R_{3j} - \overline{R}_3)\right]^2$$

Squaring the right- hand side yields

$$\sigma_P^2 = E[\; X_1{}^2 (R_{1j} - \overline{R}_1)^2 + X_2{}^2 (R_{2j} - \overline{R}_2)^2 + X_3{}^2 (R_{3j} - \overline{R}_3)^2$$
$$+ 2X_1 X_2 (R_{1j} - \overline{R}_1)(R_{2j} - \overline{R}_2) + 2X_1 X_3 (R_{1j} - \overline{R}_1)(R_{3j} - \overline{R}_3)$$
$$+ 2X_2 X_3 (R_{2j} - \overline{R}_2)(R_{3j} - \overline{R}_3) \;]$$

Applying the properties of expected return yields

$$\sigma_P^2 = X_1^2 E(R_{1j} - \overline{R}_1)^2 + X_2^2 E(R_{2j} - \overline{R}_2)^2 + X_3^2 E(R_{3j} - \overline{R}_3)^2$$
$$+ 2X_1 X_2 E\left[(R_{1j} - \overline{R}_1)(R_{2j} - \overline{R}_2)\right] + 2X_1 X_3 E\left[(R_{1j} - \overline{R}_1)(R_{3j} - \overline{R}_3)\right]$$
$$+ 2X_2 X_3 E\left[(R_{2j} - \overline{R}_2)(R_{3j} - \overline{R}_3)\right]$$

Utilizing σ_i^2 for variance of asset i and σ_{ij} for the covariance between assets i and j, we got

$$\sigma_P^2 = X_1^2 \sigma_1^2 + X_2^2 \sigma_2^2 + X_3^2 \sigma_3^2 + 2X_1 X_2 \sigma_{12} + 2X_1 X_3 \sigma_{13} + 2X_2 X_3 \sigma_{23} \tag{10}$$

This formula can be extended to any number of assets. Examining the expression for the variance of a portfolio of three assets should indicate how. The variance of each asset is multiplied by the square of the proportion invested in it. Thus, the first part of the expression for the variance of a portfolio is the sum of the variances on the individual assets times the square of the proportion invested in each, or

$$\sum_{i=1}^{N} (X_i^2 \sigma_i^2) \tag{11}$$

Conclusion

The risk of the portfolio of assets can be very different from the risk of the individual assets comprising the portfolio. The covariance and correlation coefficient analysis for different securities show us that when good and bad returns of two assets tended to occur at opposite times, a portfolio can always be constructed with zero risk. Also, we derive this formula that can be extended for any number of assets. Also, this paper analyze and describe with examples the formula for the return on a portfolio of assets as weighted average of the return on the individual assets.

Bibliography

Elton, Edwin J., and Gruber, Martin J. "Risk reduction and portfolio size :An analytical Solution." *Journal of Business*, 50, No.4 (1977).

Markowitz, Nanry. "Markowitz revisited," *Financial Analysts Journal*, 32, No. 4 (1976)

Ross, Stephen A. "Adding risks: Samuelson's Fallacy of Large Number Revisited" *Journal of Financial and Quantitative Analysis*, (1999).

Rubinstein, Mark. "The fundamental theorem of Parameter-Preference Security Valuation", *Journal of Financial and Quantitative Analysis.* (1973)

Wagner, W. and Lau, S. "The Effects of Diversification on Risk," *Financial Analysts Journal*, (1971)

Withmore, G.A. "Diversification and the Reduction of the Dispersion: A Note" *Journal of Financial and Quantitative Analysis* (1970)